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THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

Cambridge, Massachusetts, U.S.A.
August 30–September 6, 1950

As already announced in the July, 1948, issue of this Bulletin, an International Congress of Mathematicians will be held in Cambridge, Massachusetts, in 1950 under the auspices of the American Mathematical Society. The Society originally planned to act as host for a Congress in September, 1940, which was also scheduled to meet in Cambridge. Plans for the 1940 Congress were practically completed when the outbreak of World War II in September, 1939, made it necessary for the Society to postpone the Congress to a more favorable date.

The 1950 Congress will be the third International Congress of Mathematicians to be held on the continent of North America. The first was held at Northwestern University in 1893 and the second at the University of Toronto in 1924. International Congresses were held at intervals of approximately four years, except when war intervened, until 1936. There has been no international gathering of mathematicians since that time and it is the sincere hope of the Organizing Committee that the gathering in 1950 will be a truly international one, that American mathematicians will attend in large numbers, and that all other countries will be well represented. The Council of the American Mathematical Society has voted unanimously to hold a Congress to which mathematicians of all national and geographical groups will be invited.

Time and Place. The dates for the Congress have been fixed as August 30–September 6, 1950. Harvard University will be the principal host institution. A number of other institutions in metropolitan Boston will join in the entertainment of Congress visitors by arranging special features on their campuses. Mathematicians so desiring will be housed at a modest charge in the Harvard University dormitories and meals will be served in the University dining rooms. There will be accommodations for members of families, special provision being made for the care of children. Those who prefer to live in hotels can be comfortably provided for in Cambridge or Boston.

Organization. The Organizing Committee has invited more than a score of outstanding mathematicians to deliver stated addresses. There will be Conferences in four different fields; these are more fully described below. There will be seven Sections for the presentation of short contributed papers and by invitation of the Section Chairmen there will be a small number of half-hour addresses as part of the Section programs.

In recent years mathematicians have been much impressed by the success of the conference method for presenting research in fields in which vigorous advances have just been made or are in progress. There will accordingly be a coordinated program of formal lectures and informal open discussion, and the stated addresses will be integrated as far as possible with the work of the Conferences. The following list of topics gives an indication of the nature of each of the Conferences:

Algebra (Chairman, A. A. Albert): 1, Groups and universal algebra; 2, Structure theory of rings and algebras; 3, Arithmetic algebra; 4, Algebraic geometry.

Analysis (Chairman, Marston Morse): 1, Algebraic tendencies in analysis; 2, Analysis and geometry in the large; 3, Extremal methods and geometric theory of functions of a complex variable.

Applied Mathematics (Chairman, John von Neumann): 1, Partial differential equations; 2, Statistical mechanics; 3, Random processes in physics and communication.

Topology (Chairman, Hassler Whitney): 1, Homology and homotopy theory; 2, Fibre bundles and obstructions; 3, Differentiable manifolds; 4, Topological groups.

The Sections for the presentation of short contributed papers will be as follows: I, Algebra and Theory of Numbers; II, Analysis; III, Geometry and Topology; IV, Probability and Statistics, Actuarial Science, Economics; V, Mathematical Physics and Applied Mathematics; VI, Logic and Philosophy; VII, History and Education. Each member of the Congress may present only one contributed paper and the time allotted for each paper will be ten minutes. Abstracts for such papers should not exceed 400 words in length and must be submitted on blanks which may be secured from the Secretary of the Congress. Abstracts must be in the hands of the Organizing Committee not later than May 15, 1950. The Organizing Committee has decided that contributed papers must be presented in person.

The official languages of the Congress are English, French, German, Italian, and Russian.

Entertainment. There will be many interesting entertainment features, including a reception, teas, a symphony concert, and a banquet.

Membership in the Congress. Membership in the Congress will be open to all qualified persons, whether they are able to be present in person or not. For regular members of the Congress the fee is $15.00;

these persons will receive the Proceedings of the Congress and will be admitted without charge to the various entertainment functions mentioned above. Members of families, accompanying Congress members and not participating in the scientific meetings, may become associate members, for whom the fee is $7.50; they will not present papers or receive the Proceedings, but will be entitled to many of the other privileges of membership. Only members and associate members of the Congress will have the privilege of residing in the Harvard dormitories.

Financial Support. Besides the support from Harvard University, generous subventions have been subscribed for the Congress by Bell Telephone Laboratories, Carnegie Corporation, General Electric Company, Institute for Advanced Study, Massachusetts Institute of Technology, National Research Council, Rockefeller Foundation, Standard Oil Development Company, and several private donors.

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THE GEOMETRY OF FINSLER SPACES

HERBERT BUSEMANN

The term "Finsler space" evokes in most mathematicians the picture of an impenetrable forest whose entire vegetation consists of tensors. The purpose of the present lecture is to show that *the association of tensors* (or differential forms) *with Finsler spaces is due to an historical accident, and that,* at least at the present time, *the fruitful and relevant problems lie in a different direction.*

Finsler spaces were discovered by Riemann in his lecture [1]:[1] *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (1854). The goal which Riemann set for himself was the *definition and discussion of the most general finite-dimensional space in which every curve has a length derived from an infinitesimal length or line element.* In modern terminology Riemann's approach is this: Let a differentiable manifold M of a certain class be given. In any local coordinate system $(x_1, \cdots, x_n) = (x)$ a length $F(x, dx)$ must be assigned to a given line element $(x, dx) = (x_1, \cdots, x_n; dx, \cdots, dx_n)$ with origin x. If $x(t)$ is a (smooth) curve in M then $\int F(x, \dot{x})dt$ is its length.

In order to insure that the length of a curve is positive and independent of the sense in which the curve is traversed, Riemann requires $F(x, dx) > 0$ for $dx \neq 0$ and $F(x, dx) = F(x, -dx)$.

Next Riemann assumes [1, p. 277] that the length of the line element remains unchanged except for terms of second order, if all points undergo the same infinitesimal change. This amounts to the condition $F(x, kdx) = kF(z, dx)$ for $k > 0$. Nowadays we rather justify this condition by requiring that a change of the parametrization of the curve does not change its length.

Riemann then turns immediately to the special case where $F(x, dx) = [\sum g_{ik}(x)dx_i dx_k]^{1/2}$, that is, to those spaces which are now called Riemann spaces. The general case is passed over with the following remarks: the next simplest case would comprise the manifolds, in which the line element can be expressed as the fourth root of a biquadratic differential form. The investigation of these more general types would not require any essentially different principles, but it would be time consuming and contribute comparatively little new to the theory of space (verhältnismässig auf die Lehre vom Raume wenig

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An address delivered before the Los Angeles Meeting of the Society on November 27, 1948, by invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings; received by the editors December 8, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

neues Licht werfen), because the results cannot be interpreted geometrically (see [1, p. 278]).

Here is one of the few instances where Riemann's feeling was wrong.[2] Nevertheless the passage had a great influence: the general case was for a long time entirely neglected, and when it was taken up the principles of Riemannian geometry were applied. The results thus obtained are not different enough to enrich geometry materially, moreover they frequently do not lend themselves to a naïve geometric interpretation.

The following is a *brief sketch of the history*. The integral $\int F(x, \dot{x})dt$ is the subject of the Weierstrass theory in the calculus of variations. Therefore it is not surprising that the first contributions are due to workers in this field. The pioneers are Bliss [2, 3] and his students (in particular Underhill [4]) in this country and Landsberg [5] in Germany. To develop a geometry it is necessary to have extremals or geodesics whose subarcs furnish locally unique shortest connections. The Legendre or Weierstrass conditions of the calculus of variations suggest to require that $F(x, dx)$ is for fixed x a convex function of dx:

$$F(x, dx) + F(x, \delta x) \geqq F(x, dx + \delta x) \tag{1}$$

or, which is the same, that $F(x, y) = 1$ is a convex surface in y-space. (1) will be assumed henceforth.[3]

Bliss' approach is as follows: the homogeneity of $F(x, dx)$ in dx permits to write $F(x, dx)$ in the form

$$F(x, dx) = \left[\sum g_{ik}(x, dx)dx_i dx_k\right]^{1/2} \tag{2}$$

where the $g_{ik}(x, dx)$ are homogeneous of degree 0 in dx.

If a field of curves is distinguished and $\delta(x)$ is tangent to the field at x, then $\sum g_{ik}(x, \delta(x))dx_i dx_k$ is the line element of a Riemann space. The methods of Riemannian geometry are applicable, but *the*

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[2] It is probable that he did not give the matter much thought. The lecture [1] was to give him the privilege to teach at the University of Göttingen. The candidate had to propose three topics, and the faculty selected one of them. Riemann had put the present topic last. He did not expect to work it out and was pressed for time, when it was selected. This is evidenced by the following passage from a letter to his brother Wilhelm, dated December 8, 1853: . . . und musste dabei drei Themata zur Probevorlesung vorschlagen, von denen dann die Fakultät eins wählt. Die beiden ersten hatte ich fertig und hoffte dass man eins davon nehmen würde; Gauss aber hatte das dritte gewählt, so bin ich wieder etwas in der Klemme, da ich dies noch ausarbeiten muss.

[3] On the other hand, many results hold without Riemann's condition $F(x, dx) = F(x, -dx)$. The present lecture supposes, however, that this relation holds. In the sequel it will be convenient to use the term line element both for $ds = F(x, dx)$ and for ds^2.

results depend on the choice of the field. For instance, the Gauss-Bonnet Theorem has been investigated by Bliss [3] from this point of view.

Finsler was the first who studied the general spaces systematically in his thesis [6] written under Carathéodory's guidance. His main idea is this: if a curve is given, then a field which contains the curve defines a Riemann metric as above. *Those results which are independent of the choice of the field are the real geometric properties of the curve.* In this way Finsler succeeded in developing a theory of curves, and also the foundations for a theory of surfaces. This is the reason why the name Finsler space was generally accepted for the general class of spaces first defined by Riemann.

Then a new line of thought developed in the geometric school at Prague with Berwald, Funk, and Winternitz as principle representatives.[4] *A Finsler space is not considered as a point space but primarily as a set of line elements in which a Riemannian metric is associated with each line element.* The main emphasis in this theory lies on the definitions and properties of parallelism and similar questions. The whole development culminated in Cartan's monograph [8], which is considered to have given the theory of Finsler spaces its final form (endgültige Gestalt, see [7, p. 1]).

Under the restriction to use nothing but Riemannian methods it may be true that the theory has reached its limits. However, as soon as this restriction is dropped it appears that the surface has hardly been scratched.

A first, but not the most interesting or decisive, step beyond the mentioned results consists in *extending the results of Riemannian geometry rather than its methods.*

As an example consider the following statement: In a Riemann space R with nonpositive curvature there is exactly one geodesic arc connecting two points within a given homotopy class. A real geometer will be convinced that such a nice geometric property must have a simple geometric core which has nothing to do with the Riemannian character of the metric. Guided by this conviction he soon finds that a Riemann space has nonpositive curvature if and only if in small geodesic triangles abc the (shortest) geodesic connection of the mid points b', c' of the sides ab and ac is at most half as long as the third side bc:

$$2b'c' \leqq bc. \tag{3}$$

This inequality makes sense in any Finsler space. If it is assumed

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[4] For the rather voluminous literature see the excellent report [7] of Berwald.

(locally) then it turns out, see [9], that not only the mentioned theorem, but *the whole beautiful theory of spaces with nonpositive curvature holds* as it was developed by Hadamard [10] and Cartan [11, Note III]. The proofs are, however, quite different, for instance properties of convex functions replace the Gauss-Bonnet Theorem. It should also be noted that (3) does not pressuppose differentiability properties, and that, in fact, the theory holds without them.

This is only one example among many. Very frequently, when the conclusion of a theorem on Riemann spaces is a simple geometric fact, the hypothesis can be freed of its Riemannian character and the theorem then carries over to Finsler spaces.

But investigations of this type yield nothing but direct analogs to Riemannian results and are not fundamentally new. Poincaré said somewhere[5] on the subject of generalizations: On ne fait pas un grand voyage pour ne trouver que ce qu'on a chez soi. In the case of Finsler spaces a long trip will pay, because they are really fundamentally different from Riemann spaces. Just now we have reached a stage where we begin to understand the problems.

Let us start with *volume and area.* The problem how to define these concepts was among the first which attracted the attention in Finsler spaces. The approach was as follows:

The volume element of a Riemann space is $|g_{ik}(x)|^{1/2} dx_1 \cdots dx_n$, where $|g_{ik}(x)|$ is the determinant of the $g_{ik}(x)$. We wish to find an expression in the derivatives of $F(x, dx)$ which reduces to $|g_{ik}(x)|^{1/2}$ in the Riemannian case. This turns out to be possible in infinitely many ways. Consequently there seemed to be many possible definitions of volume, and the problem was considered as fruitless.

Is this conclusion justified? An unprejudiced attack on the problem would doubtless begin with the question: *Why are the definitions of volume and area unique in the Riemannian case?* The answer is simple, but rests on two fundamental principles which are usually not formulated explicitly. Let G be a bounded domain in a Riemann space R whose volume is to be determined. Guided by the procedure employed in integration, we divide G into small domains G_ν. In each G_ν select a point x^ν and replace $g_{ik}(x)$ in G_ν by $g_{ik}(x^\nu)$. With this metric G_ν is euclidean. We know what volume in a euclidean space means. Its value for G_ν is $|g_{ik}(x^\nu)|^{1/2}\int\int_{G_\nu} dx_1 \cdots dx_n$. By the usual limit process of integration we find $\int\int_G |g_{ik}(x)|^{1/2} dx_1 \cdots dx_n$ as Riemannian volume of G. The principle which underlies this procedure is clearly this: A Riemannian metric is locally euclidean. *At every point*

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[5] The author forgot where he saw this remark and was unable to locate it, but would be grateful for a reference.

the volume element of the space must coincide with the volume element of the local geometry.

Turning to area we observe: If $x_i(u_1, \cdots, u_k) = x(u)$ is a k-dimensional submanifold M of R, then the metric R induces on M the metric

$$d\sigma^2 = \sum_{i,k,j,h} g_{ik}(x(u)) \frac{\partial x_i}{\partial u_j} \frac{\partial x_k}{\partial u_h} du_j du_h = \sum_{j,h} \gamma_{jh}(u) du_j du_h.$$

With this metric M becomes a k-dimensional Riemann space. Its volume must be the area of M as manifold in R. This statement is based on the principle that area *must be intrinsic*, that is, depend only on the $\gamma_{jh}(u)$.

We now try to treat volume and area in a Finsler space R with line element $F(x, dx)$ by using the same principles. We divide a given domain G in R into small domains G_ν. In G_ν we select a point x^ν. In G_ν we replace $F(x, dx)$ by $F(x^\nu, dx)$. But there it seems to end, because the space with $F(x^\nu, dx) = F(dx)$ as line elements *is not euclidean, but is a Minkowskian* (or finite-dimensional Banach) *space* S.

If dx is replaced by x the distance of two points x, y in S is given by

$$F(y - x) = |y - x| \cdot F\left(\frac{y - x}{|y - x|}\right) = |y - x| \cdot F(u) \tag{4}$$

where u is a unit vector in the direction of the vector $y - x$. Therefore the Minkowski distance originates from the euclidean distance $y - x$ by multiplying it with a factor which depends only on the direction of the segment from x to y. The triangle inequality holds because of (1). The euclidean segments are also shortest connections for the Minkowski metric, and the only ones when the unit sphere $F(x) = 1$ is strictly convex.[6] The same Minkowski space can be derived from different euclidean spaces (x) in the form (4). All these spaces are related by nondegenerated affine transformations and are called associated to the Minkowski space, compare [13, §2].

Obviously the Minkowski distance (4) is invariant under the translations $x' = x + a$. Since these operate in a simply transitive way on the space, the theory of Haar measure shows that *up to a constant factor at most one measure exists which is invariant under the translations.* One such measure is the Lebesgue measure $|M|_n^L$ of a set M in the euclidean space (x) with distance $|y - x|$. This measure itself cannot be used because it depends on the choice of the associated space. Thus the question reduces to determining λ such that

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[6] For these statements compare [12, § II.1].

$$|M|_n = \lambda |M|_n^L \text{ is an acceptable measure in } S.$$

Several possibilities suggest themselves: the distance $F(y-x)$ or one-dimensional measure can be interpreted as follows: The Minkowskian unit sphere V: $F(x, y)=1$ is convex and has O as center. For given points x, y let z_1z_2 be the diameter of V parallel to the line xy or the intersection α of the line parallel to xy through O with the solid sphere W: $F(x)\leq 1$. Then $F(z_2-z_1)=2$ and since $(y-x)/|y-x| =(z_2-z_1)/|z_2-z_1|$

$$\begin{aligned} F(y-x) &= \frac{|y-x|}{|z_2-z_1|}F(z_2-z_1) = \frac{2}{|z_2-z_1|}|y-x| \\ &= \frac{2}{|\alpha|_1^k}|y-x|. \end{aligned} \tag{5}$$

Observing that 2 is the volume of the one-dimensional euclidean unit sphere and keeping the principle of intrinsiqueness in mind, we are led to the following choice of λ: Let N be a set in a k-dimensional, $0<k\leq n$, linear space L. Let a k-dimensional space parallel to L through O intersect W in the set α and define in analogy with (5)

$$N_k = \frac{\Omega_k}{|\alpha|_k^L}\cdot |N|_k^L, \qquad \Omega_k = \pi^{n/2}\Gamma^{-1}\left(\frac{n}{2}+1\right).^7 \tag{6}$$

On the other hand Hausdorff measure could be used. In fact there are two Hausdorff measures: in one definition the set N is covered by *spheres* of diameter less than ϵ, and in the other by *arbitrary sets* of diameter less than ϵ. The equality of these measures in the euclidean case depends on the fact that the sphere maximizes the volume among all sets with a given diameter (see, for instance, [13, p. 239]).

Since the Minkowski spheres also maximize the volume among all sets of a given Minkowski diameter [13, pp. 243–246], the two k-dimensional Hausdorff measures are equal for the above set N. What is more, they also yield the value $\Omega_k/|\alpha|_k^L$ for λ. There are other cogent reasons [13, pp. 242, 243] which lead to the same choice of λ. Consequently, *measure in Minkowski spaces is just as uniquely determined and therefore as interesting as in euclidean spaces.*

The principle that the volume element of a space at a point must coincide with the volume element of the local space then determines volume in Finsler spaces. Area is defined by the same procedure as in

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[7] So that Ω_k is the volume of the k-dimensional euclidean unit sphere. It should be shown that $|N|_k$ is independent of the associated metric, but this is contained in what follows.

Riemann spaces using the principle that area is intrinsic.

Hence, there is just as little choice for the definition of volume and area in Finsler spaces as there is in Riemann space. How could this definition then have been missed by others?[8] The volume element corresponding to (6) is an integro-differential expression, and the preconceived idea that it must be a differential expression barred the way.

In order to discover what volume and area in Finsler spaces are, Minkowski spaces had to be investigated, just as euclidean geometry furnished the clue for Riemann spaces. But *euclidean geometry does more for Riemann spaces, not only volume and area, but all the basic concepts of Riemannian geometry like angle, curvature of curves and surfaces are dictated by the corresponding euclidean concepts.* Cartan's approach to Riemannian geometry in [11] is based on this idea. Clearly the step from no geometry to euclidean geometry is incomparably much wider than that from euclidean geometry to Riemannian geometry.

The volume problem makes it more than probable that *an analogous situation exists for Finsler spaces. Therefore the study of Minkowskian geometry ought to be the first and main step, the passage from there to general Finsler spaces will be the second and simpler step.*

What has been done in Minkowskian geometry, what are the difficulties and problems, and which tools will be necessary? Little has been done, but the field is quite accessible. The main difficulty comes from our long euclidean tradition, which makes it hard (at least for the author) to get a feeling for the subject and to conjecture the right theorems.

The type of problem which faces us is clear: A Minkowskian geometry admits in general only the translations as motions and not the rotations. Since the group of motions is narrower we expect more invariants. By passing from euclidean to projective geometry, ellipses, parabolas, and hyperbolas become indiscernible. The present case presents *the much more difficult converse problem, to discern objects which have always been considered as identical.*

An example will make this clearer. A sphere $V(p, \rho)$ in the euclidean or Minkowskian space is defined as the locus of those points x whose distance px from p equals ρ. Let $W(p, \rho)$ denote the set of points x with $px \leq \rho$. In the euclidean case the sphere has the following other properties:

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[8] Actually Choquet gave this definition in [14], but only as one possibility among several, so that the uniqueness properties of this area were not noted, nor are there any theorems on this area. I also heard that C. Loewner proposed in conversations the use of this area.

(1) *$W(p, \rho)$ maximizes the volume among all sets of diameter 2ρ.*

(2) *$V(p, \rho)$ solves the isoperimetric problem.*

(3) *Spheres lead to the area $A(s)$* of a convex or sufficiently smooth non-convex simple closed surface S bounding a set K through the relation

$$A(s) = \lim \rho^{-1}(|K_\rho|_n^L - |K|_n^L) \tag{7}$$

where $K_\rho = \bigcup_{x \in K} W(x, \rho)$.

(4) $V(p, \rho)$ is envelope of planes normal to the rays with origin p.

It was already mentioned that Minkowski spheres have property (1) which is decisive for measure theory. But *a Minkowski sphere does, in general, not solve the isoperimetric problem for the area defined above.* The solutions are found by applying the Brunn-Minkowski theorem, see [15]. They are homothetic to each other. It is convenient to single out one of them $T = T(o, 1)$, by the requirement that it has o as center and that its Minkowski area is n times its Minkowski volume.[9]

The relation (7) does not hold for the Minkowski area $A_m(s)$ of S and the Minkowski spheres $W(p, \rho)$. However, if $T(p, \rho)$ denotes the surface obtained from $T(o, 1)$ by the dilation $x' = \rho x$ followed by the translation $x'' = x' + p$ and $W^*(p, \rho)$ is the closed set bounded by $T(p, \rho)$, then (7) holds in the form

$$A_m(S) = \lim \rho^{-1}(|K_\rho^*|_n - |K|_n) \tag{8}$$

where $K_\rho^* = \bigcup_{x \in K} W^*(x, \rho)$, see [15].

Thus $T(p, \rho)$ has two properties of the euclidean sphere.

Property (4) proves particularly illuminating. A line L in euclidean or in Minkowskian geometry is *normal* to a set M at a point $f \in M$ if L contains f and every point x of L has f as foot on M, that is, if $xy \geqq xf$ for $x \in L$ and $y \in M$. In euclidean geometry normality of a line to a line L' or a hyperplane H at f implies normality at f of L' to L or normality to L of every line in H through f. It is well known from the calculus of variations (where the word transversality is used) and from the theory of Banach spaces that this is not so in Minkowski spaces.

If H is a supporting plane of the Minkowski sphere $V(p, \rho)$ at a point f, then $L = pf$ is normal to H and to every line in H through f.

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[9] The area of the euclidean unit sphere is $n\Omega_n$. The corresponding relation is not true for the Minkowskian $V(o, 1)$ and $W(o, 1)$.

Therefore a Minkowski sphere is still envelope of planes to which the radii are normal.

Is it also the envelope of planes normal to the radii? Normality of a plane to a line is as yet undefined, but it is easy to see how it can be defined. That L is normal to H may be expressed as follows: if H_1, H_2 are any two planes parallel to H, then the segment intercepted by H_1, H_2 on L is not longer than a segment intercepted on any other line L^* or also: the one-dimensional measure of a set on a straight line L^* not parallel to H is at least as large as the measure of its projection parallel to H on L.

It is then natural to call the *hyperplane H normal to the line L* if the $(n-1)$-dimensional Minkowski measure (6) of a set in a hyperplane H^* not parallel to L is at least as large as the measure of its projection parallel to L on H.

The existence of a hyperplane normal to a given line L is trivial and not very interesting. But *the converse question, whether for a given hyperplane H a line L exists to which H is normal, is very interesting, since a positive answer implies among other things that hyperplanes are minimal* (in the sense of area minimizing) *surfaces.*[10]

The question turns out to be equivalent to the following problem on convex bodies: Let W be a convex body with center o. For any hyperplane H through o lay off on the normal to H at o (in both directions) the $(n-1)$-dimensional volume $|H\cap W|_{n-1}^{L}$ of $H\cap W$. Is the locus Y of the points thus obtained a convex surface?

The answer is positive, but not trivial, unless the proof in [16] can be considerably simplified. This whole discussion shows, what should have been a truism from the beginning, that *the theory of convex bodies is one of the essential tools for Finsler spaces.*

Returning to property (4) we ask: what are the envelopes of the planes normal to the rays issuing from a fixed point p? They are found to be the solutions $T(p, \rho)$ of the isoperimetric problem. The deeper reason for this surprising fact is that for $W = W(o, 1)$ the above constructed surface Y is related to $T(o, 1)$ by a polar reciprocity with respect to a euclidean sphere $\sum x_i^2 = r^2$ with a suitable radius $r > 0$.

Thus property (4) *divides into two properties, the spheres $V(p, \rho)$ have one of them and the solution $T(p, \rho)$ of the isoperimetric problem the other.*

If the definition and four properties of euclidean spheres had all led to different Minkowskian surfaces, the theory would be too disorganized to be interesting. Had they all led to the Minkowski sphere, the above quoted aphorism of Poincaré would apply. The fact that just two surfaces emerged seems to promise an interesting future.

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[10] These and some of the following statements will be proved in *The foundations of Minkowskian geometry*, which is going to appear in the Comment. Math. Helv.

The properties of Minkowskian geometry hitherto discussed are of the most primitive nature. This reflects the actual state of the theory, with one exception: *the differential geometric results connected with the sphere as carrier of the spherical image have been extended to Minkowski spaces in so-called relative differential geometry.* Since this branch is almost unknown in America, a digression on its origin and nature may be useful. Minkowski [17, §24] observed that (7) yields

$$A(S) = nV_1(K, W(o, 1)) \tag{9}$$

where $V_i(L_1, L_2)$ denotes generally the mixed volume of $n-i$ times the convex body L_1 and i times the convex body L_2 (compare [19, §§29 and 30]). Guided by (9) he defined [17, §27] for an arbitrary convex body L

$$A_L(S) = nV_1(K, L)$$

as area of S relative to L. Minkowski showed moreover [18, §4] that the mixed volumes $V_i(K, W(o, 1))$ are in E^3 closely related to the elementary symmetric functions of the principle radii of curvature of S, and that similar statements apply to $V_i(K, L)$. This theory was continued by others, in particular extended to E^n (see [19, §38]). Thus a differential geometry relative to L was developed.[11]

It was not recognized that its results contribute to Minkowskian geometry because they do not yield theorems on the Minkowski geometry $\mathfrak{M}$ with $V(o, 1)$ as unit sphere if applied to $W(o, 1)$ as L. However (8) shows that

$$A_m(S) = nV_1(K, T).$$

Therefore the differential geometry relative to T belongs to $\mathfrak{M}$.

A typical result, contained in Duschek [20] and derived under the usual unspecified differentiability assumptions of differential geometry, is the following:

Map a given surface S in a three-dimensional Minkowski space on T by associating points x of S and x' of T where the tangent planes are parallel. At any point x of S there are exactly two directions d_1, d_2 which are parallel to the corresponding directions δ_1, δ_2 at x', so that, corresponding to the formula of Olindes Rodrigues in ordinary differential geometry, $d_i x = R_i \delta_i x'$. The R_i do not depend on the associated euclidean metric and have therefore a Minkowskian meaning.

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[11] A complete list of the literature until 1934 is found in [19, p. 65].

Duschek shows that several of the usual interpretations of R_i can be extended to the present case and proves moreover: *S is a minimal surface,* that is, the first variation of the Minkowski area of S vanishes, *if and only if* $R_1^{-1}+R_2^{-1}\equiv 0$.

Reassuring as these results are, they bypass technically simpler, but much more basic, questions which ought to be answered first. The first theorem of ordinary differential geometry is the statement that the curvature determines a plane curve. More explicitly: if $p(s)$, $p'(s)$, $0\leqq s\leqq\alpha$, are arcs in E^2 with the arclength s as parameter and equal curvature for equal s, then a motion of E^2 exists which carries $p(s)$ into $p'(s)$ for $0\leqq s\leqq\alpha$. In E^2 this is equivalent to

$$p(s_1)p(s_2) = p'(s_1)p'(s_2) \qquad \text{for } 0 \leqq s_i \leqq \alpha. \tag{10}$$

In a Minkowski plane the relation (10) may hold for arcs for which no motion of the plane exists which carries $p(s)$ into $p'(s)$.

The first problem of Minkowskian differential geometry is therefore, clearly, to discuss the meaning of the statement that an arc is determined, and then to find curvature functions which determine the arc in some sense, for instance (10). But the problem has never been treated.[12]

This confirms that in spite of all the work on Finsler spaces we are now at a stage which corresponds to the very beginning in the development of ordinary differential geometry. Therefore the mathematician who likes special problems has the field. After sufficiently many special results have been accumulated someone will create the appropriate tools. At the present time it is difficult to guess what they will be beyond a vague feeling that some theory of integro-differential invariants will be essential.

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TOPOLOGICAL METHODS IN THE THEORY OF LEBESGUE AREA

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1. Introduction. In conformity with the title of this address, the principal objective is the consideration of topological techniques which have been employed in the theory of surface area initiated by Lebesgue [15][1] in his thesis.

It may appear strange that topology is required at all in a discipline classified in the field of analysis—certainly Lebesgue in 1902 gave little hint of what was to come. On the other hand, it cannot be claimed that this sort of thing is an isolated phenomenon, and perhaps one may be permitted to argue that one of the beauties of mathematics is the presence of such mixtures between fields.

Historically speaking, it was de Geöcze who called attention to certain phenomena—in a series of highly significant papers written during the years 1908 to 1914—and, in attempting to fit these phenomena into the framework of the theory, first noticed the need for topology. (For a partial list of these papers see de Geöcze [12]. A more detailed bibliography at this point, and elsewhere throughout the paper, is found in Radó [26].)

In discussing this matter let it be said, even at the risk of considerable over-simplification, that *the significant problems in Lebesgue area have their source in the attempt to find 2-dimensional analogues for four classical results in the theory of length.* In any event, with an eye on cohesion, it is here planned to consider only those topological techniques which have bearing on what might be called the *analogy problem.*

In consequence of this general plan of attack it is not only appropriate to sketch these classical results, but fortunately in so doing one automatically gains a measure of intuition by analogy and adequate motivation is provided for the 2-dimensional theory.

Before plunging into the discussion, however, a remark should be made as to the manner of presentation in contradistinction to the plan of attack. Matters of analogy leading to intuitive understanding are considered important enough to develop so as to be practically self-contained. (For example, during the first part of this paper every

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

effort will be made to define the principal concepts involved.) Consequently, it is to be hoped that a substantial portion of the subject matter will be followed readily by all. This plan must, of course, be abandoned in those sections dealing with specific topological techniques—this is certainly not the place to consider such matters in any detail.

2. **Coordinate functions.** For the purpose of considering the previously mentioned four classical results for curves, suppose f is a mapping, that is, a continuous transformation, from the unit interval X^1 into Euclidean 3-space E^3. It is assumed that *coordinate axes* Y^1, Y^2, and Y^3 have been selected in E^3. If $p^i: E^3 \to Y^i$ is the notation for the *perpendicular projection* of E^3 onto Y^i, while f^i is the composition $p^i f$, then $f^i: X^1 \to Y^i$, $i = 1, 2, 3$, and in analysis the mapping f is usually given by the so-called *coordinate functions*:

$$y^1 = f^1(x), \qquad y^2 = f^2(x), \qquad y^3 = f^3(x), \qquad x \in X.$$

3. **Classical formula.** The classical integral formula $\int_{X^1}[(J^1)^2 + (J^2)^2 + (J^3)^2]^{1/2}dx$, where $J^i = df^i/dx$, $i = 1, 2, 3$, is to be designated by $I^1(f)$. This formula will *not* generally provide the length of the "*curve given by f.*" Indeed, even if Lebesgue integration is used—and this will invariably be the case—the formula will generally be a meaningless collection of marks, since there is no guarantee that the integrand exists almost everywhere and is summable. In this connection it is to be understood, however, that *if the symbol $I^1(f)$ appears in a statement then it has meaning*; in other words, the integrand in question does exist almost everywhere and, indeed, is summable.

4. **Representation.** Before going any farther, it is advisable to make some comment on the phrase "curve given by f" used in §3.

The mapping f is simply a *representation* for a curve and the concept of changing the representation is common, even in elementary mathematics, though it is subject to many interpretations. For a number of reasons the most convenient interpretation for present purposes is due to Fréchet [11]. Accordingly $f: X^1 \to E^3$ and $g: X^1 \to E^3$ are said to *represent the same curve*, or, to use the modern terminology, are said to be *Fréchet equivalent* (notation: $f \sim g$), if and only if for every $\epsilon > 0$ there is a homeomorphism $h_\epsilon: X^1 \approx X^1$ such that $d\{f(x), gh_\epsilon(x)\} < \epsilon$, $x \in X^1$. A formal definition for a curve can now be offered in terms of this equivalence relation. (See Youngs [31] where the matter is considered in detail for both curves and surfaces.)

It is well known that the length of a curve is initially defined in terms of a *given representation* as the limit of the lengths of any sequence of inscribed polygonal mappings which converge uniformly to the given representation. It is easy to see, however, that the length of a curve is—as it should be—*independent* of the representation. (For greater detail see Radó [26].) The notation $L^1(f)$ is here employed for the length of the curve given by the mapping $f:X^1 \rightarrow E^3$.

5. **Classical theorems for curves and future criteria for surfaces.** It is now possible to state three of the promised classical results:

C_1: $L^1(f) < \infty \leftrightarrow f^i$ *is* BV(=*of bounded variation*), $i=1, 2, 3$.

C_2: $L^1(f) < \infty \rightarrow L^1(f) \geqq I^1(f)$, *and* $L^1(f) = I^1(f) \leftrightarrow f^i$ *is* AC (=*absolutely continuous*), $i=1, 2, 3$. (In conformity with the convention in §3 the first statement means: $L^1(f) < \infty \rightarrow I^1(f)$ *exists* and $L^1(f) \geqq I^1(f)$.)

C_3: $L^1(f) < \infty \rightarrow$ *there is a representation* $g \sim f$ *such that* $I^1(g) = L^1(g) = L^1(f)$.

6. **Interval function.** It would be presumptuous to define the concepts BV and AC except for the fact that a record of these definitions affords both an insight into the 2-dimensional case and a considerable economy in the sequel.

For convenience suppose $\phi:X^1 \rightarrow E^1$ is a mapping from the unit interval into a line. (The mapping ϕ is of the same type, therefore, as the coordinate functions f^1, f^2, and f^3 of §2.)

The key to the definitions of BV and AC is a certain *interval function* now to be considered. Suppose that r is a closed interval $a \leqq x \leqq b$ in X^1, while $\dot{r} = a \cup b$. Then $v_\phi(r)$, the interval function in question, is simply the length of the interval from $\phi(a)$ to $\phi(b)$ in E^1.

7. **An alternative interpretation.** *For the sake of insight into the 2-dimensional case another interpretation of* $v_\phi(r)$ *is important.*

Consider the components (=maximal connected subsets) of $E^1 - \phi(\dot{r})$, and to avoid trivialities assume there is a bounded component K. Then

$$v_\phi(r) = |K|$$

where $|K|$ is the 1-dimensional Lebesgue measure of K; in this case, simply the length of the open interval K.

Let Γ be the family of those mappings $\gamma:r \rightarrow E^1$ which agree with ϕ on $\dot{r}$. Then

$$\gamma(r) \supset K \cup \phi(\dot{r}), \qquad \gamma \in \Gamma,$$

but there is a mapping $\gamma_0 \in \Gamma$ (for example the unique linear mapping in Γ) such that

$$\gamma_0(r) = K \cup \phi(\dot{r}).$$

Hence

$$\bigcap_{\Gamma} \gamma(r) = \gamma_0(r).$$

Therefore, $[E^1-\phi(\dot{r})]\cap\bigcap_{\Gamma}\gamma(r)=[E^1-\phi(\dot{r})]\cap\gamma_0(r)=\gamma_0(r)-\phi(\dot{r})=K$, and one has the interpretation

$$v_\phi(r) = \left| [E^1 - \phi(\dot{r})] \cap \bigcap_{\Gamma} \gamma(r) \right|.$$

8. **Bounded variation and absolute continuity.** If R is a closed interval in X, let $\sigma(R)$ be the notation for a subdivision of R; that is, a *finite* collection of r's, whose union is R and whose interiors are disjoint in pairs. Define

$$V_\phi(R) = \sup_{\sigma(R)} \sum_{r\in\sigma(R)} v_\phi(r).$$

The number $V_\phi(R)$ is called the *variation* of ϕ on R, and

ϕ *is* BV *means* $V_\phi(X^1) < \infty$.

As for the concept of AC, to do minor violence to the customary definition:

ϕ *is* AC *means*: (a) for every $\epsilon>0$ there is a $\delta(\epsilon)$ such that if $r_1, \cdots, r_n$ is a set of closed intervals with interiors disjoint in pairs and $\sum|r_i|<\delta(\epsilon)$, then $\sum V_\phi(r_i)<\epsilon$.

(b) $$V_\phi(R) = \sum_{r\in\sigma(R)} V_\phi(r), \text{ for any } \sigma(R).$$

It is well known, however, that (b) is always true, hence it is redundant in this definition and is here displayed only for future use.

9. **Final classical theorem for curves and future criterion for surfaces.** Now that these definitions have been recorded it is important to observe that in the case of a mapping $f: X^1\to E^3$, if σ is a subdivision of X^1 and

$$g_\sigma(f) = \sum_{r\in\sigma} \left[v_{f^1}^2(r) + v_{f^2}^2(r) + v_{f^3}^2(r)\right]^{1/2},$$

then $g_\sigma(f)$ is the length of the *inscribed polygon* determined by σ. Moreover, if

$$G(f) = \sup_{\sigma} g_\sigma(f)$$

then one has the final classical result:

C$_4$: $L^1(f)=G(f)$.

(Notice that C$_4$ implies $L^1(\phi)=G(\phi)=V_\phi(X^1)$.)

10. **Surfaces.** *In now considering the situation for surfaces, the same general outline is employed* except that f is *now* a mapping from the unit square X^2 into Euclidean 3-space E^3, and in contradistinction to the earlier convention, it is *now* assumed that Y^1, Y^2, and Y^3 are *coordinate planes* in E^3. If $p^i: E^3 \to Y^i$ is the *perpendicular projection*, then let $f^i = p^i f$ as before. The mapping $f^i: X^2 \to Y^i$ is a so-called *flat transformation* from the unit square into a plane, $i = 1, 2, 3$.

The classical integral formula (see §3) remains unchanged except that J^i is *now* the Jacobean of the *flat* transformation f^i and, of course, 2-dimensional integration is understood. The formula will be designated by $I^2(f)$ and the earlier conventions concerning its use apply. (See §3.)

It has been stated that outstanding problems in Lebesgue area arise in the search for definitions of BV and AC for flat transformations which will provide results analogous to the classical theorems C_1, C_2, C_3, and C_4 (see §§5 and 9). *These theorems thus become criteria for the suitability of proposed concepts of BV and AC.*

Initial attacks on the problem were made by de Geöcze, and though he was far from being entirely successful in his venture, it came to be considered a safe judgment—even in a field where history has shown the most plausible conjectures to be fraught with disaster—that modifications of his ideas would stand the test of these criteria. In a sense, appropriate concepts have long been available—the difficulty has been in providing proofs.

In defining BV and AC for flat transformations, the development will be by analogy from the 1-dimensional case and it must be stated that considerable expository license will be taken with the ideas of Cesari [**3, 4, 5, 6, 7**], Radó [**18, 19, 20, 24, 25, 26, 27**], and Reichelderfer [**27, 28**].

In accord with the modifications introduced above, the mapping ϕ (cf. §6) is *now* considered to be a flat transformation from the unit square X^2 into a plane E^2. (It is therefore of the same type as the mappings f^i found above.) The *interval* function of the 1-dimensional case is to be replaced by a function of the closure r, of a *Jordan region* bounded by a *simple closed polygon* $\dot{r} \subset X^2$.

In the case for curves $v_\phi(r)$ was *interpreted* as the 1-dimensional measure of a certain set in $E^1 - \phi(\dot{r})$ (see §7). By direct analogy, $v_\phi(r)$ is now *defined* to be the 2-dimensional measure of the corresponding set $[E^2 - \phi(\dot{r})] \cap \bigcap_\Gamma \gamma(r)$ where Γ is the family of mappings $\gamma: r \to E^2$ which agree with ϕ on $\dot{r}$.

The variation $V_\phi(R)$ of ϕ on R is defined in exactly the same manner (see §8) *and there is no change in the definitions of BV, AC, and $G(f)$.*

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(See §§8 and 9.) In view of this fact these definitions are not recorded anew. It is to be noted, however, that item (b) in the definition of AC is *not* redundant. In other words, what was really a theorem in the 1-dimensional case now becomes an essential portion of the definition of AC. (See Cesari [6].)

It should be noted that the functional $g_\sigma(f)$ can no longer be interpreted as the area of an inscribed polyhedron even if σ is a triangulation of X^2. The functional $G(f)$, however, may be interpreted as one of the possible definitions for the *de Geöcze area* of the surface represented by f.

11. **Lebesgue area.** Prior to considering how these concepts fit into the framework of the criteria C_1, C_2, C_3, and C_4 (see §§5 and 9), it is proper to define $L^2(f)$, the Lebesgue area of the surface represented by f. A mapping $\mathfrak{p}:X\rightarrow E^3$ is said to be *polyhedral* (=quasi linear, see McShane [16]) if there is a triangulation T of X with the property that $\mathfrak{p}^i=p^i\mathfrak{p}$ is linear on each triangle of T. (See §10.) The elementary area $E(\mathfrak{p})$ of the polyhedron given by $\mathfrak{p}$ is simply the sum of the areas of the triangles $\mathfrak{p}(\Delta)$, $\Delta\in T$, and

$$L^2(f) = \inf\left[\liminf_{n\to\infty} E(\mathfrak{p}_n)\right],$$

where the infimum is taken for all sequences $\mathfrak{p}_n$ converging uniformly to f. It is not difficult to see that if f and g *represent the same surface* (see §4) then $L^2(f)=L^2(g)$. (There is an obvious lack of analogy here between the definitions of length (see §4) and area. In point of fact, length could equally well have been defined by 1-dimensional considerations precisely analogous to the 2-dimensional considerations above. This was not done since the definition of §4 is better known. It is not to be supposed, conversely, that area can be defined by considerations precisely analogous to those employed in the definition of length given in §4. For a discussion of this matter see Youngs [31].)

12. **Generalized Jacobians.** In addition to the definition of Lebesgue area, it is necessary to say a word about the classical integral formula $I^2(f)$, where the symbols J^i now stand for the Jacobian of f^i, $i=1, 2, 3$. (See §§3 and 10.) For a variety of reasons it is necessary to generalize the concept of Jacobian.

It will be recalled that the *absolute* value of the ordinary Jacobian of a flat transformation $\phi:X^2\rightarrow E^2$ measures the local "ratio of magnification" accorded by the mapping. By analogy from the 1-dimensional case (see end of §9) one can hope that $V_\phi(r)=L^2(\phi|r)$—where

$\phi | r$ is the mapping ϕ considered only on r. Hence a natural generalization is to define

$$\mathcal{J}(x) = \lim_{|s| \to 0} \frac{V_\phi(s)}{|s|}$$

where s is a square with sides parallel to the axes of X^2 and $x \in s^0$. (*If the ordinary Jacobian* $J(x)$ *exists almost everywhere then the same is true of* $\mathcal{J}(x)$, *and* $|J(x)| = \mathcal{J}(x)$ *almost everywhere*; see Cesari [4] and Radó [26].)

If the ordinary Jacobians in the classical integral formula $I^2(f)$ are *replaced* by the generalized Jacobians, then the new formula is designated by $\mathfrak{J}^2(f)$ and the earlier conventions concerning its use apply (see §§3 and 10).

13. **Results for surfaces.** *As to the four criteria* (see §§5 and 9) *it is now possible to state that these statements all hold if* I^1 *is replaced by* $\mathfrak{J}^2$ *and* L^1 *by* L^2.

This simple assertion is by no means simple to prove. It represents the concerted efforts of many mathematicians over approximately half a century and, in point of fact, C_3 has only recently been announced by Cesari [8].

In proving these theorems one must naturally work with the basic function $v_\phi(r)$, and it must be stated that an equivalent definition in terms of the *degree of a mapping* is employed. (See, for example, Cesari [3] and Radó [26].) This equivalent definition need not concern us. The point is that in consequence of this fact, it is only to be expected that standard theorems on the degree will be encountered. What is perhaps unexpected is that further topological methods, to be known in this context as *topological adventures*, must be brought to bear on the situation. In any event it is the topological adventures *alone* which will be the center of attention.

It should also be stated that in view of technical complexities involved in the proofs and, indeed, the sheer lengths of the proofs themselves, no details can be undertaken, and the *application* of the topological adventures to be considered can at best be implied.

14. **Inequality of Cesari and criterion C_1.** Of the four theorems C_1 perhaps deserves special attention in view of its difficulty and the bearing its solution has on some of the others, notably C_4.

That $L^2(f) < \infty$ implies f^i is BV was shown independently by Cesari [3] and Radó [24] during the war years. Cesari, however, proved both parts of C_1 using an argument which culminated in the *basic inequality*,

$$V_{f^i}(X^2) \leqq L^2(f) \leqq V_{f^1}(X^2) + V_{f^2}(X^2) + V_{f^3}(X^2), \qquad i = 1, 2, 3.$$

From this inequality the issue is settled at a glance. This superb result must certainly rank as an accomplishment of the first magnitude.

The right half of the Cesari inequality is much more difficult to prove than the left. On the other hand, it will follow immediately from the definition of $L^2(f)$ if it can be shown that for each integer n there is a polyhedral mapping $\mathfrak{p}_n : X \to E^3$ such that

$$\text{(a)} \qquad d(f(x), \mathfrak{p}_n(x)) < 1/n, \qquad x \in X.$$

$$\text{(b)} \qquad E(\mathfrak{p}_n) \leqq V_{f^1}(X^2) + V_{f^2}(X^2) + V_{f^3}(X^2). \qquad \text{(See §11.)}$$

To obtain (a) by itself is a triviality, but to obtain it in conjunction with (b) is no easy task and involves carefully fitting together a polygonal mapping by a highly delicate argument. A salient step deeply imbedded in the chain of reasoning is concerned with a property of Euclidean 3-space which should be of some interest.

In E^3 let M consist of the coordinate axes X, Y, Z. With $\delta > 0$, define M_δ to be the union of the four lines:

$$X_\delta : y = 0,\ z = -\delta. \qquad Y_\delta : x = 0,\ z = \delta.$$

$$Z_\delta^+ : x = \delta,\ y = \delta. \qquad Z_\delta^- : x = -\delta,\ y = -\delta.$$

Now suppose that one is given a closed path in $E^3 - M$ at a distance greater than δ from M. A lemma of Cesari [2] states that *if the path is contractible in $E^3 - M_\delta$ then it is also contractible in $E^3 - M$ and conversely.* (This form of the statement is due to Eilenberg [10] who has also considerably shortened the original proof.)

The inequality of Cesari shows, by the way, that

$$\text{(a)} \qquad L^2(f^i) = V_{f^i}(X^2), \qquad i = 1, 2, 3,$$

and hence

$$\text{(b)} \qquad L^2(f^i) \leqq L^2(f) \leqq L^2(f^1) + L^2(f^2) + L^2(f^3), \qquad i = 1, 2, 3.$$

But (a) can be proved independently, and hence the Cesari inequality follows if the same is true of (b).

The first half of (b) is obvious and the second has the deceptive air of appearing to be simpler to prove than the Cesari inequality since the functional L^2 alone appears. However, it is not known whether (b) can be proved directly or not.

15. **Middle space topology.** It is not to be supposed however that C_1 was proved without further topological adventures. For example a close scrutiny of the character and disposition of the inverse sets

$f^{-1}(y)$, $y \in E^3$, was required. Since studies of this character also appear in connection with the other theorems, and here analytic topology plays a major role, the matter cannot be dismissed with cursory mention. (For reasons which will become apparent this whole branch of the theory has aptly been called *middle space topology* by Radó [**23**].)

The basic concept involved is that of a *monotone-light factorization* of a mapping. If $f: X \rightarrow Y$ is a mapping from a compact space into a metric space, then the Eilenberg-Whyburn Factor Theorem (see Whyburn [**30**]) states that there is a pair of mappings $m: X \rightarrow \mathfrak{X}$ and $l: \mathfrak{X} \rightarrow Y$ such that:

(a) $f = lm$.

(b) m is monotone ($= m^{-1}(x)$ is a continuum, $x \in \mathfrak{X}$), and $m(X) = \mathfrak{X}$.

(c) l is light ($=$no component of $l^{-1}(y)$ consists of more than one point, $y \in Y$).

The space $\mathfrak{X}$ is called the *middle space* of the *monotone-light factorization* lm of f.

In the study of Lebesgue area it became evident that the structure of the middle space $\mathfrak{X}$—the image of the monotone factor—was of considerable importance in contradistinction to the apparently paradoxical fact that the structure of $f(X)$—the image of f—had relatively little to do with the matter.

For example, it was observed by Morrey [**17**] that *if X is a 2-cell and $L^2(f) < \infty$ then it is possible to find a new representation $g: X \rightarrow E^3$ such that the Lebesgue area is given by the classical integral formula $I^2(g)$.*

16. **Representation problem.** This is perhaps as good a time as any to state that the general problem of finding new representations for a surface had earlier attracted the attention of Kerékjártó [**14**]. He considered the case in which X is a *2-sphere* while $f_1: X \rightarrow E^3$ and $f_2: X \rightarrow E^3$ are representations of the same surface, that is to say, are Fréchet equivalent (see §4). Under these conditions, Kerékjártó showed that *there is a monotone-light factorization lm_i of f_i with common middle space $\mathfrak{X}$, $i = 1, 2$.* (Note that the factorizations also have a common light factor.)

It was initially supposed that the converse of this theorem is true. This is *not* the case (see Youngs [**32**]), and it should be mentioned that it is the converse which is important in these problems. However, Kerékjártó's work initiated the search for conditions which are both necessary and sufficient for $f_1 \sim f_2$. (See §4.) This is the *representation problem* for surfaces. The solution requires the use of algebraic topology—in this presentation the Čech cohomology theory with the additive group of integers as the coefficient group will be employed. The result in question is the following and is initially stated for the case X is a 2-cell with boundary $\dot{X}$.

$f_1 \sim f_2$ if and only if:

1. *There is a monotone-light factorization lm_i of f_i with common middle space $\mathfrak{X}$, $i=1, 2$.*

2. *If $\dot{m}_i = m_i | \dot{X}$ then there is a monotone-light factorization $\lambda\mu_i$ of $\dot{m}_i$ with common middle space Ξ, $i=1, 2$. (This implies $m_1(\dot{X}) = m_2(\dot{X}) \equiv \dot{\mathfrak{X}}$.)*

3. *If $m_i^*: H^2(\mathfrak{X}, \dot{\mathfrak{X}}) \to H^2(X, \dot{X})$ and $\mu_i^*: H^1(\Xi) \to H^1(\dot{X})$ are the homomorphisms induced by the mappings m_i and μ_i respectively, $i=1, 2$, then there is an automorphism $\eta: H^2(X, \dot{X}) \approx H^2(X, \dot{X})$ such that with respect to the following diagram*

$$\begin{tikzcd}
 & H^1(\Xi) \arrow[dl, "\mu_1^*"'] \arrow[dr, "\mu_2^*"] & \\
H^1(\dot{X}) \arrow[d, "\delta"'] & & H^1(\dot{X}) \arrow[d, "\delta"] \\
H^2(X, \dot{X}) & & H^2(X, \dot{X}) \arrow[ll, "\eta"'] \\
 & H^2(\mathfrak{X}, \dot{\mathfrak{X}}) \arrow[ul, "m_1^*"] \arrow[ur, "m_2^*"'] &
\end{tikzcd}$$

one has $\delta\mu_1^* = \eta\delta\mu_2^*$ and $m_1^* = \eta m_2^*$.

In the event X is a 2-sphere, $\dot{X}=0$, and the statement of the result is simplified in the obvious manner (Youngs [32]).

17. **Additivity theorem.** The first active use made of the solution of the representation problem was in the proof of an *additivity theorem for the Lebesgue area.* Suppose $f: X \to E^3$ represents a surface with X a 2-sphere, and lm is a monotone light factorization of f with middle space $\mathfrak{X}$. (See §15.) The collection of true cyclic elements of $\mathfrak{X}$ is at most denumerable. (See Whyburn [30].) Let them be represented by $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \cdots$. It is well known that there is a unique monotone retraction $r_n: \mathfrak{X} \to \mathfrak{C}_n$, $n=1, 2, 3, \cdots$.

If $f_n = lr_n m$, $n=1, 2, 3, \cdots$, then this class of mappings is said to be the *cyclic decomposition* of f, and the *cyclic additivity theorem* for Lebesgue area states that

$$L^2(f) = \sum L^2(f_n),$$

where the symbols on the right are interpreted to mean 0 if there are no true cyclic elements.

It should be stated that the theorem is also true in the event X is a 2-cell as has been shown by Radó [26] and Helsel [13].

The cyclic additivity theorem shows that $L^2(f)=0$ *if the middle space has no true cyclic elements*, that is to say, if $\mathfrak{X}$ is a dendrite. Radó [22] has also proved the *converse* of this theorem thus providing an elegant characterization for surfaces of zero area.

18. **Criterion C$_4$.** It can be shown that the Geöcze area is also cyclicly additive, and hence the equality of the two definitions of area, that is, a solution of C$_4$ (see §9) follows if $G(f_n)=L^2(f_n)$, $n=1, 2, 3, \cdots$. This has been done independently by Cesari [5, 7] and Radó [24, 26] in the event $L(f_n)<\infty$—it is topologically interesting to note that the proof depends heavily on the fact that the surface represented by f_n can also be represented by a mapping g_n which is light on the interior of X^2, $n=1, 2, 3, \cdots$.

In the event $L^2(f_n)=\infty$ Cesari completed the solution of this celebrated problem by employing his inequality (see §14) together with the obvious fact that

$$V_{f_n}^i(X^2) \leqq G(f_n), \qquad i = 1, 2, 3.$$

For now

$$L(f_n) \leqq V_{f_n}^1(X^2) + V_{f_n}^2(X^2) + V_{f_n}^3(X^2) \leqq 3G(f_n).$$

Consequently $L^2(f_n)=\infty$ implies $G(f_n)=\infty$.

19. **Criteria C$_2$ and C$_3$.** The lack of récent specific mention of C$_2$ and C$_3$ (see §5) may lead one to suppose that here, at any rate, no topological adventures are required. While, relative to the other theorems, there is a certain element of truth to this surmise, it is not to be supposed that C$_2$ or C$_3$ can be proved in the body of analysis. For example, a basic step in the proof of C$_2$ is the fact that C$_4$ is true in particular if $L^2(f)<\infty$. (See Cesari [4] and Radó [26].) Hence indirectly at least the most powerful methods of middle space topology are required. As for C$_3$, Cesari's announcement shows that he proves this by first generalizing the result of Morrey already stated in part. (See §15 and Cesari [8, 9].) The argument again may be based upon middle space topology. Hence in a certain sense the most that can be said is that no topological adventures are required which are spectacularly different from those already employed in the proofs of C$_1$ and C$_4$.

Relative to C$_3$, a portion of the results of Cesari shows that *if* $L^2(f)<\infty$ *then there is a closed set* $H\subset X$ *and a representation* g *of the surface such that* $L^2(g)=\int_H[(J_g^1)^2+(J_g^2)^2+(J_g^3)^2]^{1/2}dx$. (See §3 and

Cesari [8].) The subscript s is here employed to indicate that the ordinary Jacobians are taken relative to the representation g but that the concept of differentiation is appropriately generalized using the work of Stepanoff.

To date this result is the closest approach to an unsolved problem of obvious interest. *If* $L^2(f) < \infty$ *is there a representation* $g \sim f$ (see §4) *such that* $L^2(g) = I^2(g)$? In other words is every surface of finite Lebesgue area so representable that the ordinary classical integral formula applied to this representation yields the Lebesgue area? Radó [21] has shown that if $I^2(f)$ has meaning, then $L^2(f) \geqq I^2(f)$, and it is known that every surface, regardless of its Lebesgue area, has a representation g for which $I^2(g) = 0$. (See Youngs [33].) If the above question is answered in the affirmative it will certainly provide a beautiful link between the classical ideas of the last century and the modern theory of Lebesgue area. (Added in proof: Cesari has recently shown that such representations exist.)

20. **Generalizations.** The statement of this unsolved problem concludes this survey of some of the topological adventures forced by these classical problems. However, a few concluding remarks will be made on generalizations in two directions:

1. The problem of mappings from cells of dimension higher than 2.
2. The problem of mappings from compact connected 2-manifolds.

The first problem has attracted the attention of Federer who has reported (in correspondence) a fruitful generalization of the concept of BV for mappings from an n-cell into Euclidean n-space. His concept is, in point of fact, a modification of the method by which Radó defines BV in the 2-dimensional case; namely by using the concept of essential multiplicity which he brought to bear on these problems twenty years ago. (See Radó [18].)

Without going into details, Federer's ideas may be sketched as follows:

Suppose $\phi: X^n \rightarrow E^n$ is a mapping from an n-cell into Euclidean n-space. For $y \in E^n$ let $S_\epsilon(y)$ be the open sphere of radius ϵ about y, and $U_1, U_2, U_3, \cdots$ be those components of $\phi^{-1}(S_\epsilon(y)) = W$ having the property that $U_i \cap \dot{X}^n = 0$, $i = 1, 2, 3, \cdots$ ($\dot{X}^n$ is the boundary of X^n).

If $\bar{} = \phi | W$ then $\bar{\phi}: W \rightarrow S_\epsilon(y)$ and induces a homomorphism $\bar{\phi}^*: H^n(S_\epsilon(y)) \rightarrow H^n(W)$ between the Lefschetz groups. (See Cartan [1].) But $H^n(W)$ is naturally isomorphic to the weak direct product $PwH^n(U_i)$. If this isomorphism is denoted by η, and generators s and u_i are selected for the infinite cyclic groups $H^n(S_\epsilon(y))$ and $H^n(U_i)$ respectively, $i = 1, 2, 3, \cdots$, then

$$\phi_\epsilon^*(s) = (k_1u_1,\ k_2u_2,\ k_3u_3,\ \cdots)$$

where k_i is an integer and different from zero for only a *finite* number of subscripts $\kappa_\epsilon(y)$.

Now $\kappa_\epsilon(y)$ is independent of the selection of generators and does not decrease as $\epsilon \to 0$. If $\kappa(y) = \lim \kappa_\epsilon(y)$, then in case $n=2$, Radó defines it to be the *essential multiplicity* of y under ϕ. He further defines $V_\phi(X^2) = \int_{E^2}\kappa(y)dy$ and shows that

$$L^2(\phi) = \int_{E^2} \kappa(y)dy.$$

If $n > 2$ then $L^n(\phi) \geqq \int_{E^n}\kappa(y)dy$ and the equality need not hold even if $n=3$. ($L^n(\phi)$ is the n-dimensional Lebesgue "area" of the "surface" represented by ϕ and is defined by obvious modifications of §11.)

On the other hand, Federer defines $\lambda_\epsilon(y) = \sum |k_i|$, and $\lambda(y) = \lim \lambda_\epsilon(y)$. He then proves the fundamental equality:

$$L^n(\phi) = \int_{E^n} \lambda(y)dy.$$

The publication of this beautiful result will certainly be awaited with interest.

It should be mentioned, under this first direction of generalization, that results for mappings—even from 3-cells—comparable in scope and importance with those results in the 2-dimensional case which are concerned with middle space topology lead immediately to basic unsolved problems; for example, the characterization of a 3-cell. One is perhaps permitted, therefore, to take a dim view of the possibility of startling developments along the lines of middle space topology in the immediate future.

The second direction along which generalizations may be made, especially insofar as the corresponding representation problem is concerned (see §16), has been of personal interest during the past few years. This general representation problem has been solved and uses the fundamental work done by Roberts and Steenrod [29] in characterizing the monotone image of a 2-manifold.

This is hardly the place for a discussion of the solution, but it may be of interest to note that a theorem corresponding to the cyclic additivity theorem goes through, and yields a generalization of the theorem of Radó concerning surfaces of zero area. (See §17.) Specifically, *if* $f: X \to E^3$ *is a mapping from a compact connected 2-manifold and* lm *is a monotone-light factorization of* f *with middle space* $\mathfrak{X}$ (*see*

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§15), *then* $L^2(f)=0$ *if and only if* $H^2(\mathfrak{X})=0$; *or alternately*, $L^2(f)=0$ *if and only if* dim $\mathfrak{X}<2$. A certain "slenderness" of the middle space is thus a necessary and sufficient condition for the Lebesgue area to be zero.

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Indiana University

THE OCTOBER MEETING IN NEW YORK

The four hundred fiftieth meeting of the American Mathematical Society was held at Columbia University on Saturday, October 29, 1949. The attendance was about 270, including the following 230 members of the Society.

Milton Abramowitz, M. I. Aissen, E. J. Akutowicz, C. B. Allendoerfer, R. D. Anderson, T. W. Anderson, R. G. Archibald, L. A. Aroian, Frederick Bagemihl, Joshua Barlaz, P. T. Bateman, F. P. Beer, E. G. Begle, Stefan Bergman, Lipman Bers, Garrett Birkhoff, D. W. Blackett, A. L. Blakers, A. A. Blank, S. G. Bourne, C. B. Boyer, Paul Brock, A. B. Brown, F. H. Brownell, L. J. Burton, Jewell H. Bushey, J. H. Bushey, W. R. Callahan, H. E. Campbell, P. G. Carlson, Jeremiah Certaine, K. T. Chen, Y. W. Chen, W. L. Chow, L. W. Cohen, A. J. Coleman, T. F. Cope, Natalie Coplan, L. M. Court, M. D. Darkow, Martin Davis, B. V. Dean, H. F. DeBaggis, J. L. Doob, Arnold Dresden, Aryeh Dvoretzky, Samuel Eilenberg, M. P. Epstein, R. M. Exner, J. M. Feld, F. A. Ficken, N. J. Fine, R. S. Finn, A. D. Fleshler, R. M. Foster, R. H. Fox, Gerald Freilich, Orrin Frink, G. N. Garrison, Hilda Geiringer, H. A. Giddings, J. B. Giever, B. P. Gill, G. H. Gleissner, H. E. Goheen, W. H. Gottschalk, Bernard Greenspan, Harriet Griffin, Emil Grosswald, E. J. Gumbel, V. H. Haefeli, F. C. Hall, F. S. Hawthorne, G. A. Hedlund, M. H. Heins, Alex Heller, Erik Hemmingsen, A. H. Henry, L. H. Herbach, A. A. Herschfeld, Einar Hille, Abraham Hillman, Joseph Hilsenrath, Banesh Hoffmann, T. R. Hollcroft, L. A. Hostinsky, E. M. Hull, Witold Hurewicz, L. C. Hutchinson, M. A. Hyman, Eugene Isaacson, Nathan Jacobson, Wenceslas Jardetzky, Børge Jessen, S. A. Joffe, Fritz John, R. A. Johnson, F. E. Johnston, Shizuo Kakutani, Edward Kasner, W. H. Keen, M. E. Kellar, L. S. Kennison, J. F. Kiefer, H. S. Kieval, S. C. Kleene, J. R. Kline, E. G. Kogbetliantz, E. R. Kolchin, Horace Komm, B. O. Koopman, Jack Laderman, A. W. Landers, J. P. LaSalle, V. V. Latshaw, Solomon Lefschetz, Marguerite Lehr, Benjamin Lepson, Howard Levi, D. C. Lewis, M. A. Lipschutz, S. R. Lipsey, L. H. Loomis, E. R. Lorch, Lee Lorch, Janet McDonald, Brockway McMillan, L. A. MacColl, H. M. MacNeille, Irwin Mann, A. J. Maria, M. H. Maria, W. T. Martin, W. S. Massey, A. E. Meder, Paul Meier, A. N. Milgram, K. S. Miller, W. H. Mills, Don Mittleman, E. E. Moise, Deane Montgomery, Marston Morse, H. H. Mostafa, G. D. Mostow, T. S. Motzkin, F. J. Murray, David Nelson, C. V. Newsom, A. V. Newton, P. B. Norman, I. L. Novak, C. O. Oakley, J. C. Oxtoby, T. E. Peacock, A. J. Penico, I. D. Peters, B. J. Pettis, Everett Pitcher, E. L. Post, Walter Prenowitz, M. H. Protter, Hans Rademacher, Hans Rådström, G. N. Raney, H. E. Rauch, G. E. Raynor, M. S. Rees, Helene Reschovsky, Moses Richardson, J. F. Ritt, I. F. Ritter, J. E. Robinson, P. C. Rosenbloom, H. D. Ruderman, Raphael Salem, H. E. Salzer, Arthur Sard, Samuel Schecter, Henry Scheffé, Eugene Schenkman, M. M. Schiffer, Pincus Schub, Abraham Schwartz, G. E. Schweigert, Wladimir Seidel, Atle Selberg, D. B. Shaffer, Daniel Shanks, C. E. Shannon, H. N. Shapiro, I. M. Sheffer, Seymour Sherman, James Singer, M. H. Slud, L. L. Smail, Ernst Snapper, Andrew Sobczyk, D. C. Spencer, George Springer, Fritz Steinhardt, Wolfgang Sternberg, J. J. Stoker, R. R. Stoll, M. M. Sullivan, Olga Taussky, D. L. Thomsen, John Todd, P. M. Treuenfels, H. C. Tsang, Oswald Veblen, S. I. Vrooman, H. V. Waldinger, J. L. Walsh, Alan Wayne, Alexander Weinstein, Louis Weisner, M. E. White, P. M. Whitman, D. V. Widder, Albert Wilansky, W. G. Wolfgang, Jacob Wolfowitz, Y. K. Wong,

M. A. Woodbury, D. M. Young, J. W. Young, J. A. Zilber, H. J. Zimmerberg, Leo Zippin.

At 1:30 p.m. there was a business meeting of the Society. At this time the Secretary presented certain changes in the By-Laws which had been recommended by the Council at the Summer Meeting. These changes were concerned with the time of the annual meeting of the Board of Trustees, the method of arranging for temporary replacements on editorial committees, the requirements for implementation of the publication of the Proceedings of the Society and the Memoirs of the Society. The changes were unanimously approved. President J. L. Walsh presided at the business meeting and the address which followed.

At 1:45 p.m. Professor R. H. Fox of Princeton University gave an address on *Covering spaces.*

There were two sessions for contributed papers at 3:00 p.m., one for papers in analysis, applied mathematics, and logic in which Dr. Mina S. Rees presided and one for papers in algebra, geometry, and topology in which Professor G. E. Schweigert presided.

Abstracts whose numbers are followed by the letter "*t*" were presented by title. Paper number 15 was read by Professor Seidel, and paper number 36 was read by Professor Cohen. Professor Milkman was introduced by Professor L. T. Wilson.

ALGEBRA AND THEORY OF NUMBERS

1. H. E. Campbell: *An extension of the "principal theorem" of Wedderburn.*

An ideal I of a linear associative algebra A over any field is called a *first zero-trace ideal* if the first trace of every element of I is zero. Every such I is contained in a unique maximal first zero-trace ideal T_1 called the *first liberal.* The difference algebra A/T_1 is separable and A is separable if and only if $T_1=0$. If $A\neq T_1$ there exists $S_1\cong A/T_1$ such that $A=S_1+T_1$. Similar results follow for second traces. Let N be the radical and T' the first (and second) liberal of A/N. Then the unique ideal T of A such that $T/N=T'$ is the *liberal.* Both traces of every element of T are zero and A/T is separable. If $A\neq T$ there exists $S\cong A/T$ such that $A=S+T$. The decomposition based on T_1 is the same as the well known theory based on N for fields of characteristic $p=0$ or $p>n$ (n order of A). If $p\leqq n$ there may be decomposition on T_1 but none on N or vice versa. Since $T=N$ when A/N is separable, the decompositions on T and N are the same when there is one on N. Also there may be a decomposition on T when there is none on N. Moreover, if $\overline{T}$ is an ideal of smallest order such that $A/\overline{T}$ is separable and $\overline{S}\cong A/\overline{T}$ such that $A=\overline{S}+\overline{T}$, then $\overline{T}$ is the liberal. (Received June 15, 1949.)

2*t*. A. H. Clifford: *Extensions of semigroups.*

If S is an ideal of a semigroup Σ, that is, $S\Sigma\subseteq S$ and $\Sigma S\subseteq S$, D. Rees (Proc. Cambridge Philos. Soc. vol. 36 (1940) pp. 387–400) defines the difference semigroup $\Sigma - S$ to be essentially that obtained by collapsing S into a single zero element, while the remaining elements of Σ retain their identity. Starting with given semigroups S and T, the latter having a zero element, the corresponding "extension problem" is to find every possible semigroup Σ containing S as an ideal such that $\Sigma - S \cong T$. A solution is given in terms of homomorphisms of T into the semigroups of left and right translations of S. (Received August 16, 1949.)

3. Benjamin Lepson: *Certain best possible results in the theory of Schnirelmann density.*

The $\alpha+\beta$ hypothesis, conjectured by Khintchine and proved by Mann, is discussed. This states that $\gamma \geqq \min(1, \alpha+\beta)$, where α, β, and γ are the densities of the sets A, B, and $A+B$ respectively, whenever both A and B contain zero. It is shown that, for any pair (α, β), there are sets A and B, each containing zero, for which the equality holds. If the above inequality holds for all sets A and B, where A and B contain fixed finite sets E and F respectively, then E and F both contain zero, in which case the same inequality is still best possible for every pair (α, β). (Received September 13, 1949.)

4*t*. F. I. Mautner: *Irreducible infinite-dimensional representations of certain groups.*

There are groups every one of whose elements has only a finite number of distinct conjugates, which have infinite-dimensional irreducible unitary representations. (Received July 12, 1949.)

5*t*. Eugene Schenkman: *A theory of subinvariant Lie algebras.* Preliminary report.

A theory of subinvariant Lie algebras over a field of characteristic 0 is developed analogous to Wielandt's theory (Math. Zeit. vol. 45 (1939) pp. 209–244) of subinvariant subgroups (that is, subgroups that occur in a composition series of a given group). The concept of subinvariance is used to prove the following main theorem: Let A be a Lie algebra over an arbitrary field with center zero; let $B = \bigcap_k A^k$ where $A^i = [A^{i-1}, A]$, and let d denote the dimension of the derivation algebra $D(B)$ of B, c the dimension of the center of B. Consider the chain of algebras $D_0 = A$, $D_1 = D(D_0), \cdots, D_n = D(D_{n-1}), \cdots$. Then for all n, $\dim D_n \leqq d+c$. Since D_n has no center for all n, the dimension of D_n is nondecreasing as a function of n and it is immediate from the theorem that for some n the only derivations of D_n are inner. This latter result was announced by Chevalley for characteristic 0 (Proc. Nat. Acad. Sci. U.S.A. vol. 30 (1944) p. 274). It is also evident that the main theorem is applicable to the tower of automorphism groups of a Lie group. (Received September 12, 1949.)

6. Ernst Snapper: *Completely primary rings.* I and II.

Ring means commutative ring with unit element. A completely primary ring R is a ring whose radical $N(R)$ is a maximal ideal. Algebraic and trasncendental extensions $R \subset R'$ have been defined and investigated. The definitions are generalizations of notions studied by A. Frankel and W. Krull. The investigation is based on the ideal theory of $R[x]$. First, all rings A with the property that $A/N(A)$ is a principal ideal ring are studied. ($R[x]$ has this property.) It is shown that the not nilpotent elements of A permit a unique factorization. Then, polynomial rings $B[x]$, where B is any ring, are investigated. To every nondivisor of zero f of $B[x]$ an integer $O(f)$, called the *order* of f, is associated. $O(f)$ is at least as important as the degree of f; a side result is that the radical and Jacobson radical of $B[x]$ always coincide. The theories of the rings A and $B[x]$ are then applied to $R[x]$. The ensuing theory of algebraic and transcendental extensions will be used in further papers to derive structure theorems for completely primary rings. (Received October 29, 1949.)

ANALYSIS

7t. P. R. Garabedian (National Research Fellow): *A remark on the moduli of Riemann surfaces of genus 2.*

Let S be a closed Riemann surface of genus 2, and let $w_1(s)$ and $w_2(s)$ be a pair of normal integrals of the first kind on S. These integrals have about one set of canonical cuts on S the periods 0 or πi, and about the remaining canonical cuts they have the periods a_{11}, a_{12}, a_{21}, a_{22}, with $a_{12}=a_{21}$. It is shown that the three complex quantities a_{11}, a_{12}, a_{22} are a set of conformal moduli of the Riemann surface S. The proof is based on Riemann's relations $V_1=w_1(s)$, $V_2=w_2(s)$, $s\in S$, for the zeros V_1, V_2 of the theta-function $\theta(u_1, u_2)$ associated with S. Special symmetric cases are treated which are related to a paper of Ahlfors and Beurling (Comptes Rendus du Dixième Congrès des Mathématiciens Scandinaves, Copenhagen, 1947, pp. 341–351). (Received July 25, 1949.)

8t. E. R. Lorch: *The fundamental theorem for self-adjoint transformations.*

A brief proof is given of the fundamental theorem for self-adjoint transformations H (most general case) in Hilbert space $\mathfrak{H}$. The structure theorem is established in the following form: Let $\{\lambda_n\}$ be a monotone increasing set of real numbers, $n=0, \pm 1, \pm 2, \cdots$, without limit point; hence the set "covers" the real axis. Then there exist in $\mathfrak{H}$ closed linear manifolds $\mathfrak{M}_n$ which are orthogonal in pairs, which span $\mathfrak{H}$, and such that on $\mathfrak{M}_n$, H is bounded and satisfies $\lambda_n I \leqq H \leqq \lambda_{n+1} I$. The key to the proof is the improper Cauchy integral $K_{\lambda\mu}(m, n)=(2\pi i)^{-1}\int_C(\zeta-\lambda)^m(\mu-\zeta)^n(\zeta I-H)^{-1}d\zeta$. Here λ and μ are real numbers, $\lambda<\mu$; m and n are positive integers; and C is a simple closed contour containing λ and μ. The properties—principally of a multiplicative character—of the transformation $K_{\lambda\mu}(m, n)$ are derived with the help of the functional equation of the resolvent of H and the elements of the classic Cauchy theory. (Received July 26, 1949.)

9. Joseph Milkman: *Note on the functional equations*: $f(xy)=f(x)+f(y)$, $f(x^n)=nf(x)$.

Augustin-Louis Cauchy (*Cours d'analyse de l'école Royale Polytechnique*, part I, 1821, p. 109) gives a beautiful proof that the only continuous solution of the functional equation $f(x)+f(y)=f(xy)$, where $f(x)$ is defined for all real numbers x, is the function $f(x)=a \ln x$. Paul Erdös (Ann. of Math. (1946)) proved by analytic number theoretic methods that if $f(m)$ is additive and $f(m+1)\geqq f(m)$, $f(m)=c \ln m$. This implies the theorem, "If $f(m+1)>f(m)$ and $f(mn)=f(m)+f(n)$ hold for all positive integers m and n, then $f(m)=c \ln m$," which the author proves by showing that any solution $f(m)$ can be imbedded in a continuous solution $f(x)$, thus rendering the theorem a corollary of Cauchy's solution of $f(xy)=f(x)+f(y)$. The author also proves that if $f(r)$ is a monotonically increasing function on a set S which is everywhere dense among the positive real numbers and contains all positive integral powers of all members of the set S, for example, the rational numbers, and $f(r^p)=pf(r)$ for any prime p, then $f(r)=k \ln r$. (Received August 17, 1949.)

10. K. S. Miller: *On iterative methods in linear differential equations.*

The problem treated in this paper is that of estimating rates of convergence in solving homogeneous linear differential equations by an interative process. Let L be an ordinary linear differential operator of order n. It is shown that two operators M and N exist such that $L=M+N$ and (under certain restrictions on the boundary conditions) $T=-MN^{-}$ is an integral operator with a symmetric kernel. [N^{-} is a "partial inverse" of N defined in terms of a Green's function.] In Theorem 2, it is proved that if $\epsilon_m(x)=Lu_m(x)$ where $u_m(x)$ is the mth iterant, then $\epsilon_m(x)=T\epsilon_{m-1}(x)$. Theorem 3 states: as m approaches infinity, $u_m(x)$ approaches a solution of the equation $Lu=0$ satisfying the specified boundary conditions. Considered as a linear transformation in Hilbert space, T is self-adjoint with finite norm. Hence if $\{\psi_m(x)\}$ is a complete orthonormal set associated with T with characteristic values $\lambda_1, \lambda_2, \cdots$ (multiplicities included) and if $\epsilon_0(x)=\sum_{\alpha=1}^{\infty} b_\alpha \psi_\alpha(x)$ then $\|\epsilon_m(x)\|^2=\sum_{\alpha=1}^{\infty} b_\alpha^2 \lambda_\alpha^{2m}$. Estimates of the rapidity of convergence can be drawn from the values of the λ_α. Applications to machine methods are discussed. (Received June 15, 1949.)

11. M. H. Protter. *On a boundary value problem for an equation of mixed type.* Preliminary report.

Consider the equation $K(y)u_{xx}-u_{yy}=0$ (*) with $K(y)$ a nondecreasing function defined for $y\leqq 0$, continuous at $y=0$ and with $K(0)=0$. Let D be the domain bounded by the interval $[a, b]$ along the x-axis and the characteristic curves, $x=g_1(y)$, $x=g_2(y)$, through the end points of $[a, b]$. Suppose $F_0(x)$, $a\leqq x\leqq b$, and $G_0(y)$ are given sufficiently well-behaved functions. The existence and uniqueness of the solution $u(x, y)$ of (*) for (x, y) in D, satisfying the conditions $u(x, 0)=F_0(x)$, $u(g_1(y), y)=G_0(y)$, are shown. The method consists of approximating $K(y)$ by a step-function and then solving the corresponding purely hyperbolic equations. Bounds are obtained for the solution in terms of the given data; proceeding to the limit yields the result. (Received July 20, 1949.)

12*t*. Herman Rubin and M. H. Stone: *Postulates for generalizations of Hilbert space.*

The aim of this paper is to reduce the postulates of Jordan and von Neumann for Hilbert space. This is done by eliminating the triangle inequality from the assumed properties of the norm-function. (Received August 29, 1949.)

13*t*. I. E. Segal: *The class of functions which are absolutely convergent Fourier transforms.*

The set of functions on a locally compact abelian group G which are absolutely convergent Fourier transforms is dense (in the uniform topology) in the space of continuous functions on G which vanish at infinity (a function does this if the set where its absolute value exceeds any given positive number has compact closure), and is either all of that space, or of first category in it, according as G is finite or not. A corollary which is classical, but which is obtained here in a purely existential and noncomputational fashion, is that not every continuous function on the reals which tends to zero at infinity is an absolutely convergent Fourier transform, and similarly for Fourier series. (Received September 14, 1949.)

14t. I. E. Segal: *The two-sided regular representation of a unimodular locally compact group.*

If G is a unimodular locally compact group, then every bounded linear operator on the space H of functions on G square-integrable relative to Haar measure which commutes with all left translations $f(x)\to f(ax)$ ($a\in G$; $f\in\mathcal{H}$) on $\mathcal{H}$ is in the weak closure of the algebra generated by the right translations $f(x)\to f(xa)$ on $\mathcal{H}$. The special case of this theorem in which G is discrete was established by Murray and von Neumann, whose proof made essential use of discreteness. It follows that the lattice of all closed linear subspaces of $\mathcal{H}$ which are invariant under both left and right translations is a Boolean algebra. A second corollary is that if G is separable, then there exists an element of $\mathcal{H}$ whose (left and right) translations span $\mathcal{H}$. (Received September 14, 1949.)

15. Wladimir Seidel and Otto Szasz: *On positive harmonic functions and ultraspherical polynomials.*

In a forthcoming paper E. F. Beckenbach, W. Seidel, and Otto Szasz have studied recurrent determinants of ultraspherical polynomials and have shown, among other things, that the quadratic forms $\sum_{\mu,\nu=0}^{n} [P_{\mu+\nu}^{(\lambda)}(x)/P_{\mu+\nu}^{(\lambda)}(1)]u_\mu u_\nu$, where $P_n^{(\lambda)}(x)$ is the ultraspherical polynomial of degree n and order λ, are positive definite for $\lambda>0$, $n=0, 1, 2, \cdots$, and $x>1$. In the present paper it is shown that the hermitian forms $\sum_{\mu,\nu=0}^{n} [P_{|\mu-\nu|}^{(\lambda)}(x)/P_{|\mu-\nu|}^{(\lambda)}(1)]u_\mu \bar{u}_\nu$ are positive definite for $\lambda>0$, $n=0, 1, 2, \cdots$, and $-1<x<1$, a result conjectured by G. Szegö. This result follows from the fact that the harmonic function, $1/2+\Re\{\sum_{n=1}^{\infty} [P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)]z^n\}$ is positive in the unit circle $|z|<1$ for $-1\leqq x\leqq 1$ and $\lambda>0$. For the special case $\lambda=1$ this result was proved earlier by G. Szegö. (Received September 16, 1949.)

16. I. M. Sheffer: *On the solution of sum-equations.*

Consider the system of sum-equations (1) $\sum_{j=0}^{\infty} a_{nj}x_{n+j}=c_n$ $(n=0, 1, \cdots)$. Let (2) $A_n(t)=\sum_0^{\infty} a_{nj}t^j$ $(n=0, 1, \cdots)$ be analytic in $|t|<R$. There is a sequence of polynomials $\{H_n(1/t)\}$ in $1/t$, biorthogonal to the set $\{t^nA_n(t)\}$. Let (3) $\Phi(u, 1/t)=\sum_0^{\infty} H_n(1/t)u^n$, (4) $\Psi(1/u, t)=\sum_0^{\infty} (t/u)^nA_n(t)$, (5) $\Lambda(u, 1/t)=\sum_0^{\infty} H_n(1/t)u^nA_n(u)$. It is shown how convergence properties of series (3), (4), (5) and of series $\sum_0^{\infty} c_nH_n(1/t)$ determine solutions of system (1). In particular, if the analytic continuation of (3) (in the variable u) assumes the form (6) $\Phi(u, 1/t)=\sum_{j=1}^{k} R^j(1/t)\Theta_j(u)+\sum_0^{\infty} H_n^*(1/t)u^n$, with appropriate conditions (too lengthy to cite here) on the functions that appear on the right-hand side, then the homogeneous system (1) (that is, with $c_n=0$) has solutions determined by the functions $R_j(1/t)$. Examples are given in the way of illustration. (Received September 9, 1949.)

17t. G. M. Wing: *The mean convergence of orthogonal series.*

The mean convergence of Fourier-Bessel series is studied. The functions $\phi_n(x)=2^{1/2}J_\nu(\mu_n x)/J_{\nu+1}(\mu_n)$, where $\nu\geqq 1/2$ and $\{\mu_n\}$ is the sequence of successive positive roots of $J_\nu(x)$, form an orthonormal set on $(0, 1)$ with weight function x. Let $\int_0^1|f(x)|^p x\,dx<\infty$ for some p, $4/3<p<4$, and define $a_n=\int_0^1\phi_n(x)f(x)x\,dx$. It is proved that $\lim_{N\to\infty} \int_0^1\left|f(x)-\sum_{n=1}^{N} a_n\phi_n(x)\right|^p x\,dx=0$. The methods used are similar to those of M. Riesz (*Sur les fonctions conjugées*, Math. Zeit. vol. 27 (1927) pp. 218–244) and H. Pollard (*The mean convergence of orthogonal series*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 387–403; vol. 63 (1948) pp. 355–367). An example is given to show that the theorem fails for $1 \leq p < 4/3$ (or $p > 4$). It is also proved that if $\int_0^1 |f(x)|^p dx < \infty$ for some $p > 1$ and $b_n = \int_0^1 f(x)\phi_n(x)x^{1/2}dx$ then $\lim_{N\to\infty} \int_0^1 |f(x) - \sum_{n=1}^{N} b_n\phi_n(x)x^{1/2}|^p dx = 0$. These results are used to study the more general problem in which the μ_n are the zeros of $xJ'_\nu(x) + HJ_\nu(x) = 0$ (H a real number). Similar theorems are established in this case. Polynomials orthogonal on the interval $(-1, 1)$ are also investigated and some results of Pollard on mean convergence of Jacobi series are generalized. (Received August 17, 1949.)

APPLIED MATHEMATICS

18t. Hilda Geiringer: *Linear differential equations of the plane stress problem for a perfect plastic body.*

For the perfect plastic body the state of stress is restricted to a manifold $F(\sigma_1, \sigma_2, \sigma_3) = 0$ (σ_i principal stresses). The nonlinear plane stress problem, $\sigma_3 = 0$, determined by two equilibrium conditions and $F(\sigma_1, \sigma_2, 0) \equiv F(\sigma_1, \sigma_2) = 0$ has been linearized by v. Mises. With ξ, η denoting principal stress directions he introduced $\theta = (x, \xi)$ and some parameter s for which $F(\sigma_1(s), \sigma_2(s)) = 0$ as coordinates in a "stress graph" and derived (if $j = \partial(s, \theta)/\delta(\xi, \eta) \neq 0$) linear differential equations for x and y dependent on s and θ. Much simpler equations hold for $X = x \cos \theta + y \sin \theta$, $Y = y \cos \theta - x \sin \theta$. With $\sigma'_i = d\sigma_i/ds$, $f(s) = (\sigma_2 - \sigma_1)/\sigma'_1$, $g(s) = (\sigma_2 - \sigma_1)/\sigma'_2$, and subscripts denoting partial differentiation: $X_\theta - Y = gY_s$, $Y_\theta + X = fX_s$, $X_{\theta\theta} - fgX_{ss} = -X + X_s \cdot (f'g + f - g)$, and $Y_{\theta\theta} - fgY_{ss} = -Y + Y_s \cdot (g'f + f - g)$. These equations which are hyperbolic, parabolic, elliptic according as fg is greater than, equal to, or less than 0, form the basis of the linearized theory. After choice of a specific yield condition all methods for linear equations, for example, Bergman's operator method, or expansion in series of functions are available. For the slope of the characteristics in the "physical plane" (the images of the fixed characteristics in the stress graph for which $ds/d\theta = (fg)^{1/2}$) there holds $dy/dx = \tan(\theta \pm \phi)$ with $\tan^2 \phi = f/g$. By means of this mapping the initial value problem is solved approximately by a simple graphic procedure. (Received September 8, 1949.)

19t. Hilda Geiringer: *Parabola-yield-condition for the perfect plastic body.*

For the "parabola-limit" (v. Mises, 1949) $F(\sigma_1, \sigma_2) = (\sigma_1 + \sigma_2)^2 - 4aK(\sigma_1 - \sigma_2) - 4a^2K^2 = 0$, $a = 1 + 2^{1/2}$, the *plane stress* problem becomes hyperbolic throughout, exhibiting remarkable mathematical simplifications. With notations of the preceding abstract, and $g(s)h'(s) + h(s) = 0$, $X/gh = \psi_s$, $Y/h = \psi_\theta$, $j = \partial(s, \theta)/\partial(\xi, \eta) \neq 0$, a *linear* equation of the problem is $L = \psi_{ss} - fg\psi_{ss} - \psi_s \cdot (fg' - f - g) = 0$, satisfied, for the parabola-limit, by products of trigonometric and hypergeometric functions. The fixed characteristics in the "stress graph" are circles (congruent sin-lines) for s and θ polar (rectangular) coordinates; for the characteristics in the "physical plane" $dy/dx = \tan(\theta + \phi)$, where $\tan^2 \phi = (a - s)/(a + s)$. With $s = a \sin t$, L reduces to $\psi_{\theta\theta} - \psi_{tt} + \psi_t \cdot (\cos t)^{-1} = 0$, $|t| < \pi/2$, to which, for example, Bergman's operator method can be applied. With a view to more general use a space generalization is derived: With $J_1 = \sigma_1 + \sigma_2 + \sigma_3$, $J_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$, $F(\sigma_1, \sigma_2, \sigma_3) = (4a^2K^2 - J_1)^2 - 16a^2K^2(J_1^2 - 4J_2)$ for $|J_1| \leq 2aK$, while $F = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 - 4a^2K^2$ for $|J_1| \geq 2aK$.

For $\sigma_3=0$ this reduces to $F(\sigma_1, \sigma_2)=0$, it intersects each "ray" in centric symmetric points and approximates well the v. Mises (1913) limit. (Received September 8, 1949.)

20*t*. Hilda Geiringer: *Simple wave solutions for the plane stress problem of the perfect plastic body.*

Solutions of the plane plasticity problem (two equilibrium conditions, one yield condition, $F(\sigma_1, \sigma_2)=0$) are studied such that each straight line of a continuous one-dimensional set in the x-y-plane carries a constant value of the stress tensor Σ. It follows that such solutions exist only in the hyperbolic domain, that these straight lines form a family of characteristics, $\{C_1\}$, that each C_1 is mapped onto a point (s, θ) of the "stress graph," all points lying on a characteristic curve Γ, and that all "cross characteristics," $\{C_2\}$—and consequently the entire "simple wave"—are likewise mapped onto Γ. It can be shown that Σ is the same function of the slope $m=\tan \delta$ for all sets of straight lines. For the "elliptic limit," $F=\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2-4K^2$, the wave extends over 180°. The solution is (upper sign for σ_1, lower for σ_2): $\sigma_{1,2}=K(3m\mp(4+m^2)^{1/2})/(3+3m^2)^{1/2}$, $\theta=\pi/4+\arctan m-2^{-1}\arctan m/2$ $(-\pi/2\leqq\delta\leqq\pi/2)$. The cross characteristics, $\{C_2\}$, in the case of the "centered wave," have the polar equation $r^2\cos\delta=r_0^2$. For the "parabola-limit" $F\equiv(\sigma_1+\sigma_2)^2-4aK(\sigma_1-\sigma_2)-4a^2K^2$ the wave extends over 270°: $\sigma_{1,2}=2^{-1}Ka$ $(2\cos 2\delta/3\mp\sin^2 2\delta/3)$, $\theta=2\delta/3$ $(0\leqq\delta\leqq 3\pi/2)$. For the centered wave the $\{C_2\}$ have the equation $r=r_0$ $(\sin 2\delta/3)^{-3/2}$, while for the principal stress lines $r=r_0(\sin\delta/3)^{-3}$ and $r=r_0$ $(\cos\delta/3)^{-3}$ respectively. (Received September 8, 1949.)

21. Fritz John: *On the motion of floating bodies.* II.

Uniqueness and existence proofs are given for the simple harmonic motion of a heavy liquid of constant depth h in the presence of a partly immersed obstacle. Let R be the region in xyz-space bounded by the average free surface S_F, the average immersed obstacle surface S_I, and the bottom surface S_B. It is assumed that no point of S_F lies vertically above a point of S_I. The motion is described by the complex velocity potential W. W satisfies Laplace's equation in R, $W_n=0$ on S_B, $W_n=\lambda W$ on S_F, and $W=W'+W''$, where W' is the "primary" wave which is regular everywhere, and W'' satisfies Sommerfeld's radiation condition. It is proved that W is uniquely determined by W' and by the normal derivative W_n of W on S_I. Conversely, there exists a motion for given W_n on S_I and given W', if it is assumed that S_I is convex and that S_I and S_F intersect at right angles. (Received September 8, 1949.)

22*t*. R. R. Reynolds: A note on orthogonalization.

Let f and v be column matrices of functions f_m, which are complete and orthonormal with respect to a metric [ ] in a space S, and v_m, which are only complete in S. A triangular matrix Γ exists such that $f=\Gamma v$; from the orthonormality relation $[ff^*]=I$ (f^* is the transposed conjugate of f) it follows that $I=\Gamma[vv^*]\Gamma^*$, or (a) $[vv^*]=D=\Delta\Delta^*$, (b) $\Gamma=\Delta^{-1}$, (c) $D^{-1}=\Gamma^*\Gamma$. Thus the orthonormalization coefficients are determined from an auxiliary matrix Γ appearing in the so-called square root method of inverting D; this is accomplished in the steps (a), (b), (c), and was noted by Schur (J. Reine Angew. Math. (1917)) and applied by Banachiewicz (Bulletin International de l'Académie Polonaise des Sciences et des Lettres. Ser. A (1938)). (Received September 16, 1949.)

23. D. M. Young: *Iterative methods for solving the finite difference analogue of the Dirichlet problem.*

The finite difference analogue of the Dirichlet problem with N interior mesh points can be solved systematically either by: (i) replacing the trial value at each point by the average of the values at four adjacent points and iterating $(u^{i+1}(u_1, u_2, \cdots, u_N) = R(u^i))$; or (ii) traversing the points in a prescribed order σ and averaging as in (i) except that new values are used as they are obtained $(u^{i+1} = L_\sigma(u^i))$. It is proved for the symmetric transformation R that if $R(v) = \mu v$ there exists v^* such that $R(v^*) = (-\mu)v^*$. Although the matrix of the Liebmann operator L_σ is non-symmetric, under the usual choice of σ the N-dimensional function space S consists of: an invariant subspace S_0, associated with the eigenvalue 0, of dimension $N - s/2$ (where s is the number of nonzero eigenvalues of R); and a complementary subspace S_1 such that the normal form of the corresponding submatrix is diagonal. To each pair v, v^* corresponds $v_1 = \mu^{p+q}v = (-\mu)^{p+q}v^*$ such that $L_\sigma(v_1) = \mu^2 v_1$, where $x = ph, y = qh, h =$ mesh size, and the ordering σ is taken with increasing $(x+y)$. The rate of convergence of L_σ is twice that of R. Moreover the eigenvalues of L_σ are all real. A counterexample is given to show that there exists an ordering σ for which the results are not valid. (Received September 14, 1949.)

GEOMETRY

24*t*. M. C. Foster: *Transformations on rectilinear congruences.*

Each line of a congruence is associated with that point on the unit sphere at which the normal is parallel to the line. Accordingly, a congruence is defined by the coordinates (a, b) of the point in which an arbitrary line pierces the tangent plane at the associated point. A second congruence (A, B) is associated with the original by various transformations on (a, b), such as the affine transformation and inversion. The conditions are developed whereby a congruence of one type is transformed into another. (Received September 16, 1949.)

25. T. S. Motzkin: *The dual curve for* $p \neq 0$. Preliminary report.

While much of the classical theory of algebraic plane curves subsists for an algebraically closed coordinate field of nonzero characteristic, three main new features appear. 1. Duality is no longer a contact transformation. Every (irreducible) curve is the dual C' of infinitely many curves C. $C'' = C$ if (I): the duality $C \rightarrow C'$ is birational. (I) holds if and only if $p = 2$ and C has not an inflexion everywhere. 2. If (I) is not true then the order of contact of the tangent P' at a general point P is some $p^e - 1$, and P' may have more than one point of contact. Plucker's formulae change accordingly. 3. The tangent $P_0' = \lim P'$ of a branch with origin P_0 may differ from its quasitangent $P_0^* = \lim P_0P$, and P_0 from the quasiorigin $P_0''^* = \lim P_0'P'$. A branch has four independent characteristic numbers and, for curves fulfilling (I), together with the dual branch six or five (for $p = 0$ two, its cusp number s and the dual s'). For $p = 3$, $s = s'$; the dual of a cusp is a cusp, of an inflexion a triple cusp. For $p = 2$, $s = 0$; every cusp is to be counted twice. (Received September 6, 1949.)

26*t*. H. P. Mulholland: *On Geöcze's problem for nonparametric surfaces.*

While previous solutions of this problem, by A. Mambriani (Annali della Scuola

Normale Superiore di Pisa vol. 13 (1944) pp. 1–17) and by the author (Proc. London Math. Soc., forthcoming) are independent of earlier restricted solutions, in the present paper the author settles the problem comparatively shortly by following up an earlier approach due to T. Radó. Let $f(x, y)$ be everywhere continuous, R be an oriented rectangle, $A(R)$ the Lebesgue area and $A_Y(R)$ the W. H. Young area of the surface $S(R)$: $z=f(x, y)[(x, y)\in R]$, $A^*(R)$ the lower limit of the areas of polyhedra $z=p(x, y)[(x, y)\in R]$ *inscribed* to $S(R)$ as the maximum face-diameter tends to 0, $2^{-1}hk\Phi(x, y; h, k)$ the area of the triangle with vertices $(x, y, f(x, y))$, $(x+h, y, f(x+h, y))$, $(x, y+k, f(x, y+k))$, and $I^*(R, q)$ be lim sup $\iint_R \Phi(x, y; h, k)dxdy$ as $h, k\to 0$, $h/k=q$ (a constant). The desired proof that $A^*(Q)=A(Q)$, where Q is $[0, 1; 0, 1]$, rests on lemmas: (i) $A^*(R)\leqq I^*(R, q)$ (cf. T. Radó, *Length and area*, New York, 1948, V. 3.53, pp. 548–549); (ii) $I^*(R, q_R)-A(R)\leqq\{8A(R)[A(R)-Y(R)]\}^{1/2}$, where $Y(R)$ is the magnitude of the vector-area of the curve bounding $S(R)$ and q_R depends on its direction; (iii) $A^*(R)$ almost always increases by subdivision; (iv) $A(R)=A_Y(R)$ (a known result). (Received September 6, 1949.)

27. Walter Prenowitz: *Spherical geometries and multigroups.*

Spherical geometries are defined by postulates similar to those of J. R. Kline (Ann. of Math. (2) vol. 18 (1916)) in terms of ***point*** and ***between***, but involving no continuity or dimensional restriction. Let an abelian multigroup be called ***regular*** if (1) it has an identity 0 satisfying $a+0=a$, (2) each element a has an inverse $-a$ satisfying $a+(-a)\supset 0$, (3) subtraction is related to addition in the usual way, $a-b=a+(-b)$. The theory of regular multigroups which bears very strong analogies to that of abelian groups is outlined. A spherical geometry G is converted into a regular multigroup M by adjoining an "ideal point" o and properly defining $+$, the join of two points. The geometry of G is reduced to the algebra of M, for example spherical subspaces correspond to submultigroups, halfspaces to cosets. Spherical geometries are characterized by the fact that their associated multigroups satisfy (4) the idempotent law $a+a=a$ and (5) each element other than 0 has order 3. (Received September 15, 1949.)

28. W. G. Wolfgang: *On the construction of the equation of the n-dimensional analogue of the conic sections.*

The author generalizes to n dimensions a method given by Brink in his book, *Essentials of analytic geometry*, of constructing the equation of the conic passing through five given points. Given $[n(n+3)]/2$ points in n-space such that no $n+1$ of them lie in an $(n-1)$-space, construct the equation of the n-dimensional analogue of the conic (the locus of an equation of the second degree in n variables) passing through them as follows. (i) Choose any $2n$ of the given points and group them in $n(n-1)+2$ groups of n points each, then write the equations, $\Pi_\nu=0$, of the n-dimensional linear forms passing through each of these groups. (ii) Pair these equations so that neither member of a pair intersects the other member in any of the given points. Relabel the equations so that these pairs are denoted by: $\Pi_{2\nu}\cdot\Pi_{2\nu+1}=0$. (iii) Form the equation: $\sum_{\nu=0}^{n(n-1)/2}\lambda_\nu(\Pi_{2\nu}\Pi_{2\nu+1})=0$, $\lambda_0=1$. (iv) Substitute the remaining $n(n-1)/2$ points in this equation and obtain $n(n-1)/2$ equations in $n(n-1)/2$ unknowns. (v) Solve these equations simultaneously and denote the values by α_ν. (vi) Then $\sum_{\nu=0}^{n(n-1)/2}\alpha_\nu(\Pi_{2\nu}\cdot\Pi_{2\nu+1})=0$, $\alpha_0=1$, is the equation of the nondegenerate analogue of the conic passing through the given points. (Received October 5, 1949.)

Logic and Foundations

29. S. C. Kleene: *A symmetric form of Gödel's theorem.*

It has been remarked that recursively enumerable sets behave surprisingly similarly to analytic sets and general recursive sets to Borel sets. There is a theorem which says that two disjoint analytic sets can always be separated by a Borel set. In this note, two disjoint recursively enumerable sets C_0 and C_1 are constructed which cannot be separated by a general recursive set. Hence there is no exact parallelism between the two theories. Given any two disjoint recursively enumerable sets D_0 and D_1 such that $C_0 \subset D_0$ and $C_1 \subset D_1$, a number f can be found such that $f \notin D_0 + D_1$. The sets C_0 and C_1 are obtained by rearranging Rosser's proof of Gödel's theorem, in which the existence of an undecidable proposition is inferred from simple consistency only instead of ω-consistency. The rearrangement makes the treatment of the proposition and of its negation symmetrical. (Received September 18, 1949.)

30*t*. A. R. Schweitzer. *An analysis of Sir James Jeans' Physics and philosophy.* I.

Jeans' treatise, *Physics and philosophy* (Cambridge and New York, 1943; reprinted, 1946) has to do with nature, events of nature and their pattern; pictorial representation and mathematical description of this pattern. Two such descriptions which, in Jeans' view, are believed to be "complete and perfect" are provided by Einstein's generalized theory of relativity and the new quantum theory due to Heiseinberg; de Broglie and Schrödinger, and Dirac (ibid. pp. 12, 68, 118; pp. 158, 174). Relativity is approached by discussing space, time and space-time historically and critically. As a basis for criticism Jeans introduces (ibid. pp. 42, 43) a doctrine of three worlds of modern science: the world of the electron or the minute-scaled world of atomic physics; the "man-sized" world; and the world of the nebulae or the vast-scaled world of astronomy. Jeans assumes (ibid. pp. 43, 70) that the same laws of nature prevail throughout the range of phenomena of the three worlds. (Received September 13, 1949.)

31*t*. A. R. Schweitzer: *An analysis of Sir James Jeans' Physics and philosophy.* II.

Jeans' view of space associated with the concepts "right" and "left" (ibid. p. 65) is discussed by the author in detail by referring to the author's foundations of geometry. Transition to philosophy is effected by Jeans by recognizing that in modern science certainty must be replaced by probability. Much use is made by Jeans of analogy in the course of his exposition and prominence is given to simplicity as a working hypothesis in the evolution of physics. Philosophically, Jeans is committed to positivism. The author refers to Sir A. S. Eddington's combination of a modification of relativity ("intermediate" relativity) and quantum theory: *Fundamental theory* (posthumous volume, Cambridge, 1946). (Received September 13, 1949.)

32*t*. A. R. Schweitzer: *An examination of Whitehead's Process and reality.* I.

Whitehead's *Process and reality* (New York, 1929) representing the "philosophy of organism" is classed among treatises having a heraclitean motive including Schleiermacher's philosophy of dependence and Bergson's philosophy of movement. More specifically, Whitehead's philosophy is interpreted as a theory suggested by mathematics as a science of extension (space, time, space-time) and by mathematical physics (including atomic physics). The latter subject is considered mainly in Part II, Chapter III: The order of nature; the former is treated in Part II, Chapter II: The extensive continuum and in Part IV: The theory of extension. Both aspects are recognized by Whitehead in his theory of prehension (Part III: pp. 337, 365; compare Part II, p. 227 and Part IV, p. 433). This generalization leads to Whitehead's cosmology (ibid. p. 365). (Received September 13, 1949.)

33*t*. A. R. Schweitzer: *An examination of Whitehead's Process and reality*. II.

Geometry is described by Whitehead philosophically as the "investigation of the morphology of nexus"; subsequently geometry is discussed in a sense familiar to mathematicians. Arithmetical theorems seem to Whitehead the "most obviously metaphysical" among propositions. Historically, Whitehead regards his philosophy as a continuation of Plato and Locke and the inversion of Kant's philosophy. A concluding part of the author's paper is concerned with Whitehead's theory of orientation and his doctrine of relations. Whitehead's references to orientation include opposites; discrimination between positive and negative applied to prehension; polarity applied to actual entity (actual occasion), becoming, concrescence and creative urge; order applied to society and nexus. Whitehead describes relation as a genus of contrasts or modes of synthesis of entities in one prehension. The author discusses Whitehead's essay, *An enquiry concerning the principles of natural knowledge* (Cambridge, 1919) as a preliminary to *Process and reality*. (Received September 13, 1949.)

STATISTICS AND PROBABILITY

34. E. J. Gumbel: *Orthogonal least squares applied to probability papers*.

A probability paper for a continuous unlimited statistical variate x which permits a linear reduction $x=u+y/\alpha$ is a rectangular grid for the observed variate x, and the reduced variate y, where the probability $F(y)$ is written instead of y. The simplest plotting procedure is the use of the mean frequency $m/(n+1)$ for the mth among n observations. The tiresome calculation of the cross product $(\overline{xy})_n$ may be avoided by minimizing the orthogonal distances which leads to the equations $u=\bar{x}-\bar{y}_n/\alpha$ and $\alpha=\sigma_n/s_x$, where $\bar{x}$ and s_x are the observed mean and standard deviation, and $\bar{y}_n$, σ_n are the corresponding theoretical values. These two moments may be calculated as functions of n from a table of $y(F)$. For symmetrical distributions, $\bar{y}_n$ vanishes, and u is estimated as the observed mean. For the standardized normal distribution, the expression $z_{2,n}$ defined by $F(z_{2,n})=\sigma_n$ approaches quickly a linear function of $\breve{y}_n$, where $\breve{y}_n$ is defined as the solution of $F(\breve{y}_n)=n/(n+1)$. For the asymptotic distribution of extreme values of the exponential type $z_{1,n}$ defined by $F(z_{1,n})=\bar{y}_n/\gamma$ and $z_{2,n}$ defined by $F(z_{2,n})=\sigma_n 6^{1/2}/\pi$ are also linear functions of $\breve{y}_n$. These relations simplify the calculations of $\bar{y}_n$ and σ_n as functions of n. (Received September 19, 1949.)

35. B. O. Koopman: *Necessary and sufficient conditions for Poisson's distribution*.

An infinite sequence of sets of n independent trials ($n=1, 2, 3, \cdots$) is considered; $p_{n,k}$ is the probability that, in the nth set of trials, the kth trial shall succeed. It is proved in this paper that a necessary and sufficient condition for obtaining the Poisson distribution (of given total numbers of successes in the nth set of trials) as a limit ($n \to \infty$) is that: (a) $p_{n,1}+ \cdots +p_{n,n}$ approach a limit, and (b) the maximum $p_{n,k}$ ($1 \leqq k \leqq n$) approach zero (as $n \to \infty$). The special case in which the probabilities are in fixed ratios ($p_{n,1}:p_{n,2}:p_{n,3}: \cdots$) independent of n is considered and connected with certain divergent series of non-negative terms. Applications are given, in some of which the ratio of the maximum to minimum $p_{n,k}$ in the set ($1 \leqq k \leqq n$) increases as any given positive power of n as $n \to \infty$. (Received October 3, 1949.)

TOPOLOGY

36. L. W. Cohen and Casper Goffman: *The metrization of uniform space.*

It is shown that a necessary and sufficient condition for a uniform space S to be metrizable with distances in an ordered abelian group is that the neighborhood system $U_\xi(x)$, $\xi < \xi^*$, $x \in S$, satisfy the axioms 1, 2, 3, 4′ in the authors' paper *A theory of transfinite convergence*, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 65–74. (Received August 15, 1949.)

37. G. D. Mostow: *A theorem on locally euclidean groups.*

The *derived subgroup* of a topological group is defined as the closure of the commutator subgroup. If G is a locally compact, connected topological group, then its derived series, that is, series of successive derived subgroups, becomes stationary at some finite stage. The *core* of such a group is defined as the subgroup in the derived series which coincides with its derived subgroup. *Theorem.* Let G be a locally euclidean group and let C denote its core. Then G/C is a Lie group. In the special case that the core consists of the identity element alone, this theorem asserts that a solvable locally euclidean group is a Lie group. (Received September 16, 1949.)

38. Everett Pitcher: *A model for the homotopy theory of spheres.* Preliminary report.

A model for the homotopy theory of the n-sphere is constructed by use of deformations inherent to the theory of critical points of the length integral. Methods based on exact homomorphism sequences are used to prove theorems which have the following corollaries: (1) it is false that $\pi_{2n}(S^n)=0$ for two consecutive values of $n \geqq 2$; (2) it is false that $\pi_{2n+2}(S^n)=0$ for two consecutive values of $n \geqq 3$; (3) it is false that $\pi_{2n+4}(S^n)=0$ for two consecutive values of $n \geqq 5$. Of special interest is a class of homomorphisms of $\pi_r(S^n)$ into $\pi_{r-2}(S^{p(n-1)-1})$, defined for all r, with p determined so that $p(n-1)-1 \leqq r \leqq (p+1)(n-1)-2$ (and believed to be equivalent to the generalized Hopf homomorphism of G. W. Whitehead when $p=2$). (Received September 16, 1949.)

T. R. HOLLCROFT,
Associate Secretary

THE NOVEMBER MEETING IN CHICAGO

The four hundred fifty-first meeting of the American Mathematical Society was held at the Illinois Institute of Technology on Friday and Saturday, November 25–26, 1949.

The total attendance was about 175 including the following 151 members of the Society:

A. A. Albert, W. R. Allen, W. F. Atchison, W. L. Ayres, Reinhold Baer, R. G. Bartle, S. F. Bibb, K. E. Bisshopp, L. M. Blumenthal, D. G. Bourgin, A. J. Brandt, Richard Brauer, R. H. Bruck, J. R. Buchi, J. W. Butler, S. S. Cairns, A. P. Calderon, R. H. Cameron, C. S. Carlson, S. S. Chern, Herman Chernoff, H. M. Clark, E. H. Clarke, M. D. Clement, Harvey Cohn, B. H. Colvin, E. G. H. Comfort, J. J. Corliss A. R. Craw, H. J. Curtis, M. M. Day, John DeCicco, R. F. Deniston, D. M. DeWitt, Flora Dinkines, N. J. Divinsky, J. M. Dobbie, R. J. Duffin, John Dyer-Bennet, D. O. Ellis, R. L. Erickson, M. H. M. Esser, H. P. Evans, R. L. Evans, D. A. Flanders, L. R. Ford, Abraham Franck, Evelyn Frank, D. M. Friedlen, C. G. Fry, R. E. Fullerton, Albert Furman, J. W. Gaddum, Abe Gelbart, D. G. Gore, S. H. Gould, L. M. Graves, Harold Greenspan, M. M. Gutterman, P. R. Halmos, Charles Hatfield, E. D. Hellinger, I. N. Herstein, G. P. Hochschild, C. C. Hsiung, H. K. Hughes, Herbert Jehle, Ralph Hull, W. E. Jenner, M. M. Johnsen, L. H. Kanter, Irving Kaplansky, L. M. Kelly, D. E. Kibbey, Jacob Korevaar, W. C. Krathwohl, M. Z. Krzywoblocki, H. G. Landau, E. P. Lane, R. E. Langer, K. B. Leisenring, Howard Levin, D. J. Lewis, Saunders MacLane, H. M. MacNeille, W. H. Marlow, M. S. Matchett, C. W. Mathews, A. E. May, Karl Menger, J. M. Mitchell, J. T. Moore, C. W. Moran, K. A. Morgan, D. R. Morrison, Leopoldo Nachbin, M. J. Norris, Daniel Orloff, E. H. Ostrow, Gordon Pall, M. C. Peixoto, M. M. Peixoto, Sam Perlis, D. H. Potts, G. B. Price, L. E. Pursell, A. L. Putnam, W. P. Reid, W. T. Reid, Haim Reingold, L. G. Riggs, A. E. Ross, J. M. Sachs, R. G. Sanger, A. C. Schaeffer, K. C. Schraut, W. T. Scott, I. E. Segal, B. R. Seth, S. S. Shü, J. H. Siedband, G. F. Simmons, I. M. Singer, M. L. Slater, D. M. Smiley, M. F. Smiley, Albert Soglin, Daniel Sokolowsky, E. H. Spanier, E. J. Specht, C. E. Springer, Marvin Stippes, E. B. Stouffer, A. F. Strehler, T. T. Tanimoto, C. J. Thomas, E. F. Trombley, H. L. Turrittin, J. L. Ullman, F. A. Valentine, Bethuhne Vanderburg, Robert Wasserman, André Weil, George Whaples, L. R. Wilcox, A. W. Wundheiler, J. E. Yarnelle, J. W. T. Youngs, R. K. Zeigler, Daniel Zelinsky, Antoni Zygmund.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor R. H. Bruck delivered an address entitled *An extension theory for a certain class of loops.*

Sessions for the presentation of contributed papers were held at 10:30 A.M. on Friday and Saturday and at 3:15 P.M. on Friday.

Presiding officers for the various sessions were Professors Reinhold Baer, L. M. Blumenthal, L. R. Ford, Gordon Pall, and Dr. H. M. MacNeille.

On Friday afternoon there was a tea for the Society in the Metallurgical and Chemical Engineering Building.

Abstracts of papers read follow below. Papers read by title are indicated by the letter "*t*." Paper number 46 was read by Mr. Calderon and paper number 51 by Dr. Slater.

ALGEBRA AND THEORY OF NUMBERS

39. Richard Brauer: *On the zeta functions of algebraic number fields.* II.

Consider algebraic number fields K of degree n over the field of rational numbers and denote the discriminant by d, the class number by h, and the regulator by R. In an earlier paper (Amer. J. Math. vol. 69 (1947) pp. 243–250) the asymptotic relation $\log (hR) \sim \log |d|^{1/2}$ was proved for fields of a fixed degree n. This result is now extended to sequences of fields normal over the field of rational numbers for which $\log |d| / n \log n \rightarrow \infty$ (where n is not fixed). A weaker result is obtained for fields which are not normal. (Received October 13, 1949.)

40. M. M. Day: *Amenable groups.* Preliminary report.

The definitions of amenable and strongly amenable groups were given in Bull. Amer. Math. Soc. Abstract 55-11-507. Theorem 1. If G is (strongly) amenable, so is every subgroup of G and every factor group of G. Theorem 2. G is (strongly) amenable if and only if every finitely generated subgroup of G is (strongly) amenable. Corollary. No amenable group has a free subgroup on more than one generator. (Received October 14, 1949.)

41. D. O. Ellis: *Autometrized Boolean algebras.*

Let B be a Boolean algebra with meet, join, and complement denoted by ab, $a+b$, and a' respectively. Defining "distance" of a and b by $d(a, b) = a'b + ab'$, we call such a system an autometrized Boolean algebra. The notions of betweenness, congruence, superposability, and free mobility are defined in terms of this "distance" analogously to the definitions of these concepts in a metric space. It is shown that betweenness is equivalent to general lattice betweenness (Pitcher and Smiley, Trans. Amer. Math. Soc. vol. 52 (1942) pp. 95–114), and that B has the property of free mobility. A B-metrized space is any abstract set with symmetric "distances" in B which satisfy the vanishing condition. It is shown that if each three points of an arbitrary B-metrized space are congruently contained in B then the space is congruently contained in B, and that three is the smallest number with that property. (Received September 30, 1949.)

42*t*. Marshall Hall: *A basis for free Lie rings and higher commutators in free groups.*

E. Witt has found the number $\psi n(q)$ of linearly independent elements of degree n in the free Lie ring with q generators but his method gives no way of finding a basis. By an isomorphism, $\psi n(q)$ is also the number of independent commutators of weight n modulo weight $n+1$ in the free group with q generators. An explicit construction is given in this paper for a basis of the Lie ring and the corresponding higher commutator groups. The commutators are precisely those which arise in Philip Hall's collecting process as given in *A contribution to the theory of groups of prime power order*, Proc. London Math. Soc. vol. 36 (1934) pp. 29–95. (Received September 25, 1949.)

43t. Irving Kaplansky: *Topological representation of algebras.* II.

The previous paper by Arens and the author was devoted mainly to the study of commutative algebraic algebras. In the present paper commutativity is weakened to the assumption of a polynomial identity. Such algebras are shown to be built out of certain uniquely determined homogeneous parts, and the latter are determined if a countability assumption is satisfied. There are applications to Kurosch's problem of determining whether algebraic algebras are locally finite: (1) the answer is affirmative for the case of a polynomial identity, (2) if the answer is affirmative for primitive algebras and nil algebras, then it is affirmative for all algebraic algebras. (Received September 20, 1949.)

44t. B. E. Meserve: *A note on polynomial equations and inequalities.*

Methods for the determination of the number r of n-tuples of real numbers satisfying the system (A) $0<f_i$ $(i=1, 2, \cdots, p)$, $0=f_i$ $(i=p+1, \cdots, m)$ where the f_i are polynomials in $x_1, x_2, \cdots, x_n$ with real coefficients have been given by A. Tarski (*A decision method for elementary algebra and geometry*, Report 109, Project Rand) when r is finite. If $p=m$, the resulting system (B) $0<f_i$ $(i=1, 2, \cdots, m)$ is either inconsistent or r is infinite. In the latter case the system is satisfied for n-tuples lying in a certain number k of n-dimensional maximal open segments. These segments form the *content* of the system (B). The determination of k constitutes an extension of Sturm's Theorem to polynomial inequalities and has not in general been solved. Solutions have been given by the writer for the case of one unknown (Amer. J. Math. (1947) pp. 357–370) and, in this paper, for the system (C) $0<b_iE_i$ $(i=1, 2, \cdots, n)$ where E_i is the elementary symmetric polynomial of degree i in $x_1, x_2, \cdots, x_n$. The system (C) is consistent if and only if $0\neq b_1b_2\cdots b_n$ and if consistent has $k=n!/[V!(n-V)!]$ where V is the number of variations of sign in the sequence $1, b_1, b_2, \cdots, b_n$. Using extensions of Descartes' rule of signs the content of (C) is explicitly given for all cases in which $k=1$ or n. (Received October 11, 1949.)

45. Gordon Pall: *Simultaneous representation by adjoint quadratic forms.*

An algorithm is given for constructing all simultaneous solutions of $a=f(x_1, \cdots, x_n)$, $0=x_1y_1+\cdots+x_ny_n$, $b=g(y_1, \cdots, y_n)$, in integers x_i, y_i such that $(x_1, \cdots, x_n)=(y_1, \cdots, y_n)=1$. Here f is any real nonsingular quadratic form, and g is its adjoint form. In general, there is a precise formula connecting the number of "sets" of solutions by the system of classes of a genus, and certain classes of quadratic forms in $n-2$ variables along with the numbers of solutions of certain systems of quadratic congruences. The results are especially simple if f is in a genus of one class, and if a and b are relatively prime integers. Thus, to state the simplest example, if a and b are positive and relatively prime, the number of solutions of $a=x_1^2+x_2^2+x_3^2$, $0=x_1y_1+x_2y_2+x_3y_3$, $b=y_1^2+y_2^2+y_3^2$, in integers such that x_1, x_2, x_3 are relatively prime and y_1, y_2, y_3 are relatively prime, is equal to $24kk'$, where k denotes the number of solutions of $t^2\equiv -b \bmod a$, and k' is the number of solutions of $u^2\equiv -a \bmod b$. (Received October 11, 1949.)

ANALYSIS

46. A. P. Calderon and Antoni Zygmund: *Note on the boundary values of functions of several complex variables.*

(i) Let $f(z_1, \cdots, z_k)$ be a function regular in the polycylinder $|z_1| < 1, \cdots, |z_k| < 1$, and satisfying the condition $\int_0^{2\pi} \cdots \int_0^{2\pi} \log^+ |f| \, d\theta_1 \cdots d\theta_k = O(1)$, where f stands for $f(r_1 e^{i\theta_1}, \cdots, r_k e^{i\theta_k})$. Then f has a finite limit as $(z_1, \cdots, z_k)$ tends nontangentially to almost every point $(e^{i\theta_1^0}, \cdots, e^{i\theta_k^0})$, provided all the ratios $(1-r_j)/(1-r_k)$ are contained between two positive numbers. (The nontangential approach just quoted means that each z_j tends nontangentially to $e^{i\theta^0}$.) The proof of (i) is based on the following theorem. (ii) Let $f(z_1, \cdots, z_k)$ be regular for $|z_1| < 1, \cdots, |z_k| < 1$, and suppose a set E situated on the distinguished boundary (D_k) $z_1 = e^{i\theta_1}, \cdots, z_k = e^{i\theta_k}$ of the unit polycylinder has the following property: as $(z_1, \cdots, z_k)$ approaches any point $(e^{i\theta_1^0}, \cdots, e^{i\theta_k^0}) \in D_k$ nontangentially and so that all the ratios $(1-r_j)/(1-r_k)$ are contained between two positive numbers, the function f remains bounded. Then, under the same circumstances, the function f tends to a finite limit for almost every point $(e^{i\theta_1^0}, \cdots, e^{i\theta_k^0}) \in E$. (Received October 13, 1949.)

47. R. H. Cameron: *The first variation of an indefinite Wiener integral.*

It is shown in this paper that if $G(u)$ is the Wiener integral of the functional $F(x)$ over the set $S(u)$ of continuous functions $x(t)$ on $0 \leqq t \leqq 1$ which vanish at $t=0$ and satisfy for all t the inequality $x(t) \leqq u(t)$, then under suitable smoothness conditions on $F(x)$ and $u(t)$ and $\delta u(t)$ we have $\delta G(u)$ equal to the Wiener integral over $S(u)$ of the expression $\delta F(x) - 2F(x)\int_0^1 \delta u'(t) dx(t)$. Here it is understood that the variation $\delta F(x)$ is to be based on the independent variation $\delta x(t)$ when the latter is put equal to $\delta u(t)$. (Received October 13, 1949.)

48. R. L. Evans: *Asymptotic solutions of some linear ordinary differential equations containing a parameter.* Preliminary report.

A method of solution used earlier by R. E. Langer (Trans. Amer. Math. Soc. vol. 36 (1934) pp. 90–106) is generalized. Asymptotic solutions with error terms of the order $(1/\lambda^{p+k_0})^{1-\epsilon}$ have been found for the equations $y'' - \lambda^{2p}(\sum_{j=1}^{m} (\lambda^p x)^j \cdot \phi_{pj} + \sum_{j=1-2p-k_0}^{J} \lambda^j \cdot \phi_j + \psi_{k_0}(\lambda, x))y = 0$ in which the indices are integers such that $m=1$, $p=2$, $J=1$, and $k_0 \geqq 0$ or that $m=2$, $p=1$, and $J=k_0=0$, in which latter case a specific relation must exist among the coefficients of the ϕ's. The ϕ's are analytic functions of x, $\phi_{mp}(0) \neq 0$, and ψ_{k_0} is such that $\lambda^{2p+k_0} \cdot \psi_{k_0}$ is an analytic function of x for sufficiently small $|x|$ and bounded for all sufficiently large $|\lambda|$. The method can be extended to certain equations in which the order is larger and/or the indices m and p have values other than 1 and 2. As m is increased the number of supplementary conditions on the ϕ coefficients is also increased. (Received October 22, 1949.)

49. Abraham Franck: *On analytic functions of bounded type.*

Extending results of A. Wishard (Duke Math. J. vol. 9 (1942) p. 663) the author obtains criteria for an analytic function to be of bounded type (B.T.) in $\Re(z) = x > 0$: $f(z)$ is of B.T. in $x>0$ if and only if there exists either (i) a sequence of circles C_n which exhausts $x>0$ such that $\iint_{C_n}(\log^+ |f(z)| / |1+z|^2) dxdy \leqq MR_n$, $M=$const., R_n radius of C_n; or (ii) a sequence of circles C_n^* center on x-axis passing through $x=\rho_n$, $x=1/\rho_n$, $\lim \rho_n = \infty$ such that $\iint_{C_n^*}(\log^+ |f(z)| / |1+z|^2 (dxdy/x) \leqq M \log \rho_n$. By means of these results, extensions of criteria of A. Ostrowski and R. Nevanlinna are obtained for an analytic function to be of B.T. in a simply-connected region R possessing the following property: if $w = \phi(z)$ maps R conformally onto $|w| < 1$, two positive constants c_1 and c_2 exist such that $c_1 \leqq |\phi'(z)| \leqq C_2$. The analytic function $f(z)$ is of B.T.

in R if and only if there exists either (i) some sequence of regions D_n which exhausts R and a point $z=a$ on the boundary of R such that $\iint_{D_n}(\log^+|f(z)|/|a-z|^2)dxdy \leq M/d_n$, d_n the Fréchet distance of the boundary of D_n to the boundary of R; or (ii) some sequence of regions D_n^* which exhausts R such that $\iint_{D_n^*}(\log^+|f(z)|/d_n(z))dxdy \leq M/d_n$, $d_n(z)$ is the distance of the point z from the boundary of R. Several applications are given. (Received October 13, 1949.)

50. Jacob Korevaar. *A theorem on power series with partial sums which are free of zeros in a sector.*

If no partial sum of a formal power series (1) $\sum a_n z^n$ has a zero in the sector $|\arg z| \leq \alpha$ $(\alpha > 0)$, then (1) represents an entire function of order zero. This theorem extends and improves results of L. Weisner for $\alpha > \pi/2$ (Bull. Amer. Math. Soc. vol. 47 (1941) pp. 160–163) and of O. Szász for $\alpha = \pi/2$ (Bull. Amer. Math. Soc. vol. 49 (1943) pp. 377–383, Theorem II). It is first proved that (1) converges for some $z \neq 0$. Suppose that on the contrary $\limsup (C_k = |a_k|^{1/k}) = \infty$. For the values of n for which $C_n \geqq C_k (k < n)$, $f_n(t) = (1 + \cdots + a_{n-p}a_n^{-1}C_n^p t^p + \cdots + a_0 a_n^{-1} C_n^n t^n)^{1/n} \rightarrow 1$, boundedly and hence uniformly in $|\arg t| \leqq \alpha - \eta$, $\delta < |t| < d$ $(\eta, \delta, d > 0)$. In particular $|\{f_n(t)\}^n < 2^n (n > n_0)$ on the interval $1 \leqq t \leqq 10$ of length 9. Hence by Tchebycheff's inequality $\inf |g^{(n)}(t)| \leqq 2^{-1} n! \{4/(b-a)\}^n \sup |g(t)|$, applied to $g(t) = \operatorname{Re} \{f_n(t)\}^n$ and $\operatorname{Im} \{f_n(t)\}^n$, $a=1$, $b=10$, a_0 must be zero, which is absurd. But if (1) converges for some $z \neq 0$, R. Jentzsch's theorem on the zeros of partial sums implies that (1) converges everywhere. Finally, F. Carlson's results on the zeros of the partial sums of entire functions of positive or infinite order (Arkiv. för Matematik, Astronomi och Fysik vol. 35A (1948) no. 14) imply that (1) represents an entire function of order zero. The theorem remains true if for every n there is a sector $|\arg z - \phi_n| \leqq \alpha$ $(\alpha > 0$ independent of $n)$ in which $\sum_{k \leqq n} a_k z^k$ has no zero. (Received October 13, 1949.)

51. M. J. Norris, M. L. Slater, and C. J. Thomas: *Functional inequalities suggested by game theory.* Preliminary report.

The main results are: 1. If (i) $f(x)$ is nondecreasing and continuous in $(-\infty, \infty)$, (ii) $p(t)$ is nondecreasing in $[0, 1]$, (iii) $\alpha(t) \in L(0, 1)$, (iv) $\int_0^1 \alpha dt = k$, and $F(x)$ denotes $\int_0^x f(x)dx$, then $\int_0^1 \alpha f(\int_0^t \alpha p dt)dt \geqq F[p(0+)k]/p(0+)$ when $p(0+) \neq 0$ and $\geqq kf(0)$ when $p(0+) = 0$. 2. If (i) $f(x)$ is continuous and $\int_0^x f dx/x$ is nondecreasing in $(-\infty, \infty)$, (ii) $p(t)$ is nondecreasing in $[0, 1]$ and $\geqq 0$, (iii) $\alpha(t) \in L(0, 1)$, (iv) $\int_0^1 \alpha dt = 0$, then $\int_0^1 \alpha f(\int_0^t \alpha p dt)dt \geqq 0$. 3. If (i) $f(x, y)$ is continuous in the (x, y)-plane (ii) $f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y) \geqq 0$ for all x, y, h, k, (iii) $p(s)$ nondecreasing in $[0, 1]$ and $q(t)$ nondecreasing in $[0, 1]$, (iv) $\alpha(s) \in L(0, 1)$, $\beta(t) \in L(0, 1)$, (v) $\int_0^1 \alpha ds = \int_0^1 \beta dt = 0$, then $\int_0^1 \int_0^1 \alpha(s)\beta(t) f(\int_0^s \alpha p ds, \int_0^t \beta q dt) ds dt \geqq 0$. Continuity restrictions on $f(x)$ and $f(x, y)$ can in general be weakened, but with a loss in compactness in stating the final results. Various partial converses have been proved. (Received November 2, 1949.)

52. W. T. Scott: *The reciprocal of a continued fraction.*

In this note it is shown that the power series-continued fraction correspondence $P(x) = 1 + \sum c_n x^n \sim 1 + K(a_n x^{\alpha_n}/1)$, where $\alpha_{2n-1} = \alpha_{2n}$, a positive integer, implies the correspondence $1/P(x) \sim 1 + K(b_n x^{\alpha_n}/1)$ if and only if there exist parameters h_p such that $a_1 = 2h_1$, $a_{2n} = h_{2n-1}(h_{2n} - 1)$, $a_{2n+1} = (h_{2n}+1)h_{2n+1}$. In this case the b_n are obtained from the a_n by replacing each h_p by $-h_p$. The result, which includes a theorem of Stieltjes, yields several new continued fraction expansions. (Received October 13, 1949.)

53. I. E. Segal: *An extension of Plancherel's formula to separable unimodular groups.*

Any complex-valued function on a separable unimodular locally compact group which is square-integrable relative to Haar measure has the integral of the square of its absolute value equal to the integral over a certain measure space of the square of the relative norm (at each point of the space) of the Fourier transform of the function. The "relative norm" referred to is that defined by von Neumann for operators in factors, and the Fourier transform is defined through the use of the von Neumann reduction theory, together with a limiting process. The measure space in question has its measure ring isomorphic to the Boolean ring of closed linear manifolds in $L_2(G)$ invariant under both left and right translations, and the measure on it is unique modulo the normalization of the relative norms. Use is made of a certain countably-additive unitarily-invariant non-negative-valued function w defined, on the projections in the weak closure L of the algebra generated by the left regular representation, by the equations $w(P)=\|f\|^2$ if P is convolution by the element f of $L_1(G)$, and $w(P)=\infty$ otherwise. Any projection in L is the least upper bound of projections in L on which w is finite, and it results that factors of type III do not occur in the reduction of L with respect to its center. (Received October 12, 1949.)

54. E. F. Trombley: *Extension of M. Riesz's theorem on Hilbert transforms to functions of two variables.*

Here the methods of K. Sokół-Sokołowski (Fund. Math. vol. 34 (1947) pp. 170–177) are used to show that if $f(x, y)$ is a function in $L_p(-\infty<x, y<\infty)$ $(p>1)$, then the Hilbert transform Hf of f satisfies a relation of the form $\|Hf\|_p\leqq K_p\|f\|_p$ where K_p is a function of p alone. In the course of the proof it is shown that: If $f(x)$ is a function in $L_p(-\infty<x<\infty)(p>1)$, then the function $\tilde{f}(x)=\sup_{\sigma>0}\left|\left(\int_{-\infty}^{-\sigma}+\int_{\sigma}^{\infty}\right)(f(x+t)/t)dt\right|$ satisfies the inequality: $\|\tilde{f}\|_p\leqq A_p\|f\|_p$ where A_p is a function of p alone. This result is extended to functions of two variables and then used to obtain the above result on Hilbert transforms. (Received October 13, 1949.)

55. J. L. Ullman: *A mapping property of schlicht functions.*

The mapping property that is proved in the article holds for the normalized exterior mapping function of a simple analytic curve. Let C be a simple analytic curve in the z plane and designate its exterior by D. The normalized exterior mapping function of C is the analytic function $w=f(z)$ which is uniquely determined by conditions that (i) it is regular in D except for a simple pole at $z=\infty$, (ii) its power series expansion about $z=\infty$ has the normalization $w=z+a+b/z+\cdots$; and (iii) it maps D in a 1–1 manner onto the exterior of a circle Σ, $|w|=\rho$. Theorem I. Let C be a simple analytic curve, and designate its exterior by D. Let $f(z)$ be the normalized exterior mapping function of C. Let σ be a circle whose interior lies in D and with center at z_0. Then $F(z)=f(z)/(z-z_0)$ maps σ onto a curve in the w plane that is star-shaped from the point $w=0$. If σ is a circle containing C, then the function $f(z)$ maps σ onto a curve in the w plane that is starshaped from the point $w=0$. The first part of this theorem gives part of the domain in which the ratio of two schlicht functions is again schlicht. The second part provides a complement to the result stated in problem 317, section III of *Aufgaben und Lehrsätze aus der Analysis* by Pólya and Szegö. (Received October 13, 1949.)

56. Bethuhne Vanderburg: *A linear combination of Cesàro and Hölder summability methods*. Preliminary report.

Let H_α, C_α denote the Hölder and Cesàro summability methods of order α, H_n^α, C_n^α the nth Hölder and Cesàro transformations of any sequence $\{s_n\}_0^\infty$. The following are proven. First, for real values of $\alpha > -1$ excepting 0 and 1, the summability method $[C_\alpha - \Gamma(\alpha+1)H_\alpha][1-\Gamma(\alpha+1)]^{-1}$ is equivalent to the method $H_{\alpha+1}$. Secondly, if $\alpha > -1$ and if $\{s_n\}_0^\infty$ is limitable-$H_{\alpha+1}$ to s, then $\limsup_{n\to\infty}$ (inf) $[C_n^\alpha - s] = \limsup_{n\to\infty}$ (inf) $\Gamma(\alpha+1)[H_n^\alpha - s]$. (Received November 18, 1949.)

Applied Mathematics

57. Herbert Jehle: *Two component wave equations.*

In the case of vanishing electromagnetic potentials $\phi_\kappa = 0$, the relativistic second order Schroedinger-Klein-Gordon equation can be satisfied by iterating the equation $\gamma^\kappa(\partial/\partial x^\kappa - i\phi_\kappa)\psi = \mu\psi^*$, ψ^* = conjugate complex of ψ, μ is a real constant, where the γ^κ have to satisfy $(\gamma^{\kappa*}\gamma^\lambda + \gamma^{\lambda*}\gamma^\kappa)/2 = -g^{\kappa\lambda}1$ with $g^{00} = +1$, $g^{11} = g^{22} = g^{33} = -1$. The latter equation has, apart from equivalence transformations, two essentially different types of solutions by 2×2 matrices, that is, γ^0, the matrix with first row 0, -1 and second row 1, 0; γ^1, the matrix with first row i, 0 and second row 0, i; γ^2, the matrix with first row 0, 1 and second row 1, 0; γ^3, the matrix with first row 1, 0 and second row 0, -1; and the conjugate complex of these matrices. If the potentials $\phi_\kappa \neq 0$, such a Lorentz covariant two-component wave equation is no longer covariant with respect to coordinate reflections. If the first type of solutions γ^κ describes the motion of a positive charge in a given potential ϕ_κ, the second type describes that of a negative charge. The equation permits the definition of a real current four vector whose divergence vanishes identically by virtue of the wave equation. Gauge invariance is satisfied, and the superposition principle holds, but with arbitrary real coefficients. The relation of this equation to a real 4 component equation is analyzed by J. Serpe, Physical Review vol. 76 (1949) (in print). (Received October 25, 1949.)

58. M. Z. Krzywoblocki: *On the extension of the Kolmogoroff theory of turbulence to compressible fluids.*

Kolmogoroff's theory of locally isotropic turbulence proposed originally for an incompressible fluid was extended to flows of a compressible fluid. The following four equations were taken into account: equation of motion, conservation of mass, state, and energy. The first and second similarity hypotheses were extended to cover the domain of compressible fluids. The conclusions derived from the reduced equations of motion and energy show a full agreement with the similarity hypotheses. (Received October 10, 1949.)

59. B. R. Seth: *Finite elasto-plastic rotation of a cylinder.*

In a previous paper it has been shown that, for a first approximation, both the plastic flow and deformation theories give similar results for the torsion of a circular cylinder. A second approximation, obtained by using finite deformation, has revealed that, unlike the plastic flow theory, the deformation theory gives discontinuities in the stresses across the elasto-plastic boundary, and hence cannot be expected to give satisfactory results. This method is now extended to treat the finite elasto-plastic rotation of a circular cylinder. (Received October 7, 1949.)

60. S. S. Shü: *On Taylor-Maccoll's equation. II. An analytic formula of the solution in a series form.*

The existence theorem of conical flows with an attached conical shock wave was established mathematically some time ago by the author. The present note is to give an analytic solution of Taylor-Maccoll's equation when the tangential component of the flow velocity at the solid cone is prescribed. Because of the singularity inside the solid cone, a series expansion in terms of the semi-apex angle from the axis of the solid cone in general does not converge in the whole domain concerned. A transformation is performed so that the solution in series form is valid for the whole domain between the solid cone and the shock wave. Explicit formulas up to seven terms are worked out. Results are compared with Kopal's recent refined numerical calculations for the cone at Massachusetts Institute of Technology in excellent agreement. (Received October 13, 1949.)

61*t*. L. A. Zadeh: *An application of Fourier transformation to the solution of linear differential equations with variable coefficients.*

A method developed by the author for the analysis of linear variable networks (*Frequency analysis of variable networks*, Dissertation, Columbia University, 1949) can be applied with advantage to the solution of a wide class of linear differential equations. Consider $L(p;t)y(t)=K(p;t)x(t)$ where $p=d/dt$; L and K are linear operators; $x(t)$ is a prescribed function of time ($x(t)=0$ for $t<0$), and $y(t)$ and all of its derivatives are zero for $t<0$. Define $W(t, \xi)$ as the solution corresponding to $x(t)=\delta(t-\xi)$ (Dirac's delta function). We have $W(t, \xi)=K^*(p; \xi)G(t, \xi)$ ($p=d/d\xi$, $K^*=$ adjoint of K) where $G(t, \xi)$ is Green's function for $L(p; t)$ corresponding to the above conditions on y. Define the system function of the given equation as $H(jw; t)=\int_{-\infty}^{\infty}W(t, \xi)e^{-jw(t-\xi)}d\xi$. Define the bi-frequency system function as $\Gamma(jw;ju)=\int_{-\infty}^{\infty}H(jw;t)e^{-jut}dt$. It is shown that $y(t)=(2\pi)^{-1}\int_{-\infty}^{\infty}H(jw; t)X(jw)e^{jwt}dw$ and $Y(jw)=\int_{-\infty}^{\infty}\Gamma(jw; ju)X(ju)du$ where $X(jw)$ and $Y(jw)$ are the Fourier transforms of $x(t)$ and $y(t)$, respectively. It is also shown that $H(jw; t)$ satisfies a linear differential equation in t of the same order as $L(p; t)$. This equation is formally solved for cases where the coefficients of the given equation are either slowly varying functions of time or where they oscillate in the neighborhood of their mean values. (Received October 10, 1949.)

GEOMETRY

62. M. H. M. Esser: *Self-dual postulates for projective geometry.*

Usually projective geometry is founded on non dual postulates, in which points and lines are fundamental concepts, whereas planes are defined as certain loci of points. Thus the duality between points and planes, which is to be so important later on, is not apparent in the postulates. The author gives a self-dual system of five postulates for n-dimensional geometry, in which points and hyperplanes are fundamental, whereas lines and planes of less than $n-1$ dimensions are defined in a self-dual definition. For a space of three dimensions, the number of postulates reduces to four, and they form a system equivalent to the system of postulates found on pages 16 and 24, vol. I, of *Projective geometry* by O. Veblen and J. Young. The author has simplified certain results by Karl Menger. The details will be published in Duke Math. J. (Received October 13, 1949.)

LOGIC AND FOUNDATIONS

63t. A. R. Schweitzer: *Concerning instruments and instrumentalism.*

The author describes "instrument" as a means used to attain an end. Discrimination is made between natural and artificial instruments (extension of natural instruments). The hand is a natural instrument for grasping (prehension). From the hand as an instrument of prehension transition is made to discrimination between right and left hands with application, by abstraction, to the foundations of geometry. Any theory dealing with instrument as a leading concept is termed "instrumentalism." In this connection reference is made to Bergson's creative evolution, Whitehead's process and reality, and John Dewey's empiricism as revealed in his *Studies in logical theory* (Chicago, 1903), *Essays in experimental logic* (Chicago, 1916), *Creative intelligence* (New York, 1917), and *The quest for certainty* (Gifford Lectures, New York, 1929). Reference is also made to the Bridgewater Treatises, Treatise IV: *The hand* by Sir Charles Bell (London, 1834). (Received October 17, 1849.)

64t. A. R. Schweitzer: *On a relation between relativity and quantum theory.*

After a survey of treatises on relativity and quantum theory the author concludes: relativity and quantum theory represent, at least in their present state, divergent lines of development in the evolution of physics; these lines may, however, be so interpreted as to permit motivation by a common concept of energy. (Received October 7, 1949.)

TOPOLOGY

65t. R. H. Bing: *A necessary and sufficient condition that a topological space be metrizable.*

A regular topological space S is metrizable if and only if there is a sequence G_1, $G_2, \cdots$ such that (1) G_i is a collection of open subsets of S such that the sum of the closures of any subcollection of G_i is closed and (2) for each point p and each open set D containing p there is an integer $n(p, D)$ such that an element of $G_{n(p,D)}$ contains p and each element of $G_{n(p,D)}$ containing p is a subset of D. An example is given showing that condition (2) above cannot be changed to (2′) for each point p and each open set D containing p there is an integer $n(p, D)$ such that an element of $G_{n(p,D)}$ contains p and is a subset of D. The above result is a generalization of the notions of perfect screenability [Bull. Amer. Math. Soc. Abstract 55-7-421] and perfect separability. (Received October 11, 1949.)

66t. S. S. Chern and S. T. Hu: *Parallelisability of principal fibre bundles.*

A necessary and sufficient condition is given that a principal fibre bundle is parallelisable. The method is based on the examination of the parallelisability of the parts of the bundle over the skeletons of successive dimensions of the base space. If the bundle is $(n-1)$-parallelisable, the n-dimensional obstruction elements form a coset $W^n(\mathfrak{F}) \in H^n(B, \pi_{n-1}(G))/P^n(B, \pi_{n-1}(G))$, where B is the base space, G the reference group, and $P^n(B, \pi_{n-1}(G))$ the presentable subgroup of the cohomology group. $W^n(\mathfrak{F})=0$, $n=2, \cdots$, dim B, are then the conditions in question. (Received September 25, 1949.)

67t. S. S. Chern and Y. F. Sun: *The imbedding theorem for fibre bundles.*

The imbedding theorem of Whitney-Steenrod for sphere bundles is generalized to fibre bundles as follows: Let $\mathfrak{F}$ be a fibre bundle over a finite polyhedron B of dimension n as base space. If a principal fibre bundle over A is such that the homotopy groups of its total space vanish up to the dimension n inclusive, the bundle is universal in the sense that the bundles over B are in one-one correspondence with the homotopy classes of maps of B into A. This imbedding theorem is also true, if B is a compact metric ANR. It is also extended to the product of fibre bundles, useful for the description of the position of one fibre bundle in another. Discussions are given for the cases that the reference group is a classical group. (Received September 25, 1949.)

68. R. F. Deniston: *On the theory of limits.* Preliminary report.

Application of Zorn's lemma to a partially ordered set, X, which is inductive and has the composition property of Moore-Smith (Amer. J. Math. vol. 44, p. 102) assures that X has at least one maximal element, W_X. A set X having only one maximal element will be called an eventual set and said to converge to the limit, W_X. If for each point x of a set E is given a nonvoid family of eventual sets $\mathfrak{X}=\{X\}$ having x as the W_X point, a structure is defined by using as an arbitrary neighborhood $N_{\mathfrak{X}}^{\alpha^0}(x)$ the set of all points y for which $y=\alpha_X>\alpha_X^0$ for each α_X^0 and $X\in\mathfrak{X}$. If all eventual sets on E are isomorphic each to each, the structure given is a uniform structure. Derived eventual sets not originally given may be constructed as follows: For families of points $\{a_s\}$ in E and such that to any a_s are related an X_1 and X_2 such that $\exists a_{s'}\in X_1$, $a_{s''}\in X_2$, $a_{s''}>_{X_2}a_s>_{X_1}a_{s'}$. Then the eventual set Y given by $\{\cdots>a_{s''}>_{X_2}a_s>_{X_1}a_{s'}>\cdots\}$ has a W_Y (by Zorn's lemma), which may correspond to no point in E. Regarding a point as given by the collection of eventual sets having it as a W-point an extension of the original space is obtained. (Received October 22, 1949.)

69. F. A. Valentine: *Arcwise convex sets.*

An arc is said to be convex if it lies in the boundary of its convex hull. A set S is said to be arcwise convex, if each pair of points in S can be joined by a convex arc in S. Theorem 1. *Each component of the complement of a bounded closed arcwise convex set in E_2 (the plane) is arcwise convex.* Let x be a point in a set $T\subset E_2$ which has the property that each point in T can be joined to x by a convex arc lying in T. Designate the set of all such points x by K. Theorem 2. *If T is a simply connected continuum in E_2, then K is a simply connected arcwise convex set.* Let C be a continuum in E_2 such that each pair of points x and y in C lies in a subcontinuum of C contained in one of the half-planes determined by the line passing through x and y. Then Theorem 1 holds for C. Theorem 1 is a generalization of a corresponding theorem proved by Horn and Valentine for L sets. (Received October 12, 1949.)

J. W. T. YOUNGS,
Associate Secretary

THE NOVEMBER MEETING IN PASADENA

The four hundred fifty-second meeting of the American Mathematical Society was held at the California Institute of Technology, Pasadena, California, on Saturday, November 26, 1949. Approximately 130 persons attended, including the following 98 members of the Society:

W. G. Bade, M. M. Beenken, E. F. Beckenbach, E. T. Bell, Richard Bellman, H. F. Bohnenblust, S. G. Bourne, Herbert Busemann, W. D. Cairns, A. B. Coble, F. E. Cothran, F. G. Creese, J. H. Curtiss, Tobias Dantzig, P. H. Daus, J. E. Denby-Wilks, C. R. De Prima, R. J. Diamond, S. P. Diliberto, R. P. Dilworth, Avron Douglis, Roy Dubisch, C. L. Dunsmore, L. K. Durst, R. H. Edwards, Arthur Erdélyi, Harley Flanders, G. E. Forsythe, A. L. Foster, W. B. Fulks, W. H. Glenn, G. E. Gourrich, J. W. Green, Hugh Hamilton, A. R. Harvey, E. M. Henderson, M. R. Hestenes, P. G. Hoel, Alfred Horn, D. G. Humm, H. D. Huskey, D. H. Hyers, Rufus Isaacs, E. H. Jacobs, Glenn James, R. C. James, P. B. Johnson, William Karush, Samuel Karlin, A. J. Kempner, R. M. Lakness, Cornelius Lanczos, G. E. Latta, M. B. Lehman, D. H. Lehmer, Jack McLaughlin, A. B. Mewborn, C. B. Morrey, J. W. Odle, L. J. Paige, R. P. Peterson, R. S. Phillips, W. T. Puckett, H. R. Pyle, E. S. Quade, W. C. Randels, I. S. Reed, Julia Robinson, R. M. Robinson, E. B. Roessler, J. B. Rosser, Herman Rubin, Abraham Seidenberg, Max Shiffman, Harold Shniad, I. S. Sokolnikoff, R. D. Specht, D. V. Steed, Robert Steinberg, E. G. Straus, Irving Sussman, J. D. Swift, A.E. Taylor, F.G. Tricomi, A.W. Tucker, R. L. Vaught, S. S. Walters, Morgan Ward, W. R. Wasow, W. W. Weber, P. A. White, R. L. White, A. L. Whiteman, W. M. Whyburn, R. L. Wilder, G. M. Wing, František Wolf, J. R. Ziegler.

There was a general session in the morning for contributed papers and for the invited address *On the stability of solutions of differential equations*, by Professor Richard Bellman of Stanford University. Professor Arthur Erdélyi presided. In the afternoon there were two sectional sessions, at which Professor D. H. Hyers and A. L. Foster presided.

Following the meetings, those present were guests of the Department of Mathematics of the California Institute of Technology at a tea at the Athanaeum.

Abstracts of papers presented at the meeting follow. Those abstracts whose numbers are followed by the letter "*t*" were presented by title. Mr. Schweitzer, whose abstract was introduced by Professor Gabor Szegö, died on January 28, 1945.

ALGEBRA AND THEORY OF NUMBERS

70. Roy Dubisch: *A note on isotopy.*

The purpose of this note is to point out that isotopy of algebras may be handled from the point of view of a type of equivalence of matrices $X=(X_{ij})$ where X_{ij} is a linear form in $x_1, \cdots, x_n$ with coefficients in F. X and Y are called isotopic if X can be obtained from Y by a finite sequence of the usual three operations of ordinary equivalence theory plus the interchange of x_i with x_j, the replacement of x_i by $x_i+\xi x_j$, and the multiplication of x_i by any α in F. This technique is applied to show that every algebra which is not a zero algebra has an isotope with an idempotent. Also, all isotopically commutative and anti-commutative algebras (that is, algebras all of whose isotopes are commutative or anti-commutative respectively) are determined. (Received October 6, 1949.)

71. L. K. Durst: *Apparition of primes in elliptic sequences.* Preliminary report.

A study of the rank of apparition of primes in elliptic divisibility sequences (Morgan Ward, Amer. J. Math. vol. 70 (1948) pp. 31–74) was undertaken using lemniscate functions. These are the simplest elliptic functions admitting a complex multiplication. Computation indicates that for primes of the form $p=4k+3$ the rank is a divisor of $p+1$. This behavior is like that for Lucas' sequences. But for primes $p=4k+1$ no simple arithmetical restriction on the rank has yet been found. (Received October 14, 1949.)

72. Harley Flanders: *Unification of local class field theory.* Preliminary report.

The local class field theory is known to be valid for two types of fields: p-adic fields and formal power series fields over a finite constant field. In the first case, the theory has been developed by both commutative and non-commutative methods; however, in the second case, no commutative method has been given. The author shows how certain results of Hasse on higher ramification can be used to give a common treatment of the fundamental inequality for both cases. Having this, he applies the method of Chevalley's thesis to obtain the desired unified treatment. The methods are entirely commutative. (Received November 25, 1949.)

73. Alfred Horn: *Sentences which are true of direct unions of algebras*

For the terminology see J. C. C. McKinsey, *The decision problem for some classes of sentences without quantifiers*, Journal of Symbolic Logic vol. 8 (1943) pp. 61–76. Every sentence T corresponding to an algebra may be put in prenex form $Q^1_{x_1} \cdots Q^n_{x_n} S(x_1, \cdots, x_n)$ where the Q^i are quantifiers, and the matrix S results from a formula F in the propositional calculus by replacing in F the variables by equations. We say T is in prenex conjunctive form if F is in conjunctive normal form. Let A be a class of similar algebras and suppose Γ_i is an algebra in A for each $i \in I$. Denote by Γ the direct union of the Γ_i. Theorem: Let T be a sentence which has a prenex conjunctive form T' in which all equations are negated. Then if T is true of Γ_i for some $i \in I$, it is true of Γ. If the matrix of T' is a disjunction of inequalities, then T is true of Γ_i for some $i \in I$ if and only if it is true of Γ. Theorem: Let T have a prenex conjunctive form T' in whose matrix each disjunction contains at most one non-negated equation. Then if T is true of Γ_i for each $i \in I$, it is true of Γ. If the matrix of T' is a conjunction of equations, then T is true of Γ_i for each $i \in I$ if and only if it is true of Γ. Theorem: If for all $i \in I$, the algebras Γ_i are the same algebra Γ' and if T has a prenex form in which no universal quantifiers occur, then if T is true of Γ' it is true of Γ. These are the only results of their kind. (Received October 14, 1949.)

74. Robert Steinberg: *A geometric approach to the representations of the full linear group over a Galois field.* II.

In part I of this paper, a basic set of $p(n)$ irreducible representations of $LH(n, q)$, the group of all n-ary linear homogeneous nonsingular substitutions in the Galois field $GF(q)$, was determined through some properties of the associated finite geometry. In this part, through an application of the generating function technique used by Frobenius (G. Frobenius, Berliner Berichte, 1900, pp. 516–534) in his work on the symmetric group, the characters of one of these representations, that of degree $q^{n(n-1)/2}$ are determined explicitly. (Received October 12, 1949.)

75*t*. Robert Steinberg: *The representations of $LH(3, q)$, $LH(4, q)$, $F(3, q)$, and $F(4, q)$.*

In this paper, the representations and characters of $LH(3, q)$ and $LH(4, q)$, the groups of all third and fourth degree linear homogeneous nonsingular substitutions in the Galois field $GF(q)$, are determined, and, from them, those of $F(3, q)$ and $F(4, q)$, the corresponding fractional or collineation groups, are deduced. The method consists of the application of Frobenius' method of induced representations to linear representations of suitably chosen subgroups of $LH(3, q)$ and $LH(4, q)$. (Received October 12, 1947.)

76. A. W. Tucker: *Extensions of theorems of Farkas and Stiemke.*

Let S denote a linear subspace of an n-dimensional Euclidean vector space, S^* the orthogonal complement of S, and P the "positive orthant" consisting of all vectors with non-negative components. Then the following theorem and its converse hold: If S intersects P only in vectors whose first k components are zero, then S^* contains some vector of P whose first k components are all positive. For $k=1$ this reduces to a fundamental theorem of J. Farkas concerning the dependence of one homogeneous linear inequality on others [J. Reine Angew. Math. vol. 124 (1902) pp. 1–27] and for $k=n$ to a theorem of Erich Stiemke concerning positive solutions of a system of homogeneous linear equations [Math. Ann. vol. 76 (1915) pp. 340–342]. Let C denote the polyhedral convex "cone" consisting of all vectors that can be expressed as linear combinations of certain given vectors (finite in number) using non-negative coefficients only, and $-C^*$ the negative "polar" cone consisting of all vectors that make a non-acute angle with every vector of C. Then the theorem stated above and its converse hold also when C and $-C^*$ are substituted for S and S^*, respectively. This theorem on cones is closely related to papers on solutions of two-person games by H. F. Bohnenblust, S. Karlin, and L. S. Shapley, and by David Gale and Seymour Sherman [in a forthcoming Annals of Mathematics Study]. It also has application to econometric problems treated by G. B. Dantzig and T. C. Koopmans [in forthcoming *Proceedings of Linear Programming Conference*, Cowles Commission]. (Received October 12, 1949.)

77. A. L. Whiteman: *Theorems on quadratic partitions.*

Let $f(k)$ and $g(k)$ denote the number of distinct solutions of the congruences $x^2+x\equiv k \pmod{p}$ and $x^3+x^2\equiv k \pmod{p}$, respectively. In the present paper theorems of the following type are proved. (1) If p is a prime of the form $3n+1$, then the value of a in the quadratic partition $4p=a^2+3b^2$ is given by $a=-p+\sum_{s=0}^{p-1} f(s^3)$. (2) If p is a prime of the form $7n+1$, then the value of u in the quadratic partition $4p=u^2+7v^2$

is given by $3u = -p + \sum_{s=0}^{r-1} g(s^r)$. The formula $f(k) = 1 + ((4k+1)/p)$, where (m/p) denotes the Legendre symbol, leads at once to Chowla's formula $a = 1 + \phi_3(4)$, where $\phi_q(h) = \sum_{m=1}^{p-1} (m/p)((m^q + h)/p)$ is the Jacobstahl sum. The formula for $g(k)$ is more complicated and does not lead to a simple expression for u in terms of the Jacobstahl sum. The methods of this paper are based on the theory of cyclotomy. (Received September 20, 1949.)

ANALYSIS

78. E. F. Beckenbach: *On characteristic properties of harmonic functions.*

A lemma of L. V. Ahlfors is used in proving the following result: If $u(x, y)$ is of class C' in a domain D, and for each (x_0, y_0) in D we have $\int_0^{2\pi}\int_0^r[(\partial u/\partial x)^2 + (\partial u/\partial y)^2]\rho d\rho d\theta - \int_0^{2\pi} u(\partial u/\partial n) r d\theta = o(r^2)$, where the integrals are taken over the area and circumference, respectively, of the circle with center at (x_0, y_0) and radius r, then $u(x, y)$ is harmonic in D. (Received October 15, 1949.)

79. S. P. Diliberto: *Systems of ordinary differential equations.* II. Preliminary report.

The following theorem is established: *Let* (1) $dx_i/dt = P_i(x_1, x_2, x_3)$ $(i = 1, 2, 3)$ *where the* P_i *are of class* C^2 *in a neighborhood of* T, *a solid torus whose surface,* S, *is of class* C^2. *Assume* (H$_1$) *the vector* $V = (P_1, P_2, P_3)$ *has no singular points inside* T (*or on* S); (H$_2$) *at each point of* S, V *points inside* T; (H$_3$) *there exist surfaces* $F(x_1, x_2, x_3) =$ constant, *whose Gaussian curvature is strictly positive in a neighborhood of* T, *such that* grad $F = V$. *Then there exists a unique periodic solution* (*closed trajectory*) *of* (1) *inside* T; *further if* T *is the topological product* $S^1 \times E^2$ *of the* 1-*sphere* S^1 *with the disc* E^2 *of center* p, *then the closed trajectory is unknotted and homotopic* (*even isotopic*) (*inside* T) *to the "center-line"* $S^1 \times P$ *of the torus.* This is a partial generalization of the Poincaré-Bendixson Theorem. Proof—sketch: (1) The curvature condition implies that the distance of nearby trajectories decreases (but not that the flow decreases distance of representative points on the trajectories). (2) This implies all trajectories entering T have the same limit set, which in turn consists of a single trajectory. (3) The trajectories themselves are used to construct the homotopy. (Received October 26, 1949.)

80. R. C. James: *Bases and reflexivity of Banach spaces.*

It can be easily shown that a Banach space having a basis $\{x_n\}$ is reflexive if and only if: (1) $\sum_1^\infty a_i x_i$ converges if $\|\sum_1^n a_i x_i\|$ is bounded, and (2) $\lim_{n\to\infty} \|f\|_n = 0$, where $\|f\|_n$ is the norm of f on $x_n \oplus x_{n+1} \oplus \cdots$. By use of these conditions, it is shown that a Banach space with an unconditionally convergent basis is reflexive if no subspace is isomorphic with either (c) or $l^{(1)}$ and that the linear functionals $\{f_n\}$ which with the basis elements $\{x_n\}$ form a biorthogonal set is a basis for B^* if no subspace of B is isomorphic with $l^{(1)}$. A Banach space with a basis is reflexive if relative to any norm equivalent to the norm of B each linear functional attains its maximum on the unit sphere. (Received October 19, 1949.)

81. R. M. Lakness: *Unnecessary singularities of superharmonic functions.*

Let T be a domain in E^3 and let F be a closed bounded set of zero capacity contained in T. Let $u(M)$ be a function superharmonic and bounded below in $T' = T - F$. Then there exists a unique function $V(M)$ superharmonic in T and agreeing with $u(M)$ in T'. The values of $V(M)$ on F are given by $V(M) = \lim \inf_{M' \to M} u(M')$, for M' on T', M on F. The above theorem is an extension of the classical theorem on removable singularities of bounded harmonic functions. It states a corresponding property for superharmonic functions which are bounded below. The main tool used in the proof is the extension of Kellogg's uniqueness theorem, which has been proved by Professor G. C. Evans (Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) p. 272). (Received October 13, 1949.)

82*t*. M. Schweitzer: *On the partial sums of second order of the geometric series.*

Szegö proved (Duke Math. J. vol. 8 (1941) p. 559) that the Cesàro sums of second order of the geometric series $\sum z^n$ have a monotonically decreasing real part for $z = e^{i\phi}$, $0 \leqq \phi \leqq \gamma$, where $\sin^2 (\gamma/2) = 0.7$, $\pi/2 < \gamma < \pi$. That is, the sine polynomial $\sum_0^n C_{n+2-\nu,2}\, \nu \sin \nu\phi$ is positive in the interval mentioned. In the present paper it is shown that the number γ can be replaced by $2\pi/3$; this is the largest number of this kind. (Received October 6, 1949.)

83*t*. Otto Szasz: *On a summation method of O. Perron.*

The following transform of a series $\sum_0^\infty a_\nu$ has been studied by Perron: $\phi_\lambda(x) = a_0 + \sum_0^\infty a_\nu(x/(x+\lambda)) \cdot ((x+1)/(x+\lambda+1)) \cdots ((x+\nu-1)/(x+\lambda+\nu-1))$. If $\phi_\lambda(x) \to s$ as $x \to \infty$, then it is said that the series $\sum a_\nu$ has the sum s. The main result of this paper is that this summability method is more powerful than Cesàro summability of order λ. The essential tool is an integral representation of the function $\phi_\lambda(x)$. (Received October 12, 1949.)

84*t*. Gabor Szegö: *On certain special sets of orthogonal polynomials.*

F. Pollaczek has introduced in three Comptes Rendus notes (vol. 228 (1949) pp. 1363, 1553, 1998) various remarkable sets of orthogonal polynomials. In the simplest case they can be defined by the generating function $(1 - ze^{i\theta})^{-1/2+i\phi}(1 - ze^{-i\theta})^{-1/2-i\phi}$ where $\phi = \phi(\theta) = (2 \sin \theta)^{-1}(a \cos \theta + b)$, $a \geqq |b|$. They are orthogonal in the interval $-1 \leqq x \leqq 1$, $x = \cos \theta$, with the weight function $(\cosh \pi\phi)^{-1} \exp \{(2\theta - \pi)\phi\}$. This weight becomes zero like $\exp(-\theta^{-1})$ as $\theta \to 0$ and the asymptotic behavior of the corresponding orthogonal polynomials is "irregular" in a certain sense. A new proof for the orthogonality relation is given and asymptotic properties are studied. (Received September 28, 1949.)

85. A. E. Taylor: *Banach spaces of analytic functions.* I.

The following notations are used. Δ: the set $|z| < 1$ in the plane; $\mathfrak{A}$: the class of functions f analytic in Δ; r: a variable, $0 \leqq r < 1$; $u_n(z) = z^n$; $U_x f(z) = f(ze^{ix})$ (x real); $T_r f(z) = f(rz)$; $\gamma_n(f) = (1/n!) f^{(n)}(0)$; $B(f, g; z) = \sum_0^\infty \gamma_n(f)\gamma_n(g) z^n$; B: a normed linear space whose elements are members of $\mathfrak{A}$. B may satisfy one or more of the axioms $P_1: \gamma_n \in B^*$ (the conjugate space) and $\|\gamma_n\|$ is bounded; $P_2: u_n \in B$ and $\|u_n\|$ is bounded; $P_3: U_x$ maps B isometrically into B; $P_4: T_r$ maps B continuously into B, and $\|T_r\|$ is bounded in r. If B satisfies P_i, $1 \leqq i \leqq k$, B is said to be of type $\mathfrak{A}_k$. For B of type $\mathfrak{A}_3$ define $N(g; r) = \sup_f |B(f, g; r)|$, $\|f\| = 1$, $f \in B$, $g \in \mathfrak{A}$. Define B' as the class of $F \in \mathfrak{A}$ with $N(F) = \sup_r N(F; r) < \infty$. B' is a Banach space of type $\mathfrak{A}_4$ with norm $N(F)$. B' and B''' coincide. Define B^0 as the class of $F \in \mathfrak{A}$ for which $\lim_{r \to 1} B(f, F; r)$ exists for each $f \in B$. If B is complete of type $\mathfrak{A}_3$, B^0 is complete of type $\mathfrak{A}_4$ and a subspace of B'. Let B be complete of type $\mathfrak{A}_4$ such that $\lim_{r \to 1} \|T_r f - f\| = 0$ for each f. Then $B' = B^0$ and B' is equivalent to B^*. (Received October 12, 1949.)

86. S. S. Walters: *The space H^p with* $0 < p < 1$.

The space H^p is defined to be the class of all functions $f = f(z)$ which are regular on the interior of the unit circle and such that $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ is bounded in r on $0 \leqq r < 1$. For arbitrary $f \in H^p$ the "norm of f," $\|f\|$, is defined to be $\sup_{0 \leqq r < 1} ((1/2\pi)\int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{1/p}$. For the case $1 \leqq p$, it is known that said "norm" is a norm in the true sense, and that H^p is a Banach space. When $0 < p < 1$ it is shown that H^p is a complete, perfectly separable, linear topological space under the topology: $U \subseteq H^p$ is open in case for arbitrary $f_0 \in H^p$ it is true $\exists r > 0 \ni E_f[\|f - f_0\| < r]$ lies in U. In fact H^p is linearly homeomorphic to a closed subspace of $L^p[0, 2\pi]$. Thus, since $(L^p[0, 2\pi])^*$, the space of linear functionals defined on $L^p[0, 2\pi]$, has no elements save the zero element, one might suspect the same to be said of the conjugate space of H^p, namely $(H^p)^*$. This, however, is not the case, and in fact there exists a countable set in $(H^p)^*$ which serves to distinguish elements of H^p, whence weak convergence in H^p makes sense when $0 < p < 1$. It is shown that if a sequence of elements in H^p converges weakly, then the sequence converges to its weak limit uniformly on all compact subsets of the unit circle. (Received October 7, 1949.)

87. Morgan Ward: *Note on real continuous iteration.*

The investigations of Fuller and Ward on the continuous iteration of all functions steadily increasing in $0 \leqq x < \alpha$ (Bull. Amer. Math. Soc. vol. 40 (1934) pp. 688–690 and vol. 42 (1936) pp. 393–396) and in C^s are extended to functions in C^k and C^α. All continuous iterations for functions in these classes are specified, and it is proved that no analytic continuous iterations exist for the sub-class of functions regular in $z = x + iy$ along the positive axis $0 \leqq x < \alpha$ which are also regular along the positive axis. (Received October 14, 1949.)

88. František Wolf: *Analytic perturbation of operators in Banach spaces.* Preliminary report.

Let $A(\theta)$ be a bounded linear operator in a Banach space $\mathfrak{B}$ which depends analytically on θ in a neighborhood N of $\theta = 0$. Suppose that $A(0) = A_0$ has μ_0 for an eigenvalue with an m-dimensional eigenspace $\mathfrak{S}(0)$. Then there exists an analytic idempotent $E(\theta)$ such that the range of $E(0)$ is $\mathfrak{S}(0)$. For small enough θ, $A(\theta)$ has eigenvalues $\mu_k(\theta)$ in an arbitrary small neighborhood of $\theta = 0$. Their corresponding eigenspaces generate a space $\mathfrak{S}(\theta)$ which is also m-dimensional and is the range of $E(\theta)$. It is possible to find $f_1(\theta), \cdots, f_m(\theta) \in \mathfrak{B}$ analytic in θ which generate $\mathfrak{S}(\theta)$. In general, for $m > 1$, it is not possible to define them so that all $f_i(\theta)$ are all eigenfunctions of $A(\theta)$ corresponding to $\mu_k(\theta)$. The functions $\mu_k(\theta)$ have in general a branch-point at $\theta = 0$. In some special cases the $\mu_k(\theta)$ are all analytic in $\theta = 0$ and their coincidence at $\theta = 0$ is accidental. This is true in particular wherever it is known from the special character of $A(\theta)$ that $\mu_k(\theta)$ are real for real θ. For self-adjoint operators in Hilbert space this has been proved by F. Rellich (Math. Ann. vol. 113 (1936) p. 600). The author has found some conditions for this to occur in Banach spaces. (Received October 12, 1949.)

APPLIED MATHEMATICS

89*t*. H. W. Becker: *Planar and nonplanar cadences.*

Where $e^{tK}=1/(2-e^t)$ and $e^{tL}=e^{t/(1-t)}$, $K_n=L_n$ if $n<4$, otherwise $K_n>L_n$. The music theoretic interpretation is: that K_n is the number of cadences of n notes (melodies according to distinction of relative and not absolute pitch); while L_n is the number of planar cadences (having no crossovers in their Puttenham diagrams). Nonplanarities cannot arise, $n<4$; and $K_n \supset L_n$. The K_n are sums of differences of zero; the L_n are Laguerre numbers, and also sums of Stirling numbers of the third kind, Bull. Amer. Math. Soc. vol. 42 (1946) p. 826. Various other combinatory interpretations of K_n and L_n are tabulated. For example, they are the numbers of distributions of n men into externally and internally permutable crews, respectively. That is, in K_n, c crews may be ranked $c!$ ways; while in L_n, m members of a crew may be ranked $m!$ ways. Under this interpretation, the only overlap is $K_n \cap L_n = @_n$, where $e^{t@}=\exp(e^t-1)$. (Received October 12, 1949.)

90*t*. H. W. Becker: *The theory of the world series.*

"As the first game goes, so goes the series." The truth of this saying (S) in a four-win series is attacked combinatorially, and historically as of the 46 series to date. The winner of the first game can go on to win the series in 20 ways, or lose it in 15. So with the assumption (A) that the series is contested between teams of about equal strength and is won by "the breaks," (S) is 57% true in theory and 63% true on the record. However, the series durations have averaged almost a game shorter than they should be. This is attributable to departures from (A), and to "human equation" (B): any initial losers have a tendency to fall into brooding, and self-defeatism, though their remaining chances are better than they think. On the other hand, most of the series which go the limit have been *won* by the initial loser. That is ascribable to the "comeback equation" (C): an initial loser who has grimly fought back to eventually even terms, has now the greater momentum of victory. These considerations appertain to a mathematics, enriched to include all the problems of the mind: those of morale and gusto, as well as magnitude and order. (Received October 12 1949.)

91. G. E. Forsythe: *Round-off errors in numerical integration on automatic machinery.* Preliminary report.

Adapting numerical integration processes to automatic machinery ordinarily involves replacing a desired sum $\sum_{k=1}^{n} u_k$ by the computable sum $\sum_{k=1}^{n} N(u_k)$, where $N(u_k)$ is the nearest integer to u_k. Large round-off errors $R_n = \sum_{k=1}^{n} \{N(u_k)-u_k\}$ frequently accumulate near points $\{k'\}$ of "stationary phase" of $N(u_k)-u_k$. (H. D. Huskey, National Bureau of Standards Journal of Research vol. 42 (1949) pp. 57–62). The present author gives an approximate formula for R_n in terms of $\{N(u_{k'})-u_{k'}\}$ and $\{|\Delta u_{k'}|\}$, valid whenever Δu_k is a slowly changing function of k. The formula explains a posteriori most of the peculiarities of the round-off errors found in Huskey's integration of the system $\dot{x}=y$, $\dot{y}=-x$ on the ENIAC. Let $[u]$ be the greatest integer not exceeding u. The present author proposes a *random round-off*, whereby any real number $u=[u]+v$ is "rounded up" to $[u]+1$ with probability v, and "rounded down" to $[u]$ with probability $1-v$. The error r of random round-off is truly a random variable. Since $E(r)=0$ and $E(r^2)=v(1-v)\leq 1/2$, simple probabilistic bounds on the accumulated error can be given without assuming a distribution for v. Tests with I. B. M. equipment indicate that random round-off probably eliminates a priori the peculiarities of round-off found by Huskey on the ENIAC. (Received October 12, 1949.)

92. Rufus Isaacs: *Transtability flutter of supersonic aircraft panels.*

For certain aero-elastic configurations it is possible to ascertain critical flutter conditions from static considerations alone. When the air speed exceeds a certain value, statically stable equilibrium—and sometimes equilibrium itself—is no longer possible. Such is the case for an aircraft structure panel, buckled by thermal expansion; here "one-dimensionalized" to a buckled beam with clamped ends. With no air velocity, the beam is the classical Euler column with its discrete set of possible deflections of which the first alone is stable. As the velocity increases the first two deflections become more alike until at a certain critical speed they coincide. At higher speeds they do not exist. As the only possibilities for the beam to be stationary lie in the unstable higher modes, it is reasonable to assert that flutter now occurs. A rigorous proof is impossible without entanglement in the complicated dynamic analysis, but strong plausibility arguments exist. For example, if the continuous beam is replaced by a discrete model made of hinged bars with only two degrees of freedom, there are only two eigenvalues. When the critical speed is exceeded both—and hence all possibility of equilibrium—disappear. Flutter is certain and this model should behave like the continuous one. (Received August 22, 1949.)

93. Cornelius Lanczos: *An iterative solution of Fredholm's integral equation.*

Consider Fredholm's integral equation $y-\lambda Ky=\phi$. Instead of mere iterations which would result in the Liouville-Neumann series, the method of "minimized iterations" is employed. This process assigns to any given $\phi(x)$ and $K(x, \xi)$ a sequence of biorthogonal functions ϕ_i and ϕ_i^* ($\phi_i\phi_k^*=0$ for $i\neq k$), together with a sequence of polynomials $p_\nu(x)$, generated by a recursion relation between three consecutive p_ν. The roots of these polynomials converge to the reciprocal characteristic values $1/\lambda_j$ if these values exist, otherwise they converge to zero. A certain linear combination of the $\phi_\nu(x)$ converges to the solution of the given integral equation. The same combination, taken for $\lambda=1/\mu_j$ where μ_j is the jth root of the last polynomial $p_m(x)$ still evaluated (j fixed, m steadily increasing), converges to the characteristic solutions (defined by $\phi=0$) if these solutions exist. Numerical examples demonstrate the excellent speed of convergence for not too large values of λ. The convergence remains valid for arbitrarily large λ, although the number of iterations practically demanded becomes eventually excessive. (Received October 11, 1949.)

GEOMETRY

94. Abraham Seidenberg: *The hyperplane sections of normal varieties.*

Let V/k be a normal variety in n-space of dimension $r\geqq 2$ defined over an infinite ground-field k, and suppose further that the field of rational functions on V/k is separably generated. Then almost all hyperplane sections of V/k will also be normal, that is, the section by the hyperplane $a_0x_0+\cdots+a_nx_n=0$, $a_i\in k$, will be (irreducible and) normal provided the a_i satisfy a certain proper algebraic inequality over k. The above separability assumption may be removed upon restricting the a_i to k^p;

also, with appropriate formulation, the finiteness condition on k proves unnecessary. The more general case of an arbitrary linear system of V_{r-1}'s on V/k, which yields an analogue to the theorem of Bertini on the variable singular points of a linear system of varieties (obtained by replacing the word *singular* with the work *non-normal*), is left to a second paper. (Received October 10, 1949.)

LOGIC AND FOUNDATIONS

95. J. D. Swift: *Analogues of the Sheffer stroke in the three-valued case.*

There are precisely ninety commutative analogues of the Sheffer stroke (binary independent generating functions) in the three-valued case. These lie in fifteen classes each consisting of six conjugate functions. (Received October 12, 1947.)

STATISTICS AND PROBABILITY

96*t*. C. J. Everett and S. M. Ulam: *On an application of a correspondence between matrices over real algebras and matrices of positive real numbers.*

Since the representation of real numbers by pairs (a, b) of positive reals has an essentially matric character under addition and multiplication, it is possible to define a (many-one) correspondence from the set of all $2n\times 2n$ matrices consisting of n^2 blocks, each of form $a_{11}=a_{22}>0$, $a_{12}=a_{21}>0$, to the set of all real $n\times n$ matrices, which preserves $(+)$ and $(\cdot)$. Thus, ring operations on matrices whose elements belong to a real linear associative algebra admitting a faithful real matrix representation (C. C. MacDuffee, *On the independence of the first and second matrices of an algebra*, Bull. Amer. Math. Soc. (1929)) may be replaced by operations on corresponding real positive matrices of higher order. Similar remarks apply to matrix $\times$ vector operations. The advantage for characteristic vector problems lies in the fact that a positive matrix may be realized as the first moment matrix of a multiplicative system (Bull. Amer. Math. Soc. Abstracts 55-1-41, 42, 43), and as such, its "first" characteristic vector represents the limit population of the system, which may be investigated by "Monte Carlo" methods (cf., for example, N. Metropolis, S. Ulam, *The Monte Carlo method*, Journal of the American Statistical Association vol. 44 (1949) pp. 335–341). (Received October 13, 1949.)

97*t*. C. J. Everett and S. M. Ulam: *Random walk and the Hamilton-Jacobi equation.*

Let $v(x)$ be a continuous function on $(-\infty, +\infty)$ with $0<a\leqq v(x)\leqq 1/2$. For every subdivision s of norm d, a random walk is defined, in which a particle moves, every d^2 seconds (left or right with equal probability), a distance $d\cdot v(x_{i-1})$ when on the subdivision interval (x_{i-1}, x_i). The probability $W_s(z, x, t)\Delta x$ of the particle being in $(x, x+\Delta x)$, t seconds after starting from z, has limit $W(z, x, t)\Delta x=[\Delta x/v(x)(2\pi t)^{1/2}]$ $\exp(-S^2(z, x)/2t)$ as $d\to 0$, where $S(z, x)=\left|\int_z^x dy/v(y)\right|$ is the S function of optics satisfying the Hamilton-Jacobi equation $S_x^2=1/v^2(x)$. Setting $W_t=0$, we obtain $S^2(z, x)=t$. Thus, the positions x which are at maximum probability density at time t, in the limiting case of random walk with variable length of step, coincide with the positions at which light would arrive from the same origin z in time $t^{1/2}$. Extension to higher dimensions, now under way, may open problems of mechanics, via the Hamilton-

Jacobi equation, as well as optics itself, to statistical sampling by high speed machine techniques. (Received October 13, 1949.)

TOPOLOGY

98. R. L. Wilder: *Local orientability.*

Properties called *local orientability* were defined on p. 281 of the author's Colloquium book *Topology of manifolds* (hereafter referred to as "T. M." The word "connected" should be inserted before "neighborhood" in Definition 6.1 and "nonempty" before the second "open" in D''). An extensive study of these properties is made in this paper, they being found to be equivalent in an n-gm to each of six other properties (one of these being an axiom of Čech; cf. T. M., 289 Bibl. Comm. §6). Applications are made to orientability of n-gms. For example, in a locally orientable n-gm concurrent orientations of the local n-gms may be introduced so as to define orientability of an n-gm by means of an indicatrix and chains of n-gms analogous to the chains used by Poincaré in studying torsion (Proc. London Math. Soc. vol. 32 (1900) pp. 277–308) and in fashion similar to the classical orientability definition. From this definition a new proof of Begle's theorem (p. 251, T. M.) may be obtained. For connected n-gms, it should be noted that the assumption of local orientability implies conditions D and D' of T. M. (pp. 250, 254). (Received October 12, 1949.)

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THE INFLUENCE OF J. H. M. WEDDERBURN ON THE DEVELOPMENT OF MODERN ALGEBRA

It is obvious that the title of this paper is presumptuous. Nobody can give in a short article a really exhaustive account of the influence of Wedderburn on the development of modern algebra. It is too big an undertaking and would require years of preparation. In order to present at least a modest account of this influence it is necessary to restrict oneself rather severely. To this effect we shall discuss only the two most celebrated articles of Wedderburn and try to see them in the light of the subsequent development of algebra. But even this would be too great a task. If we would have to mention all the consequences and applications of his theorems we could easily fill a whole volume. Consequently we shall discuss only the attempts the mathematicians made to come to a gradual understanding of the *meaning* of his theorems and be satisfied just to mention a few applications.

For the understanding of the significance that Wedderburn's paper *On hypercomplex numbers* (Proc. London Math. Soc. (2) vol. 6, p. 77) had for the development of modern algebra, it is imperative to look at the ideas his predecessors had on the subject.

The most striking fact is the difference in attitude between American and European authors. From the very beginning the abstract point of view is dominant in American publications whereas for European mathematicians a system of hypercomplex numbers was by nature an extension of either the real or the complex field. While the Europeans obtained very advanced results in the classification of their special cases with methods that were not well adapted to generalization, the Americans achieved an abstract formulation of the problem, developed a very suitable terminology, and discovered the germs of the modern methods.

On the American side one has first of all to consider the very early paper by B. Peirce, *Linear associative algebras* (1870, published in Amer. J. Math. vol. 4 after his death). In it he states explicitly that mathematics should be an *abstract* logical scheme, the absence of a special interpretation of its symbols making it more useful in that the same logical scheme will in general reflect many diverse physical situations. Although it is true that he was actually able to introduce and treat only the general linear associative algebra over the complex field, yet he clearly had in mind much more, and it is his attitude which leads to the modern postulational method. In his treatment of algebras he gives a rational proof of the existence of an idempotent and employs the well known Peirce decomposition of an algebra relative to an idempotent. His results were put in a more readable form by H. E. Hawkes, *On hypercomplex number systems* (Trans. Amer. Math. Soc. vol. 3 (1902)). A correct definition of an associative algebra over an arbitrary field seems to be given for the first time by L. E. Dickson, *Definitions of a linear associative algebra by independent postulates* (Trans. Amer. Math. Soc. vol. 4 (1903)).

At that time several European mathematicians, Molien, Cartan, and Frobenius, had already arrived (always for the special cases of the real or complex field) at many of the results of the modern theory. The notions radical, semisimple, simple had been found and the decomposition of a semisimple algebra into simple components proved. Cartan derived the structure of the simple algebras but apparently without recognizing the possibility of stating the result in the very simple form Wedderburn discovered. It has to be borne in mind that all these authors had the complex field at their disposal and were therefore never hestitant to use roots of algebraic equations. This fact made a direct generalization of their results to arbitrary fields very difficult.

Wedderburn succeeded in a synthesis of these two lines of investigation. He extended the proof of all the structural theorems found by the European mathematicians for the special cases of the real and complex field to the case of an arbitrary field. By the effective use of a calculus of complexes (analogous to that which had been used in the treatment of finite groups) combined with the Peirce decomposition relative to an idempotent, he was able to prove his theorems within the given field and in a simpler way. He was the first to find the real significance and meaning of the structure of a simple algebra. We mean by this the gem of the whole paper, his celebrated:

"THEOREM 22—*Any simple algebra can be expressed as the direct product of a primitive algebra and a simple matric algebra.*"

In his terminology primitive algebra means the same thing as what we now call division algebra.

This extraordinary result has excited the fantasy of every algebraist and still does so in our day. Very great efforts have been directed toward a deeper understanding of its meaning.

In the first period following his discovery the work consisted mainly in a polishing up of his proofs. But the fundamental ideas of all these later proofs are already contained in his memoir.

In the meantime a great change in the attitude of the algebraists had taken place. The European school had discovered the great advantage of the abstract point of view which had been emphasized so early in the American school. The algebraists began to analyze Wedderburn's methods and tried to find an even more abstract background.

The essential point in the definition of an algebra is that it is a vector space of finite dimension over a field. This fact allows us to conclude that ascending and descending chains of subalgebras will terminate. After the great success that Emmy Noether had in her ideal theory in rings with ascending chain condition, it seemed reasonable to expect that in rings where the ascending and the descending chain condition holds for left ideals one should obtain results similar to those of Wedderburn. As one of the papers written from this point of view we mention E. Artin, *Zur Theorie der hyperkomplexen Zahlen* (Abh. Math. Sem. Hamburgischen Univ. vol. 5 (1926)). In 1939 C. Hopkins showed (*Rings with minimal condition for left ideals*, Ann. of Math. vol. 40) that the descending chain condition suffices.

Independently of Wedderburn's paper, the representation theory of groups had been developed under the leadership of Frobenius, Burnside, and I. Schur. These mathematicians had been very well aware of the connection with algebras, a connection given by the notion of a group ring. But little use was made of the theory of algebras.

It was Emmy Noether who made the decisive step. It consisted in replacing the notion of a matrix by the notion for which the matrix stood in the first place, namely, a linear transformation of a vector space.

Emmy Noether introduced the notion of a representation space—a vector space upon which the elements of the algebra operate as linear transformations, the composition of the linear transformations reflecting the multiplication in the algebra. By doing so she enables us to use our geometric intuition. Her point of view stresses the essential fact about a simple algebra, namely, that it has only one type of irreducible space and that it is faithfully represented by its operation on this space. Wedderburn's statement that the simple algebra is a total matrix algebra over a quasifield is now more understandable. It simply means that all transformations of this space which are linear with respect to a certain quasifield are produced by the algebra. This treatment of algebras may be found in van der Waerden's *Moderne Algebra*.

Recently it has been discovered that this last described treatment of simple algebras is capable of generalization to a far wider class of rings.

One considers a ring R and an additive group V with R as left operator domain—V playing the role of the representation space and called R-space for short. Chevalley and Jacobson proved a direct generalization of Wedderburn's theorem if two simple axioms are satisfied: That the ring R is faithfully represented by its action on V and that V is irreducible (this means that 0 and V are the only R-subspaces of V). In these terms the proof is essentially simple and geometrical, no idempotents being required, and no finiteness assumption on R.

In homage to J. H. M. Wedderburn we present in fuller detail this modern proof of his theorem.

Let R be a ring, V an R-space satisfying the two axioms stated above.[1] We shall show that V is naturally a vector space over a certain quasifield D and that practically all D-linear transformations of V are produced by elements of R.

To construct the quasifield is easy. Let D be the set of all homomorphisms of V into itself. D is a ring from first principles. Since the kernel of a nonzero element of D is an R-subspace of V which is different from V, this kernel is zero, and the element is an isomorphism. Since the image of V under this isomorphism is an R-subspace of V which is not zero, it is all of V, and we have an isomorphism of V onto V. Such a map has an inverse and we see that D is a quasifield. We have obtained in a natural, invariant manner the quasifield which Wedderburn obtained only in a noninvariant way as subring of R.

We denote the typical element of D by d and write these elements on the right of V so that our space V becomes now a right vector space over the quasifield D.

If W is a D-subspace of V, then the set of all elements of R which annihilate W is a left ideal of R which we shall call $W^{\#}$. If L is a left ideal of R, then the set of all elements of V annihilated by L is a D-subspace of V which we shall denote by $L^{\flat}$.

We can now state and prove the fundamental lemma:

$$(L \cap (\xi\cdot D)^{\#})^{\flat} = L^{\flat} + \xi\cdot D \tag{1}$$

for any left ideal L of R and any element ξ of V.

PROOF. The right-hand side is trivially contained in the left. If $L\xi = 0$, the equation becomes $L^{\flat} = L^{\flat}$. It remains only to prove that the left-hand side is contained in the right under the assumption that $L\xi \neq 0$. Since $L\xi$ is a subspace of V, and V is irreducible, $L\xi = V$: every element of V can be expressed in the form $l\xi$ where $l \in V$. Let η be an element of the left side of (1). It is annihilated by $L \cap (\xi D)^{\#}$, hence by every $l \in L$ which is in $(\xi \cdot D)^{\#}$. η is consequently annihilated by every $l \in L$ for which $l\xi = 0$. Let now ζ be any element of V; if we write it in the form $\zeta = l\xi$ and map it onto the element $l\eta$ of V we have before us a well defined map. If indeed $\zeta = l\xi = l_1\xi$ then $(l - l_1)\xi = 0$ hence $(l - l_1)\eta = 0$ or $l\eta = l_1\eta$. That this map $l\xi \to l\eta$ is a homomorphism of V into V follows from the fact that L is a left ideal; as such it is a certain element d of D and satisfies $(l\xi)d = l\eta$ for all $l \in L$. The element $\eta - \xi d$ is therefore annihilated by L and is consequently an element of $L^{\flat}$. This shows

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[1] I follow a presentation given by Mr. J. T. Tate.

$$\eta \in L^{\flat} + \xi \cdot D,$$

which is what we were trying to prove. It is of course the construction of the element d which is the heart of this method.

Let W be any D-subspace of V. If we substitute $L = W^{\#}$ in (1), the left side becomes

$$(W^{\flat} \cap (\xi D)^{\#})^{\flat}.$$

Since obviously $(A + B)^{\#} = A^{\#} \cap B^{\#}$ for any two D-subspaces A and B of V we obtain from (1)

$$(W + \xi D)^{\#\flat} = W^{\#\flat} + \xi D. \tag{2}$$

This we can use to argue in the following manner.

$$\text{If } W = W^{\#\flat} \textit{ then } (W + \xi D) = (W + \xi D)^{\#\flat}. \tag{3}$$

Combining a repeated application of (3) with

$$0^{\#\flat} = 0 \tag{4}$$

we obtain the

THEOREM.

$$W_0^{\#\flat} = W_0 \tag{5}$$

for any finite-dimensional $W_0 = \xi_1 D + \xi_2 D + \cdots + \xi_r D$.

The only gap in the argument was the proof of (4): $0^{\#\flat} =$ (trivially) $R^{\flat} =$ an R-subspace of V (which is not all of V) $= 0$, again using the irreducibility of V.

Now let $\xi_1, \xi_2, \cdots, \xi_r$ be a finite number of elements of V, linearly independent over the quasifield D. Let $W = \xi_1 D + \cdots + \xi_{r-1} D$. Since $\xi_r \notin W = W^{\#\flat}$ it follows that $W^{\#} \cdot \xi_r \neq 0$. Therefore, as usual, by the irreducibility of V we have $W^{\#} \cdot \xi_r = V$. Consequently there exists an element of $W^{\#}$ which annihilates $\xi_1, \xi_2, \cdots, \xi_{r-1}$ and sends ξ_r into any element of V. Combining such elements together we find an element of R which sends the vectors ξ_i independently into any set of r elements of V. Viewing the ξ_i as a basis for a finite-dimensional D subspace W_0 of V we see that any given D-linear map of W_0 into V can be produced by an element of the ring R. This is what we meant by the statement that "practically all" D-linear maps of V were produced by elements of R.

To specialize this result we must add the axiom that R satisfies the descending chain condition on left ideals (this is obviously true if R is Wedderburn's simple algebra). An ascending chain $W_1 \subset W_2 \subset \cdots$ of finite-dimensional D-subspaces of V leads to a descending chain $W_1^{\#} \supset W_2^{\#} \cdots$ of left ideals of R because of the statement $W^{\#\flat} = W$. Therefore V must satisfy the ascending chain condition on finite-dimensional D-subspaces. This is possible only if V itself is finite-dimensional over D. In this case our previous result shows that *every* D-linear map of V is produced by an element of R, and we have therefore obtained Wedderburn's theorem in geometric form.

As we have stated at the beginning it is not our intention to discuss the many applications Wedderburn's theorem has found, for instance, the investigations on division algebras by Wedderburn, Dickson, and others. They lead finally to a complete description of all simple algebras over an algebraic number-field by A. Albert, R. Brauer, H. Hasse, and Emmy Noether, or the theory of modular representations of algebras and groups by R. Brauer.

Let us now consider the theorem of Wedderburn concerning finite fields (*A theorem on finite algebras*, Trans. Amer. Math. Soc. vol. 6 (1905)) and its influence on the development of modern algebra. One sees immediately that the characteristic of such a field K is a prime $p>0$ and that the number of elements of K is a power p^r of p.

In 1903 E. H. Moore had determined all commutative fields of this type. The result was that to a given number p^r of elements there exists (apart from isomorphisms) only one field, namely, the Galois field of degree r and characteristic p. The proof for this fact was simplified considerably by Steinitz. It is his proof one finds in modern books on algebra.

In 1905 Wedderburn found the complete answer to our question in a paper entitled *A theorem on finite algebras*, where he proves that every field with a finite number of elements is automatically commutative (under multiplication) and therefore a Galois field.

Wedderburn introduces the center C of K and also the normalizer N_α of any element α of K. It is obvious that C and N_α are subfields and that $C \subset N_\alpha$ for each α. Denoting by q the number of elements of

C we find q^{n_α} resp. q^n for the number of elements in N_α resp. K where n_α and n are the degrees of N_α resp. K over C. Since K is an extension of N_α, the degree n_α divides n.

Wedderburn then considers the multiplicative group of K of order q^n-1. He divides it into classes of conjugate elements and obtains an identity of the form:

$$q^n - 1 = (q - 1) + \sum_{n_\alpha | n, n_\alpha < n} \frac{q^n - 1}{q^{n_\alpha} - 1} \tag{6}$$

where he unites the classes with only one element in the term $q-1$ and where the sum runs over certain divisors n_α of n, the same divisors possibly several times.

In §4 of his paper he shows the impossibility of (6) for $n>1$, making use of divisibility properties of numbers of the form a^n-b^n which are hard to establish. In §5 he gives another arrangement of this proof, again making use of these divisibility properties. A third proof by Dickson is based on similar ideas.

This result of Wedderburn has fascinated most algebraists to a very high degree and several attempts were made to simplify the proofs. Artin (Abh. Math. Sem. Hamburgischen Univ. vol. 5) gave a proof that did not make use of (6) and the divisibility properties but the proof is somewhat lengthy.

The first really simple proof of our theorem was given by E. Witt, *Über die Kommutativität endlicher Schiefkörper* in 1931 (Abh. Math. Sem. Hamburgischen Univ. vol. 8). Witt starts from (6) and makes the following simple remark:

If $\phi_n(x)$ is the nth cyclotomic polynomial, then each term in the sum on the right of (6) and also q^n-1 are obviously divisible by $\phi_n(q)$. Consequently $\phi_n(q) | q-1$. Since $\phi_n(q) = \prod(q-\epsilon)$ where ϵ runs through the primitive nth roots of unity, we have $\phi_n(q) > q-1$ if $n>1$, and this shows the impossibility of $n>1$.

In 1933 a paper by C. C. Tsen, *Divisionalgebren über Funktionenkörpern* (Nachr. Ges. Wiss. Göttingen (1933)) shed a new light on the whole question. Tsen did not investigate finite fields, but he worked with algebraic fields F of transcendency degree 1 with an algebraically closed field of constants. He proved that there does not exist any non-commutative extension field of finite degree. The method of his proof yielded really a much stronger theorem, namely:

If $N(x_1, x_2, \cdots, x_n)=0$ is an algebraic equation in F without constant term and if the total degree d is smaller than the number n of unknowns x_i, there exists a nontrivial solution in F.

If one knows this theorem for a given field F then F cannot have any noncommutative extension field E of finite degree. To see this let $\xi = x_1\omega_1 + \cdots + x_n\omega_n$ be the generic element of E ($\omega_1, \cdots$ a basis) and let $N(x_1, x_2, \cdots, x_n)$ be the reduced norm in E/F. N is a homogeneous form of $x_1, x_2, \cdots, x_n$ of a degree d which is less than n if E is noncommutative. The theorem would give the existence of a $\xi \neq 0$ whose norm is 0, which is a contradiction.

It occurred immediately to the mathematicians that possibly a Galois' field F would have the same property, so that Wedderburn's theorem would appear as a consequence of a much more general theorem on Galois fields.

In 1935 C. Chevalley (*Demonstration d'une hypothèse de M. Artin*, Abh. Math. Sem. Hamburgischen Univ. vol. 11) proved this conjecture.

Wedderburn's theorem is therefore the special case of a more general Diophantine property of fields and thus has opened an entirely new line of research.

EMIL ARTIN

BOOK REVIEWS

Cybernetics. By Norbert Wiener. The Technology Press; New York, Wiley; Paris, Hermann, 1948. 194 pp. $3.00.

As most of the readers of this review are aware, the word cybernetics has been coined to serve as the name of the new branch of science which deals with control and communication generally. The first problems that were considered in this field were, naturally, problems concerning control and communication in the inanimate systems of physics and engineering. However, it now appears, more or less clearly, that cybernetics may also have important applications in biology, psychology, and perhaps other sciences. Before beginning to comment specifically on Professor Wiener's book, it will be well to say a little about the subject of cybernetics itself.

In its present state of development cybernetics is concerned with three main themes.

First, consider the problem of control in its pure form. We desire, let us say, to make some physical variable a prescribed function of time, or, equivalently, to make some "output signal" equal to a given "input signal." In general it is impossible to achieve the desired equality exactly, but we can achieve approximate equality in various ways, depending upon the particular case in hand. Now in many cases the best way of achieving a good approximation to the desired result is by using a system which measures the difference between the input and output signals continuously or intermittently, and which operates so as to reduce the magnitude of this error. This leads us to the first main theme of cybernetics, the subject of feedback control systems. Linear feedback systems have a very attractive mathematical theory, which depends upon nothing more than the classical theories of linear differential equations and analytic functions.

The physical or other systems with which we deal in practice are always subjected to disturbances of one kind or another, and consequently the signals transmitted through the systems are accompanied by more or less troublesome "noise." The second major theme of cybernetics is concerned with the statistical properties of such noises, with means for protecting the signals against the noise, insofar as that is possible, and with the problem of correctly interpreting a received signal which is seriously perturbed by noise.

Finally, we have the part of the subject that is concerned with the nature of information itself, and with problems of coding, that is, with the selection of suitable signals for the most efficient transmission of information. Rather unexpectedly, this part of the subject turns out to have striking analogies with thermodynamics.

In Professor Wiener's book the reader will find these ideas, which we have only sketched briefly, developed at length, with ample indications of their applications to many different fields of science. In particular, it is shown that many phenomena which are familiar to biologists can very probably be explained in terms of the fundamental concepts and laws of cybernetics. Furthermore, very persuasive arguments are given for thinking that the same may be true of many psychological phenomena.

The book opens with an introduction, running to over thirty pages, in which the author gives a detailed account of the successive stages through which his own ideas on the subject have developed. If cybernetics does in fact come to have the great importance that is now predicted, this introduction will be of great interest to future historians of science.

The rest of the book consists of eight chapters, bearing the titles I. Newtonian and Bergsonian time; II. Groups and statistical mechanics; III. Time series, information, and communication; IV. Feedback and oscillation; V. Computing machines and the nervous system; VI. Gestalt and universals; VII. Cybernetics and psychopathology; VIII. Information, language, and society. These titles indicate the general character of the contents as well as is possible without a very lengthy analysis.

The most notable feature of the book is the astonishing number of diverse and highly suggestive ideas that it contains. It is clear that a rich mine has been located here, which will not be fully explored and exploited for a long time to come.

In the opinion of the reviewer the chief adverse criticism that can be made of the book is that the author seems not to have formulated any reasonable and stable concept of a class of readers for whom he is writing. He tries to make the book all things to all men. For the most part the exposition proceeds in a pleasant informal non-technical style. However, there are several occasions upon which the unsuspecting reader suddenly finds himself in the midst of a heavy downpour of mathematics, of a far from elementary character. These mathematical interludes are so closely connected with the accompanying non-technical material that they will inevitably cause considerable trouble for the non-mathematical reader. At the same time the pieces of mathematical theory are not developed far enough, or systematically enough, to be of any very direct benefit to the mathematical reader. Thus it seems that they can serve only as strong hints as to directions in which things of value are to be found.

The second chapter affords a striking illustration of this tendency to introduce bits of mathematical theory without any adequate development. Here in the short space of sixteen and a half pages the reader is hustled through a discussion which touches upon Lebesgue measure, continuous groups (including the theory of group characters), and ergodic theory. It is hard to see how anyone, unless he be an expert in this particular set of subjects, can be expected to get much of anything out of this.

Similar criticisms apply to Chapter III, which deals with the theory of information. This new theory is elusive and subtle at best; and the hurried and mathematically complicated treatment given here seems to the reviewer to verge upon utter unintelligibility.

In conclusion, simple honesty and a decent regard for good workmanship compel the reviewer to remark that someone must bear the responsibility for what is truly a wretched job of proofreading. It would be hard to find another book having as high an average number of errors per page as this one has. The errors run all the way from trivialities, which are merely occasions for exasperation, to typographical errors in the equations and formulae which seriously impair the intelligibility of the exposition. Scientists, unlike philatelists, do not value documents for the errors they contain, and therefore it is to be hoped that in a second edition we shall see most of these errors corrected.

L. A. MacColl

Topology of manifolds. By R. L. Wilder. (American Mathematical Society Colloquium Publications, vol. 32.) New York, American Mathematical Society, 1949. 10+402 pp. $7.00.

To give an idea of the scope of this book we shall begin with a brief description of a body of theorems in the topology of the euclidean plane that are referred to as the "Schoenflies program." Let M be the 2-sphere and K a closed subset of M. If K is a simple closed curve (=topological image of a 1-sphere) then $M-K$ is the union of two disjoint connected open sets A and B such that $K=\bar{A}\cap\bar{B}$ (Jordan Curve Theorem). It is further known that each of the sets $\bar{A}$ and $\bar{B}$ is homeomorphic to a closed disc. Converse theorems which give necessary and sufficient conditions on the set $M-K$ in order that K be a simple closed curve or a Peano space (=locally connected continuum) also exist.

The objective of this book is to extend this program to higher dimensions, using homology theory as the principal tool. Thus M is replaced by a generalized n-dimensional manifold (that is, an n-manifold in the sense of homology) while K is in general a closed set satisfying certain connectedness and local connectedness properties. In some cases K itself is a manifold of a lower dimension.

The first chapter reviews various concepts of general topology with special emphasis on connectedness. A topological characterization of the closed interval and the circumference are given. The second chapter begins with a discussion of local connectedness and then shifts to some properties of the n-sphere. The main theorem proved here is the Brouwer Separation Theorem which is a generalization of the Jordan Curve Theorem to the case when M is an n-sphere and K is a topological $(n-1)$-sphere. This is achieved by establishing in a very elementary manner the Alexander duality relations between the mod 2 Betti numbers of K and $M-K$. Thus the reader gets his first glimpse at homology theory. Having thus proved the Jordan Curve Theorem the author proceeds to establish Schoenflies' converse theorem. Chapter three has a more detailed discussion of Peano spaces, local connectedness, uniform local connectedness, and so on. These results are applied to obtain topological characterizations of the 2-sphere, of the 2-dimensional disc, and of 2-manifolds. The fourth chapter, after some discussion of local connectedness, contains some theorems about the positional properties of a closed subset K in the 2-sphere S^2. In particular, necessary and sufficient conditions are given which S^2-K must satisfy in order that K be a Peano continuum.

The first four chapters may be regarded as a modernization and slight enlargement of the Schoenflies program. However, in a sense, these chapters are still introductory, since the tools needed to carry out the program in all dimensions have not been yet introduced. This begins in chapter five where the Čech homology and cohomology theories for topological spaces are introduced. Throughout, the coefficient domain is assumed to be a field; this eliminates certain difficulties concerning so-called convergence properties. The chapter includes the duality between homology and cohomology and the theory of the cap-product. The Čech theory is "localized" in the next chapter, thus yielding the homological and co-homological theories of local connectedness. Uniform local connectedness and allied topics also are discussed. This machinery is applied in chapter seven to the study of continua. A substantial part of the Schoenflies program is thus given a significant generalization.

The full scale discussion of manifolds begins in chapter eight. Here an "n-dimensional generalized manifold" (abbreviated: n-gm) is defined as a locally compact n-dimensional space which is locally connected (in a sense of cohomology) in dimensions $0, \cdots, n-1$ and which at each of its points has the nth local Betti number equal to 1. For such manifolds the concept of orientability is defined, and, for orientable manifolds, the Poincaré duality theorem is proved. Under some additional assumptions the Alexander duality theorem comparing the Betti numbers of a closed subset of an n-gm and those of its complement is established. In the next chapter it is proved that a separable n-gm is a classical manifold for $n=1$, 2. For $n>2$ the result is no longer true. A comparison of the definition of an n-gm with another plausible definition is given.

The last three chapters (X–XII) are devoted to the study of positional invariants of a closed subset K of an n-gm M. In the first of these chapters the complement $M-K$ is described under the assumption that K is itself an $(n-1)$-gm. Conversely, imposing certain conditions on $M-K$, one can assert that K is an $(n-1)$-gm. The last two chapters deal with the case when K is locally connected in dimensions, $0, \cdots, k$ $(k<n)$ or is a k-gm. Roughly speaking "local duality theorems" are established which translate the smoothness property of K at a point $x \in K$ into a property of $M-K$ at x. In order to achieve such a duality one has to consider concepts of accessibility, avoidability, and so on. These chapters constitute properly the generalization of the Schoenflies program to n dimensions. An appendix on unsolved problems and a detailed bibliography conclude the book.

One of the chief features of the book is the unity of method. From the moment the Čech homology and cohomology are introduced (in chapter five) they are used consistently and systematically. In earlier chapters this unity is in places sacrificed for pedagogical reasons. An effort has been made not to present the reader with elaborate definitions and concepts without suitable introduction and motivation. The book is entirely self-contained and because of that a large portion of the space is devoted to the development of machinery. This machinery is sometimes quite delicate (for instance: a large number of various kinds of local connectedness are introduced, and they are compared) but this is necessitated by the fine nature of the problems considered. The results of the later chapters are almost entirely due to the author and only a small portion of them has been published before. Most of the results of the earlier chapters, although known, are not too readily available in the literature. The unification and systematic exposition of all these results is an immense task and should greatly enhance research in this part of topology.

SAMUEL EILENBERG

The geometry of the zeros of a polynomial in a complex variable. By Morris Marden. (Mathematical Surveys, no. 3.) New York, American Mathematical Society, 1949. 10+183 pp. $5.00.

Since the time of Gauss there has been continued interest in those problems that center around the location of the zeros of a polynomial. Short expositions of the theory have been given, and a lengthier one by Dieudonné, but now, with the publication of this work, the third in the series of Mathematical Surveys of the American Mathematical Society, a comprehensive treatise on the subject has been made available. The lengthy, 20-page bibliography attests to the aim of the author to cover the material thoroughly; and considerable work must have been involved to integrate this heterogeneous material into such unity as is possible. Even so, the book could not have been kept to 161 pages had the author not adopted the plan of putting considerable theoretical material into the problems that appear at the end of most sections. Where necessary, hints are given with the problems, a wise gesture as readers of the book will see.

In a well-written preface there is a brief account of the origin of the subject, and a statement of the two central problems that the book is to treat, together with a declaration of the principal working tools. The zeros of a polynomial $f(z)$ are functions of the coefficients. Thus one problem is to specify regions, determined by these coefficients, in which the zeros lie. Again, with $f(z)$ one may associate a second polynomial (frequently this is the derivative $f'(z)$), and thus arises the problem of relating the location of the zeros of the associate to the location of those of $f(z)$. There are also variants of these problems, of which something will be said as the individual chapters are discussed. As for the methods and tools that are used, they are classical. Thus, among other theorems, those of Cauchy, Rouché, and Hurwitz on zeros of analytic functions are of considerable use; and many theorems involve, either in proof or in statement, geometric and algebraic concepts of an elementary nature.

Chapter I is introductory. The above-mentioned classical theorems are stated and proofs given. Then various interpretations, from physics, geometry, and function-theory, are given for the zeros of the rational function

$$F(z) = \sum_{j=1}^{p} \frac{m_j}{z - z_j}.$$

One from physics goes back to Gauss: The zeros of $F(z)$ are the equilibrium points in a force field due to p masses $m_1, \cdots, m_p$ at the points $z_1, \cdots, z_p$ with the law of inverse distance. When the m_j are positive integers, $F(z)=0$ gives those zeros of $f'(z)$ that are not zeros of $f(z)$ itself, where $f(z)=\prod_{j=1}^{p}(z-z_j)^{m_j}$.

This leads at once to the Lucas theorem of Chapter II: Any convex polygon (or any circle) that contains the zeros of a polynomial $f(z)$ also contains the zeros of $f'(z)$. In this theorem the coefficients of $f(z)$ are free to lie anywhere in the complex plane; consequently one may expect to narrow the location of the roots of $f'(z)$ if one specializes the coefficients. Thus, if $f(z)$ is a *real* polynomial, the non-real zeros of $f'(z)$ are contained in the Jensen circles (theorem of Jensen). As the Lucas and Jensen theorems concern the zeros of the function $F(z)$ above, it is natural to consider a generalization to the zeros of the function $G(z)=\sum_{j=1}^{n} m_j f_j(z)$, where each $f_j(z)$ is a rational function with p finite zeros and q finite poles, and the m_j are complex constants. If the zeros and poles of each $f_j(z)$ lie in a closed convex region K, and if $\mu \leqq \arg m_j \leqq \mu+\gamma<\mu+\pi$, then the zeros of $G(z)$ are contained in the region $S(K, \phi)$, which is star-shaped relative to K and consists of all points from which K subtends an angle not less than $\phi=(\pi-\gamma)/(p+q)$. Chapter II concludes with a discussion of the zeros of the polynomials of Van Vleck and of Stieltjes that arise in a generalized Lamé differential equation.

Given a polynomial $f(z)$ of degree n, there is introduced in Chapter III the Laguerre-originated function

$$f_1(z) = nf(z) + (\zeta - z)f'(z),$$

called by the author the *polar derivative* of $f(z)$ relative to the *pole* ζ. It has the property, not shared by $f'(z)$, that under an arbitrary nonsingular linear fractional transformation, those zeros of $f_1(z)$ that are not multiple zeros of $f(z)$ and are not equal to ζ transform into zeros of the transformed function. For $f_1(z)$ there is an invariant form of the Lucas theorem, due to Laguerre: If all the zeros of $f(z)$ lie in a circular region C, and if Z is a zero of $f_1(z)$, then Z and ζ cannot both lie outside of C; and if, moreover, $f(Z)\neq 0$, any circle that passes through Z and ζ either passes through all the zeros of $f(z)$ or it separates these zeros. The Laguerre theorem is a very useful one for subsequent results. By recurrence, polar derivatives of higher order are defined at the end of the chapter.

Chapter IV deals with composite polynomials. First *apolar* polynomials are treated. Two nth degree polynomials

$$f(z) = \sum_{k=0}^{n} C(n, k)A_k z^k, \qquad g(z) = \sum_{k=0}^{n} C(n, k)B_k z^k$$

are apolar if $\sum_{k=0}^{n} (-1)^k C(n, k) A_k B_{n-k} = 0$. To a given $f(z)$ there correspond infinitely many apolar polynomials $g(z)$, and Szegö has given a useful test for apolarity: if $L[f(t)] \equiv \sum_{k=0}^{n} l_k a_k = 0$, where $a_k = C(n, k) A_k$, then $g(z) = L[(t-z)^n]$ is apolar to $f(z)$. For apolar polynomials there is the theorem of Grace: If f, g are apolar and the zeros of f lie in a circular region C, then at least one zero of g is in C. And from this follows a result of Walsh (that has beautiful application in subsequent portions of the book): Let $\Phi(z_1, \cdots, z_n)$ be a linear symmetric function of its n variables; if C is a circular region containing the points $z_1^{(0)}, \cdots, z_n^{(0)}$ then there exists at least one point ζ in C for which $\Phi(\zeta, \zeta, \cdots, \zeta) = \Phi(z_1^{(0)}, \cdots, z_n^{(0)})$. Applications of these results are then made to the problem of finding regions in which the zeros of certain polynomials lie, when these polynomials are related to others whose zeros lie in specified regions.

Chapter V opens with a two-circle theorem of Walsh that generalizes the Lucas theorem. It deals with the zeros of $(f_1 f_2)'$. A corresponding result holds for $(f_1/f_2)'$; and the general case (due to Marden) where $f_1 f_2$, f_1/f_2 are replaced by $f = (f_0 f_1 \cdots f_q)/(f_{q+1} \cdots f_p)$ is then discussed.

The results of the preceding chapters depend on knowledge of the location of (that is, knowledge of some region containing) all the zeros of a polynomial $f(z)$. Chapter VI now examines the situation when something is known of some, but not of all, of the zeros of $f(z)$. One of the fundamental results, concerning *two* zeros, is the Grace-Heawood theorem: If $f(z_1) = f(z_2) = 0$, then at least one zero of $f'(z)$ is contained in the circle of center $(z_1 + z_2)/2$ and radius $(1/2)|z_1 - z_2| \cot (\pi/n)$, where n is the degree of f. The radius is "best possible." Kakeya proposed the following problem: Given that p zeros of $f(z)$, a polynomial of degree n, are contained in a circle C of radius R, to find the radius of a concentric circle C' that will contain at least $p-1$ zeros of $f'(z)$. He showed that a function $\phi(n, p)$ exists such that $R' = R\phi(n, p)$ will satisfy the problem; but the best value of ϕ is unknown for $p > 2$. A result of Marden shows that the choice $\phi(n, p) = \csc (\pi/2q)$, $q = n - p + 1$, is permissible. Other results concerning $\phi(n, p)$ are also treated in this chapter.

Up to now the emphasis has been essentially on one of the two problems mentioned as being the subject matter of this book. The remainder of the work concerns itself largely with the other problem, namely, to obtain information on the location of the zeros of a polynomial $f(z)$ in terms of its coefficients. Chapter VII gives bounds for the absolute values of the zeros of $f(z)$ from knowledge of the moduli of the coefficients. Many names are associated with these bounds; in particular, one may mention the well known theorem of Pellet. By taking into account the arguments as well as the moduli of the coefficients, certain refinements of Pellet's result are given, and a number of applications are discussed.

Chapter VIII takes up various generalizations of Landau's theorems that every polynomial $a_0+a_1z+a_nz^n$ $(a_1\neq 0,\ n\geqq 2)$ has at least one zero in the circle $|z|\leqq 2|a_0/a_1|$, and every polynomial $a_0+a_1z+a_mz^m+a_nz^n$ $(a_1\neq 0,\ 2\leqq m<n)$ at least one zero in $|z|\leqq (17/3)|a_0/a_1|$. These are examples of the general lacunary polynomial

$$f(z) = a_0 + a_1z + \cdots + a_pz^p + \sum_{j=1}^{k} a_{n_j}z^{n_j}$$

$$(a_0a_p \neq 0,\ 1 \leqq p < n_1 < \cdots < n_k).$$

It is a result of Montel that $R=R(a_0, \cdots, a_p, k)$ exists such that the circle $|z|\leqq R$ contains at least p zeros for every such $f(z)$. Various permissible choices of R (though not the "best possible") are found in this chapter.

Chapter IX discusses the problem of determining the number of zeros of a polynomial in a half-plane, the result being expressed in terms of the number of variations in sign of certain determinants involving the coefficients of $f(z)$. Corresponding results are also obtained for the number of zeros in a given sector. In the final chapter an algorithm is given in terms of which it is possible to determine the number of zeros of a polynomial in a given circle.

The above résumé of the contents of this book gives some indication of the wealth of material to be found therein. The work should arouse interest in the theory of the zeros of polynomials, and the author is to be thanked for presenting the opportunity to learn the present state of that theory. If the reviewer may however be permitted to express a mild regret, it is that the author has not explicitly charted the possible future, for prospective workers in the field; at various stages in the work there seems to be a real need for the author to take the reader into his confidence and point the way to desirable topics as yet incompletely (or not at all) explored.

Although not every proof was checked to the last detail, enough checking leads one to conclude that the book is free of gross errors, and only slight oversights and typographical errors are to be reported. On pages 1 and 2 it is required of certain functions that they be analytic interior to a simple closed curve C and continuous on C, whereas the continuity should be interior to and on C. On page 17 the letter K in Fig. (7, 1) seems to refer to the dotted rectangle, in which case the label R should be used rather than K. From the statements of Theorem (8, 1), page 19, and line 3 of page 20, K subtends an angle less than ϕ at ζ. Consequently, the angle marked ϕ in Fig. (8, 1) seems inconsistent, as does the phrase "where, however, K subtends at ζ the angle ϕ" of line 2, page 20. On page 48, last line, the subscripts on z_1, z_2 should be removed.

In closing, it is perhaps not out of place to observe that many valuable contributions to the theory expounded in this book are due to J. L. Walsh and the author.

I. M. SHEFFER

NEW PUBLICATIONS

DELACHET, A. *L'analyse mathématique.* Paris, Presses Universitaires de France, 1949. 118 pp.

Electromagnetic theory. (Proceedings of Symposia in Applied Mathematics, vol. 2.) New York, American Mathematical Society, 1950. 4+91 pp. $3.00.

HOSSNER, A. *Einführung in die höhere Mathematik.* Vienna, Springer, 1949. 10+359 pp. $6.00.

LANGER, R. E. *Fourier's series, the genesis and evolution of a theory.* (Herbert Ellsworth Slaught Memorial Papers, no. 1.) Buffalo, Mathematical Association of America, 1949. 5+86 pp. $1.00.

DE LA VALLÉE POUSSIN, C. J. *Le potentiel logarithmique. Belayage et représentation conforme.* Louvain, Librarie Universitaire, 1949. 12+452 pp.

MAGNUS, W. See OBERHETTINGER, F.

Non-linear problems in mechanics of continua. (Proceedings of Symposia in Applied Mathematics, vol. 1.) New York, American Mathematical Society, 1949. 8+219 pp. $5.25.

OBERHETTINGER, F., and MAGNUS, W. *Anwendung der elliptischen Funktionen in Physik und Technik.* (Die grundlehren der Mathematischen Wissenschaften, vol. 55.) Berlin, Springer, 1949. 7+126 pp. 15.60 DM.

PELLETIER, J. L. *L'age des mathématiques.* Paris, L'office de Documentation Bibliographique, 1949. $3.00.

Tables of the Bessel functions of the first kind of orders fifty-two through sixty-three. (The Annals of the Computation Laboratory of Harvard University, vol. 11.) Cambridge, Harvard University Press, 1949. 12+544 pp. $8.00.

NOTES

The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Committee, who have been consulted regarding papers submitted for publication: R. P. Agnew, A. A. Albert, E. F. Beckenbach, Paul Bernays, Lipman Bers, Salomon Bochner, A. T. Brauer, R. C. Buck, Herbert Busemann, S. S. Chern, Claude Chevalley, A. B. Coble, I. S. Cohen, H. S. M. Coxeter, Anne C. Davis, J. A. Dieudonné, C. J. Everett, Herbert Federer, E. E. Floyd, R. H. Fox, J. H. Giese, L. M. Graves, Marshall Hall, P. R. Halmos, Philip Hartman, M. H. Heins, G. P. Hochschild, L. K. Hua, Leopold Infeld, R. E. Langer, Walter Leighton, Norman Levinson, N. H. McCoy, A. P. Morse, Leopoldo Nachbin, C. A. Phillips, Harry Pollard, Tibor Rado, J. H. Roberts, A. C. Schaeffer, M. M. Schiffer, I. E. Segal, Wladimir Seidel, Atle Selberg, I. M. Sheffer, M. F. Smiley, R. H. Sorgenfrey, D. C. Spencer, Otto Szasz, W. J. Thron, A. D. Wallace.

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Sherman, Edward Silverman, M. F. Smiley, P. A. Smith, I. S. Sokolnikoff, N. E. Steenrod, J. J. Stoker, Walter Strodt, Otto Szasz, A. H. Taub, Olga Taussky-Todd, William Clare Taylor, T. Y. Thomas, C. A. Truesdell, D. F. Votaw, H. S. Wall, J. L. Walsh, Morgan Ward, W. R. Wasow, Hermann Weyl, D. V. Widder, L. C. Young, Oscar Zariski, Leo Zippin, Antoni Zygmund.

The Mathematics Department of Harvard University expects to have one or two vacancies in the rank of Benjamin Peirce instructor beginning in the fall of 1950. Appointments are for a maximum period of three years. Applications should reach the Department prior to March 15, 1950.

Professor A. T. Brauer of the University of North Carolina has received one of the three inaugural Science Research Awards of the Oak Ridge Institute of Nuclear Studies for significant contributions to science in the South.

Professor Edward Kasner of Columbia University has retired with the title emeritus.

Dr. C. M. Stein has been awarded a National Research Council postdoctoral fellowship.

Dr. E. J. Akutowicz of the Institute for Advanced Study has been appointed to an assistant professorship at Pennsylvania State College.

Professor C. B. Allendoerfer of Haverford College will be on leave for the current term and has been appointed to a visiting professorship at the Massachusetts Institute of Technology.

Mr. Helmut Aulbach of Syracuse University has been appointed to an assistant professorship at Triple Cities College of Syracuse University.

Dr. Iacopo Barsotti of the University of Rome and Princeton University has been appointed to an associate professorship at the University of Pittsburgh.

Associate Professor N. R. Bryan of the University of Georgia has been appointed to an associate professorship at Clemson College.

Professor P. F. Byrd of Fisk University has accepted a position as research scientist with the National Advisory Committee on Aeronautics.

Professor S. S. Chern of the Institute for Advanced Study has been appointed to a professorship at the University of Chicago.

Dr. Herman Chernoff of the University of Chicago has been appointed to an assistant professorship at the University of Illinois.

Mr. Peter Chiarulli of Brown University has been appointed to an assistant professorship at Carnegie Institute of Technology.

Associate Professor Lester Dawson of Adams State College has been appointed to an assistant professorship at Colorado Agricultural and Mechanical College.

Associate Professor C. H. Dowker of Tufts College has been appointed visiting lecturer at Harvard University.

Mr. R. H. Edwards of the Illinois Institute of Technology has accepted a position as research physicist with the Hughes Aircraft Company, Culver City, California.

Professor F. A. Ficken of the University of Tennessee is on leave at New York University.

Mr. Emanuel Fischer of City College, New York, has been appointed to an assistant professorship at Hampton Institute.

Assistant Professor W. C. G. Fraser of Rensselaer Polytechnic Institute has been appointed to an assistant professorship at Dartmouth College.

Dr. P. R. Garabedian of Stanford University has been appointed to an assistant professorship at the University of California, Berkeley, California.

Mr. R. S. Gardner of Ohio State University has accepted a position as statistician with the U. S. Naval Ordnance Test Station, China Lake, California.

Mr. D. C. Gerneth of Sherman, Texas, has been appointed to an assistant professorship at Mississippi Southern College.

Dr. John Gurland of Harvard University has been appointed to an assistant professorship at the University of Chicago.

Mr. H. L. Harter of Purdue University has been appointed to an assistant professorship at Michigan State College.

Dr. A. R. Harvey of California Institute of Technology has been appointed to an assistant professorship at San Diego College.

Dr. L. A. Henkin of Princeton University has been appointed to an assistant professorship at the University of Southern California.

Mr. D. W. Hullinghorst has accepted a position as director of the Civil Works Investigation Project, South Pacific Division, Engineering Corps, Oakland Army Base, Oakland, California.

Dr. P. J. Kelly of the University of Southern California has been appointed to an assistant professorship at the Santa Barbara College of the University of California.

Mr. M. S. Kramer of the University of Illinois has been appointed to an assistant professorship at the New Mexico College of Agriculture and Mechanic Arts.

Dr. Knox Millsaps of the Air Materiel Command, Wright Field, has been appointed to a professorship of theoretical physics at the Alabama Polytechnic Institute.

Mr. G. D. Mostow of the Institute for Advanced Study has been appointed to an assistant professorship at Syracuse University.

Assistant Professor H. T. Muhly of the U. S. Naval Academy has been appointed to an associate professorship at the State University of Iowa.

Assistant Professor A. L. Putnam of the University of Chicago has been appointed to a professorship at New York University.

Mr. W. G. Scobert of the University of Oregon has been appointed to an associate professorship at Idaho State College.

Associate Professor Atle Selberg of Syracuse University has been appointed a permanent member of the Institute for Advanced Study.

Dr. M. M. Schiffer of Hebrew University has been appointed to a visiting professorship at Princeton University.

Dr. H. M. Schwartz of Brookhaven National Laboratory has been appointed to an associate professorship at the University of Arkansas.

Dr. Edward Silverman of the Institute for Advanced Study has been appointed to a visiting assistant professorship at Kenyon College.

Assistant Professor Marvin Stippes of Montana State College has been appointed to an assistant professorship at the University of Illinois.

Assistant Professor E. C. Stopher of Miami University has been appointed to a professorship at Fort Hays Kansas State College.

Assistant Professor Irving Sussman of the University of Dayton has been appointed to a professorship at St. Mary's College, St. Mary's College, California.

Professor Alexander Tartler of Drexel Institute of Technology has been appointed to an associate professorship at Lafayette College.

Mr. John Todd has accepted a position as chief of the Computation Laboratory, National Bureau of Standards, Washington, D. C.

Professor A. W. Tucker of Princeton University is on leave and has been appointed to a visiting professorship at Stanford University.

Dr. C. W. Vickery of the U. S. Naval Ordnance Test Station, China Lake, California, has been appointed to a professorship at Stephen F. Austin State College, Nacogdoches, Texas.

Dr. Sylvan Wallach of Johns Hopkins University has accepted a position as research engineer at the Westinghouse Electric Corp., Pittsburgh, Pennsylvania.

Professor J. A. Ward of the University of Georgia has been appointed to a professorship at the University of Kentucky.

Assistant Professor W. R. Wasow of Swarthmore College is on leave and has accepted a position as mathematician at the Institute for Numerical Analysis of the National Bureau of Standards.

Assistant Professor Euphemia R. Worthington of the University of California at Los Angeles has retired.

Mr. N. P. Yeardley of the University of Cincinnati has been appointed to an assistant professorship at Purdue University.

Dr. A. T. Yu of Lehigh University has been appointed to an assistant professorship at New York University.

Mr. R. K. Ziegler of the State University of Iowa has been appointed to an associate professorship at Bradley University.

The following promotions have been announced:

S. H. Hilding, University of Stockholm, to an assistant professorship.

W. F. Atchison, University of Illinois, to an assistant professorship.

R. H. Bing, University of Wisconsin, to an associate professorship.

M. M. Day, University of Illinois, to a professorship.

C. D. Firestone, Rutgers University, to an assistant professorship.

M. K. Fort, University of Illinois, to an assistant professorship.

F. E. Hohn, University of Illinois, to an assistant professorship.

Solomon Hurwitz, City College, New York, to an assistant professorship.

Robert Kahal, Polytechnic Institute of Brooklyn, to an assistant professorship.

W. J. Klimczak, University of Rochester, to an assistant professorship.

M. Z. Krzywoblocki, University of Illinois, to a professorship.

L. J. Paige, University of California at Los Angeles, to an assistant professorship.

J. D. Swift, University of California at Los Angeles, to an assistant professorship.

F. A. Valentine, University of California at Los Angeles, to an associate professorship.

R. W. Wagner, Oberlin College, to an associate professorship.

The following appointments to instructorships are announced: Schools of Technical Association, Copenhagen, Denmark: Mr. H. H. Hansen; Brooklyn College: Dr. M. E. Levenson; University of California, Davis: Mr. D. A. Norton; University of California at Los Angeles: Mr. G. M. Wing; Carnegie Institute of Technology: Mr. R. C. Meacham; University of Chicago: Mr. A. F. Bausch; Duke University: Mr. L. M. Fulton, Mr. Walter Rudin; University of Illinois: Mr. C. A. Phillips; Iowa State College of Agriculture and

Mechanic Arts: Mr. A. M. Feyerherm; University of Maryland: Mr. A. G. Hansen; Massachusetts Institute of Technology: Mr George Springer; University of Michigan: Mr D. A. Darling, Dr. J. L. Ullman; Montana State College: Mr. R. D. Glauz; University of Nebraska: Mr. C. C. Buck; North Dakota Agricultural College: Mr. R. L. Mentzer; Northwestern University: Mr. D. Y. Barrer; University of Pennsylvania: Mr. M. I. Aissen, Mr. H. E. Campbell; Purdue University: Mr. Gustave Rabson; University of Texas: Mr. B. J. Ball; Texas Western College: Mr. G. T. Cates; University of Utah: Miss Rachel S. Hodges; State College of Washington: Mr. P. A. Clement; Yale University: Mr. W. H. Mills.

Professor A. F. Carpenter of the University of Washington died October 16, 1949 at the age of sixty-nine years. He had been a member of the Society for forty-three years.

Professor Emeritus Peter Field of the University of Michigan died September 24, 1949 at the age of seventy-three years He had been a member of the Society for forty-nine years.

Professor Emeritus W. J. Humphreys of George Washington University died November 10, 1949 at the age of eighty-seven years. He had been a member of the Society for twenty-four years.

The note about Professor E. D. Hellinger which appeared on p. 878 of volume 55 of the Bulletin, September 1949, is in error and should read "Professor E. D. Hellinger of Northwestern University has retired with the title emeritus and has been appointed to a visiting professorship at the Illinois Institute of Technology."

BULLETIN

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PREFACE

The following pages contain a list of the additions to the Society's Library during the year 1949.

Users of this fourth Supplement are referred to the Catalogue published in 1946 (Bulletin, vol. 52, no. 5, part 2) and to the preface to Supplement No. 1 (Bulletin, vol. 53, no. 1, part 2) for an explanation of the abbreviations and for a statement of the plan followed in reporting additions to the serial publications.

ARNOLD DRESDEN
Librarian

January 1950

SECTION I
SERIAL PUBLICATIONS

Academia brasileira de ciencias, Rio de Janeiro
  Anais
    7–10, 1935–38
Academia sinica
  Science record
    1+, 1942+
Accademia della scienze di Torino
  Classe de scienze fisiche, matematiche e naturali
    Atti
      75, 1939/40
Accademia nazionale dei Lincei, Rome
  Classe di scienze fisiche, matematiche e naturali
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      s8 v1+, 1946+
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Astronomie. (Société astronomique de France) Paris
  48, 1934
Disquisitiones mathematicae et physicae. Bucarest
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Euclides, revista mensual de ciencias exactas, fisicas, quimicas, naturales y aplicaciones técnicas. Madrid
  7+, 1947+
Giornale di matematiche. (Battaglini) Naples
  4–5, 7–34, 36+, 1866+
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    s2 v1+ 1946+
Indagationes mathematicae. Amsterdam
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Matematikai és természettudományi értesítö
  58, 1939
Monatshefte für mathematik und physik, Vienna
  37, 1930
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National institute of sciences of India, Calcutta
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Portugaliae mathematica. Lisbon
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Rome
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Royal statistical society, London
Journal
Supplement
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Sapporo, Japan
Hokkaido imperial university
Faculty of science
Journal
s1. Mathematics
9+, 1940+
Sendai, Japan
Tôhoku teikoku daigaku
Science reports
s1. Mathematics, physics, chemistry
30+, 1941+
Tôhoku mathematical journal. Sendai
48+, 1941+
Unione matematica italiana
Bollettino
s2 v3, 1941; s3, v1+, 1946+
United States coast and geodetic survey, Washington, D. C.
Special publications
235, 1945
Wiskundig tijdschrift, Haarlem
4–6, 1907/09–1909/10

SECTION II

NONSERIAL PUBLICATIONS

Abascal, E. V. *See* Vidal Abascal, E.
Adler, C. F. An isoperimetric problem with an inequality. Diss., New York, 1942.
Alamo, J. B. *See* Bejar Alamo, J.
Aller, R. M. Introducción a la astronomía. Madrid, 1943.
Nuevos métodos en observaciones de pasos. Santiago de Compostela, 1944.
Ammann, A. Quelques propriétés concernant la répartition des suites de nombres module un. Diss., Geneva, 1947.
Archibald, R. C. Testimonial to R. C. Archibald on his retirement from the post of Librarian of the American Mathematical Society, December 31, 1941. fc.
Atkins, H. P. On fractional derivatives of univalent functions. Diss., Rochester, 1947.
Barlaz, J. On some triangular summability methods. Diss., Cincinnati, 1945.
Bates, G. E. Free loops and nets and their generalizations. Diss., Illinois, 1946.
Baule, B. Die mathematik des naturforschers und ingenieurs. v1–7, Zurich, 1947–48.
Beesley, E. M. II. ϕ-Cantorian functions and their convex moduli. Diss., Brown, 1943.
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LIE THEORY OF SEMI-GROUPS OF LINEAR TRANSFORMATIONS

EINAR HILLE

1. **Introduction.** In the Colloquium Lectures which I had the honor of delivering to the Society at the Wellesley meeting in August 1944, an outline was given of a theory of *one-parameter semi-groups of linear bounded operators on a complex* (*B*)*-space* $\mathfrak{X}$ *to itself.* The problem here is the study of a family of linear bounded transformations $\mathfrak{S} = \{T(\alpha)\}$, defined for $\alpha > 0$, with the product law

$$T(\alpha)T(\beta) = T(\alpha + \beta). \tag{1.1}$$

Such families arise in the most varied branches of classical and of modern analysis and are interesting for their own sake as well as for the many applications.

An extension to the n-parameter case was presented to the Society in October 1944 (abstract 51-1-15). Here the parameter $a = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a vector in n-dimensional real euclidean space E_n, the operators $T(a)$ are defined for non-negative values of the components of a, and the *product law* reads

$$T(a)T(b) = T(a + b) \tag{1.2}$$

with $a+b = (\alpha_1+\beta_1, \cdots, \alpha_n+\beta_n)$. These operators commute. If $\|T(a)\|$ is bounded for small a, if certain unions of range spaces $T(a)[\mathfrak{X}]$ are dense in the space $\mathfrak{X}$, and if $T(a)$ is a strongly measurable function of a, then $T(a)$ is actually strongly continuous for all a and $T(h)x \to x$ for each x when $h \to 0$. Further $T(a)$ is the direct product of n commuting one-parameter semi-groups

$$T(a) = T_1(\alpha_1)T_2(\alpha_2) \cdots T_n(\alpha_n). \tag{1.3}$$

It turned out later that the analysis could be extended, at least in part, to the case in which the parameter set is an *open positive cone* $\mathfrak{C}$ in a (*B*)-space $\mathfrak{P}$. Here $\mathfrak{C}$ is an open set, if a and b are in $\mathfrak{C}$ so are $\alpha a + \beta b$ for $0 \leqq \alpha$, $0 \leqq \beta$, $0 < \alpha + \beta$. The product law is still given by (1.2).

These investigations with many extensions and numerous applications have now appeared in book form ([6] in the References at the end of this address). The earliest results on continuity in the one-parameter case are due to N. Dunford [2] and extensions to the

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n-parameter case have also been found by N. Dunford and I. E. Segal [**3**]. A number of results referring to the one-parameter case with applications to stochastic processes and Brownian motion have been found independently by K. Yosida whose work is now in course of publication [**10, 11, 12**].

So far only commutative operators have been considered and the product law (1.2) is the simplest possible. The non-commutative case has resisted numerous attacks in the past and it is only a few months ago that any headway was made with this problem. I shall have the pleasure of outlining the new theory here; it is a blend of the classical theory of Lie groups with the recent theory of one-parameter semi-groups.

2. **Assumptions.** We shall be concerned with an n-parameter family $\mathfrak{S}$ of linear bounded operators $T(a)$ on a (B)-space $\mathfrak{X}$ to itself. These operators shall form a semi-group, that is, if $T(a)$ and $T(b)$ are in $\mathfrak{S}$ so are their products $T(a)T(b)$ and $T(b)T(a)$ which ordinarily are distinct.

In a first study of the problem, we are entitled to restrict ourselves to comparatively simple situations. Our assumptions will be chosen accordingly, but they will be introduced as needed so as to bring out what parts of the theory require heavy machinery and what may be proved under less restrictive assumptions. No claim is made that the assumptions have their definitive form.

We are dealing with three different spaces, the parameter, the operand, and the operator spaces, and a more or less complicated product law. Consequently we shall need assumptions of diverse nature which may be classified under the following four headings:

(1) The parameter set, that is, the values of a for which $T(a)$ is defined.

(2) The properties of $T(a)$ for fixed a.

(3) The properties of $T(a)$ as function of a.

(4) The product law, that is, the function

$$c = F(a, b) \tag{2.1}$$

which is defined by the relation

$$T(c) = T(a)T(b). \tag{2.2}$$

The assumptions relating to (1) and (2) will be kept fixed throughout the following discussion. They are:

A^1. *$T(a)$ is defined for $a = (\alpha_1, \cdots, \alpha_n)$ in E_n^+, that part of E_n in which $\alpha_j \geq 0$, $j = 1, \cdots, n$.*

A^2. *$T(a)$ is a linear bounded operator on $\mathfrak{X}$ to itself, $T(0)=I$, the identity operator, and $\|T(a)\|\leq 1$ for a in $\overline{E}_n^+$.*

The initial assumption under (3) is:

A_1^3. *$T(a)$ is a strongly measurable function of a in $\overline{E}_n^+$.*

The product law will require a number of different assumptions; we start with the following which will be used in the next section. The Euclidean length of the vector b is denoted by $|b|$.

A_1^4. *$F(a, b)$ is a continuous function on $\overline{E}_n^+\times\overline{E}_n^+$ to $\overline{E}_n^+$ and*

$$F(a, 0) = F(0, a) = a, \tag{2.3}$$

$$F(a, F(b, c)) = F(F(a, b), c). \tag{2.4}$$

A_2^4. *To every $R>0$ there is a positive $\delta=\delta(R)$ such that $F(a, h_1)\neq F(a, h_2)$ when $h_1\neq h_2$ provided $|a|<R$, $|h_1|<\delta$, $|h_2|<\delta$.*

A_3^4. *To every bounded set K whose closure is in $E_n^+(\alpha_j>0, j=1, \cdots, n)$ there is a positive $\delta=\delta(K)$ such that for $c\in K$, $|h|<\delta$, the equation $F(h, b)=c$ has a unique solution $b=\psi(c, h)$ in E_n^+ which is a continuous function of (c, h) such that for fixed values of c measurable sets correspond to measurable sets.*

Some comments are in order at this juncture. We are concerned with a full semi-group and not with a semi-group germ. Assumption A^1 is then a natural generalization from one to n dimensions, but it should be realized that *a parameter set is admissible if and only if it is closed under the product operation.* Thus A^1 implies a restriction on the product law and for other choices of $F(a, b)$ we may have to consider other configurations in E_n besides $\overline{E}_n^+$. The complex euclidean space should also be considered. Actually some portions of the theory extend without material change to the case in which the parameter set is the closure of an open positive cone in an arbitrary complex (B)-space $\mathfrak{P}$.

The boundedness assumption in A^2 is very convenient for a first study of the problem. It is a relict, however, of the days when unitary operators in a Hilbert space stood in the foreground and it has the disadvantage of obscuring the fact that the norm of a semi-group operator, while bounded on bounded sets having a positive distance from the boundary of the parameter set, may very well become unbounded when the parameter approaches the boundary. Such questions will have to be relegated to a later study, however.

Assumption A_1^3 will be discussed in the next section. Condition (2.3) expresses that $T(a)I=IT(a)=T(a)$ and (2.4) implies the *associative law* $T(a)[T(b)T(c)]=[T(a)T(b)]T(c)$. Assumption A_1^4 alone does not take us anywhere and has to be supplemented by other conditions; A_2^4 and A_3^4 suffice for questions of continuity but for the existence of one-parameter sub-semi-groups we shall need Lipschitz conditions and so on.

3. Continuity. In the case of a one-parameter semi-group boundedness of the norm together with strong measurability of the operator function $T(\alpha)$ implies strong continuity of $T(\alpha)$ for $\alpha>0$.[1] It does not imply strong continuity at the origin or continuity in the uniform operator topology for $\alpha>0$. This result extends to the present situation.

THEOREM 3.1. *Under the assumptions of* §2, *omitting* A_2^4, *the operator* $T(a)$ *is strongly continuous in* E_n^+, *that is, for* $\alpha_j>0$, $j=1, \cdots, n$.

The proof uses the same principles as in the one-dimensional case, that is, the product law plus continuity of a definite integral with respect to translations. Let D be a bounded domain the closure of which is in E_n^+. By virtue of A_3^4 there exists a positive $\eta=\eta(D)$ such that for each c in D and each $a=(\alpha_1, \cdots, \alpha_n)$ with $0\leqq\alpha_j\leqq\eta$, $j=1, \cdots, n$, the equation

$$c = F(a, b) \tag{3.1}$$

has a unique solution

$$b = \psi(c, a) \tag{3.2}$$

which is a continuous function of (c, a) in the product set in question. In particular, $|\psi(c_1, a)-\psi(c_2, a)|$ is small when $|c_1-c_2|$ is small and this holds uniformly in a.

For c_1 and c_2 in D we have then

$$T(c_1)x - T(c_2)x = T(a)[T(\psi(c_1, a))x - T(\psi(c_2, a))x].$$

The right side being independent of a, we may integrate the identity with respect to a over the cube $C(\eta)$, $\frac{1}{2}\eta\leqq\alpha_j\leqq\eta$, $j=1, \cdots, n$, obtaining

$$(\tfrac{1}{2}\eta)^n[T(c_1)x - T(c_2)x] = \int_{C(\eta)} T(a)[T(\psi(c_1, a))x - T(\psi(c_2, a))x]da$$

whence

$$(\tfrac{1}{2}\eta)^n\|T(c_1)x - T(c_2)x\| \leqq \int_{C(\eta)} \|T(\psi(c_1, a))x - T(\psi(c_2, a))x\|da.$$

---

[1] R. S. Phillips has recently shown that the boundedness assumption is superfluous and strong measurability alone is necessary and sufficient for strong continuity. [Added in proof March, 1950.]

Since $T(b)$ is a strongly measurable function of b and in the correspondence $b=\psi(c, a)$ measurable b-sets are the images of measurable a-sets, the operator $T(\psi(c, a))$ is strongly measurable in a, so that the integrand is a bounded Lebesgue measurable function of a. From the fact that $\psi(c_2, a)\to\psi(c_1, a)$ when $c_2\to c_1$, the convergence being uniform with respect to a in $C(\eta)$, one may infer that the integral tends to zero when $c_2\to c_1$. It follows that $T(c)$ is strongly continuous at $c=c_1$ and hence everywhere in E_n^+.

Ordinarily we cannot prove continuity in the uniform topology, but if we assume uniform measurability at the outset, then x may be suppressed everywhere in the proof and uniform continuity results.

Assumption A^2 is evidently used twice in the proof, but in both places it could be replaced by the weaker assumption that $\|T(a)\|$ is bounded in every bounded domain whose closure lies in E_n^+. In the case of a separable space $\mathfrak{X}$, weak measurability implies the strong kind so that "strongly" could be replaced by "weakly" in A_1^3. In this case it is also likely that we may dispense with the boundedness condition on the norm altogether, merely assuming $\|T(a)\|$ to be finite in E_n^+. This is suggested by the following considerations.

If $\mathfrak{X}$ is separable and $T(a)$ is weakly measurable in E_n^+, then $\log\|T(a)\|$ is measurable Lebesgue and different from $+\infty$. Further it satisfies the inequality

$$f(c) \leqq f(a) + f(b), \qquad c = F(a, b), \tag{3.3}$$

which is the proper generalization of the subadditive inequality

$$f(a + b) \leqq f(a) + f(b) \tag{3.4}$$

corresponding to the case $F(a, b)=a+b$. For the latter it is known that a solution, defined in E_n^+, which is measurable and different from $+\infty$, is bounded above in every bounded interior domain D whose closure lies in E_n^+ (see [6, p. 135] for the case $n=1$, the extension to arbitrary n has been given by R. A. Rosenbaum). In principle the method of the proof also extends to (3.3) but the analytical and topological difficulties are considerable so the discussion of this question has to be postponed to another occasion.

One can get fairly trivial examples indicating that $T(a)$ need not be continuous on the boundary of E_n^+ under the assumptions of Theorem 3.1, not even in the weak sense.[2] In particular, continuity is apt to fail at the origin. A common cause of such failure is that the union of the range spaces, $\bigcup_a T(a)[\mathfrak{X}]$, is not dense in $\mathfrak{X}$. In the one-parameter case it would be sufficient to add the assumption that $\bigcup_\alpha T(\alpha)[\mathfrak{X}]$ is dense in $\mathfrak{X}$ in order to obtain right-hand continuity at $\alpha=0$. It is not clear at the time of writing this if such a condition would suffice as additional assumption to ensure continuity of $T(a)$ on the boundary of E_n^+. On the other hand, if $n=1$ the assumption that $T(\alpha)x\to x$ for each x when $\alpha\to 0$ suffices for strong continuity in $\overline{E}_1^+$ and this result extends to the general case. We assume

---

[2] In the one-parameter case it may be shown that weak continuity to the right at the origin implies strong continuity. [Added in proof.]

A$_2^3$. *$T(a)$ is strongly continuous at $a=0$.*

THEOREM 3.2. *If $T(a)$ satisfies* A$_2^3$ *and the conditions of §2, except* A$_1^3$ *and* A$_3^4$, *then $T(a)$ is strongly continuous in $\overline{E}_n^+$.*

If x and $\epsilon>0$ are given, we can find an η so small that $\|T(h)x-x\|<\epsilon$ for $h\in S(0)=E(|h|<\eta,\ h\in\overline{E}_n^+)$. The mapping

$$T_a:\quad h\to F(a,h),\qquad a\in\overline{E}_n^+,$$

takes $S(0)$ into a set $S(a)$. If $|a_0|<R$ and $\eta<\delta(R)$, as we may assume, condition A$_2^4$ asserts that the mapping T_{a_0} is one-to-one so that $S(a_0)$ is an n-cell and Int $[S(a_0)]\neq\emptyset$. Thus $S(a_0)$ contains a sphere of center c_0 and radius ρ, say. But $F(a,h)$ is continuous by A$_1^4$ so there exists a δ_0 such that $|F(a,h)-F(a_0,h_0)|<\rho$ if $|a-a_0|<\delta_0$, $|h-h_0|<\delta_0$ where $h, h_0\in S(0)$. Here we choose h_0 so that $c_0=F(a_0,h_0)$. We can then be sure that $S(a)\cap S(a_0)\neq\emptyset$ if $|a-a_0|<\delta_0$. Hence if $c=F(a,h_1)=F(a_0,h_2)$ is a common point of $S(a)$ and $S(a_0)$ we have

$$\begin{aligned}\|T(a)x-T(a_0)x\| &\leqq \|T(c)x-T(a)x\|+\|T(c)x-T(a_0)x\|\\ &=\|T(a)[T(h_1)x-x]\|+\|T(a_0)[T(h_2)x-x]\|\\ &\leqq \|T(h_1)x-x\|+\|T(h_2)x-x\|\leqq 2\epsilon\end{aligned}$$

where $|a-a_0|<\delta_0$. This proves that $T(a)x$ is continuous at $a=a_0$. Hence $T(a)$ is strongly continuous in $\overline{E}_n^+$.

4. **A functional equation.** In the study of one-parameter sub-semigroups we encounter the functional equation

$$g(\rho+\sigma)=F[g(\rho),g(\sigma)] \tag{4.1}$$

where $g(\rho)$ is a function on positive numbers to $\overline{E}_n^+$. For this problem we need further information concerning $F(a,b)$.

A$_4^4$. *There exists a fixed positive constant B such that for all points in $\overline{E}_n^+$ we have*

$$\begin{aligned}|F(a_1,b)-F(a_2,b)| &\leqq [1+B|b|]\,|a_1-a_2|,\\ |F(a,b_1)-F(a,b_2)| &\leqq [1+B|a|]\,|b_1-b_2|.\end{aligned} \tag{4.2}$$

A$^4_5$. *There exists a positive monotone increasing continuous function* $\omega(\xi)$, $0<\xi<\infty$, *which tends to zero with* ξ, *such that*

$$
(4.3)\qquad \begin{gathered} F(a, b) = a + b + G(a, b), \\ |G(a, b)| \leq r\omega[|a + b|], \qquad r = \min(|a|, |b|). \end{gathered}
$$

It would be possible to combine these inequalities in such a manner that they refer to the behavior of $|F(a, b_1) - F(a, b_2) - b_1 + b_2|$ and $|F(a_1, b) - F(a_2, b) - a_1 + a_2|$. Such inequalities are basic in the study of analytical group germs in a (B)-space due to G. Birkhoff [1]. Our conditions seem to be slightly better adapted to the needs of the methods used below. They are also closely related to the assumptions of P. A. Smith [8, 9]. See further I. E. Segal [7] whose work suggests that the Lipschitz condition might be inessential but it is not clear to me at the moment how his methods could be brought to bear on the present problem. In making comparisons the reader should keep in mind that we are dealing with a fixed coordinate system, a semi-group rather than a group, and a situation in the large. All the results of the present section hold for the case in which E_n is replaced by an arbitrary (B)-space and $\bar{E}_n^+$ by the closure of an open positive cone.

With an arbitrary element $b \in \bar{E}_n^+$ we also consider its successive "powers" defined by

$$
(4.4)\qquad b^{(1)} = b, \quad b^{(m)} = F(b, b^{(m-1)}).
$$

When b and m are bulky expressions we shall write $(b; m)$ for $b^{(m)}$. A basic property of the powers is given in the following

LEMMA. *Let* τ *be defined by* $\omega(\tau) = \frac{1}{2}$. *If* $b \in \bar{E}_n^+$ *and* m *is a positive integer such that* $m|b| \leq \frac{1}{2}\tau$, *then*

$$
(4.5)\qquad b^{(m)} = mb + R_m(b), \qquad |R_m(b)| \leq m|b|\,\omega(2m|b|).
$$

The lemma holds for $m = 2$ by A$^4_5$ and is proved by induction using the inequality

$$
|R_{m+1}(b)| \leq |R_m(b)| + |b|\,\omega[(k + 1)|b| + |R_m(b)|].
$$

We can now state and prove

THEOREM 4.1. *Under the assumptions* A$^4_1$, A$^4_4$, *and* A$^4_5$ *the equation* (4.1) *with the initial condition*

$$
(4.6)\qquad \lim_{\rho\to 0} \rho^{-1} g(\rho) = a \in \bar{E}_n^+
$$

has a unique solution $g(\rho) = f(\rho a)$ *in* $\bar{E}_n^+$. *The solution is an absolutely* *continuous function of* ρ *with bounded derived numbers and for* $0 \leqq \rho \leqq \sigma$

$$|g(\sigma) - g(\rho)| \leqq B^{-1}[e^{B\sigma|a|} - e^{B\rho|a|}]. \tag{4.7}$$

Starting with an arbitrary element b of $\overline{E}_n^+$ we form a sequence of powers $(b/\nu)^{(\mu)} = (b/\nu;\ \mu)$ where μ and ν are positive integers. Using A_4^4 and induction on μ one shows that

$$|(b/\nu; \mu + 1) - (b/\nu; \mu)| \leqq \frac{|b|}{\nu}\left\{1 + \frac{B|b|}{\nu}\right\}^{\mu},$$

whence we obtain

$$|(b/\nu; \mu)| \leqq B^{-1}[e^{(\mu/\nu)B|b|} - 1] \tag{4.8}$$

for all μ, ν.

In the following $\mu = \nu$ will be a power of 2 and we shall investigate the convergence of the sequence $\{(2^{-j}b;\ 2^j)\}$. Here $(2^{-j}b;\ 2^j)$ is obviously the square of $(2^{-j}b;\ 2^{j-1})$. Using this fact, (4.8), and A_4^4 repeatedly, we obtain for $j < k$

$$\begin{aligned} \Delta_{j,k}(b) &= |(2^{-k}b;\ 2^k) - (2^{-j}b;\ 2^j)| \\ &\leqq 2^j e^{B|b|}\,|(2^{-k}b;\ 2^{k-j}) - 2^{-j}b|. \end{aligned}$$

Let R be a fixed positive number, arbitrarily large, and restrict b to the sphere $|b| \leqq R$. Suppose that j is so large that $2^{1-j}R \leqq \tau$. We can then apply the lemma with b replaced by $2^{-k}b$ and m by 2^{k-j}. After some simplification we obtain

$$\Delta_{j,k}(b) \leqq |b|\, e^{B|b|}\omega(2^{1-j}|b|). \tag{4.9}$$

It follows that the sequence $\{(2^{-j}b;\ 2^j)\}$ converges to a limit and we set

$$f(b) = \lim_{j \to \infty} (2^{-j}b;\ 2^j). \tag{4.10}$$

The convergence being uniform with respect to b for $|b| \leqq R$, we conclude that $f(b)$ is a continuous function of b.

In particular the limit exists for $b = \rho a$, uniformly with respect to ρ for $0 \leqq \rho \leqq R < \infty$ so that $f(\rho a)$ is a continuous function of ρ.

From (4.10) we conclude also that

$$f(b; m) \equiv \lim_{j \to \infty} (2^{-j}b; m2^j) \tag{4.11}$$

exists and equals the mth power of $f(b)$. But this implies that

$$\lim_{j \to \infty} (2^{-j}m^{-1}b; m2^j) = f(bm^{-1}; m)$$

exists. We want to show that the latter limit is actually independent of m and consequently equals $f(b;\ 1)=f(b)$. For this purpose we consider

$$\Delta_{k,m}(b, j) = \left| (2^{-j}k^{-1}b;\ k2^j) - (2^{-j}m^{-1}b;\ m2^j) \right|.$$

Here we use the fact that the terms on the right are squares together with A_4^4 and (4.8) to obtain the estimate

$$\begin{aligned}\Delta_{k,m}(b, j) &\leqq 2e^{\frac{1}{2}B|b|}\Delta_{k,m}(2^{-1}b, j-1) \leqq \cdots \\ &\leqq 2^j e^{B|b|}\Delta_{k,m}(2^{-j}b, 0).\end{aligned}$$

If $2^{1-j}|b| \leqq \tau$, we can apply the lemma once more and see that

$$2^j\Delta_{k,m}(2^{-j}b, 0) < 2\omega(2^{1-j}|b|) \to 0$$

when $j \to \infty$. It follows that

$$f(bk^{-1};\ k) = f(bm^{-1};\ m) = f(b;\ 1) = f(b)$$

as asserted. Hence with $b=ka$ we have $f(a;\ k)=f(ka)$ for every positive integer k.

From this it follows that $f(\rho a)$ satisfies (4.1), to start with for positive integral values of ρ and σ which implies that it also holds for rational values and finally, $f(\rho a)$ being continuous, for all positive real values.

The argument used for the convergence proof also gives the inequality

$$\begin{aligned}\left| (2^{-k}b;\ 2^k) - (2^{-k}c;\ 2^k) \right| &\leqq 2^k e^{B|c|} \left| 2^{-k}b - 2^{-k}c \right| \\ &= e^{B|c|} \left| b - c \right|\end{aligned}$$

if $|b| \leqq |c|$. Passing to the limit with k we obtain the Lipschitz condition

$$\left| f(b) - f(c) \right| \leqq e^{B|c|} \left| b - c \right|, \qquad |b| \leqq |c|. \tag{4.12}$$

It follows that $f(\rho a)$ satisfies a Lipschitz condition with respect to ρ. Formula (4.7) is an immediate consequence of (2.3), (4.1), A_4^4, and the estimate

$$\left| f(b) \right| \leqq B^{-1}[e^{B|b|} - 1] \tag{4.13}$$

which follows from (4.8).

In order to verify the initial condition we revert to the lemma once more. We have

$$(2^{-j}\rho a;\ 2^j) = \rho a + R_{2^j}(2^{-j}\rho a)$$

and if $2\rho|a| \leqq \tau$ the norm of the remainder does not exceed $\rho|a|\omega(2\rho|a|)$. When j tends to infinity, the left member tends to $f(\rho a)$. It follows that

$$|f(\rho a) - \rho a| \leqq \rho|a|\omega(2\rho|a|) \tag{4.14}$$

for $2\rho|a| \leqq \tau$ so the initial condition is satisfied.

For the uniqueness proof we shall need only A_4^4. Suppose that $g(\rho)$ and $h(\rho)$ are two solutions of (4.1) and (4.6) in $\overline{E}_n^+$. We may assume that they both satisfy (4.13) with b replaced by ρa; if necessary we restrict ρ to a fixed finite interval and replace B by a larger constant. We have then by the usual square root argument

$$\begin{aligned} |g(\rho) - h(\rho)| &= |F[g(\tfrac{1}{2}\rho), g(\tfrac{1}{2}\rho)] - F[h(\tfrac{1}{2}\rho), h(\tfrac{1}{2}\rho)]| \\ &\leqq 2e^{\frac{1}{2}B\rho|a|}|g(\tfrac{1}{2}\rho) - h(\tfrac{1}{2}\rho)|, \end{aligned}$$

whence by iteration

$$|g(\rho) - h(\rho)| \leqq 2^k e^{B\rho|a|}|g(2^{-k}\rho) - h(2^{-k}\rho)|$$

and this tends to zero when $k \to \infty$ by virtue of the common initial condition. Hence $g(\rho) \equiv h(\rho)$ and our sketch of the proof of Theorem 4.1 is complete.

This theorem may be strengthened in a direction which will be useful below.

THEOREM 4.2. *If $\omega(\xi)$ satisfies the integrability condition*

$$\int_0^1 \xi^{-1}\omega(\xi)d\xi < \infty, \tag{4.15}$$

and if $g(\rho)$ is a solution of (4.1) *in $\overline{E}_n^+$ which tends to zero with ρ, then there exists an a in $\overline{E}_n^+$ such that* (4.6) *holds.*

Suppose that ρ_0 is so small that $2|g(\rho)| \leqq \tau$ for $\rho \leqq \rho_0$. Repeated use of A_5^4 gives

$$\begin{gathered} g(\rho) = 2^k g(2^{-k}\rho) + R_k(\rho), \\ |R_k(\rho)| \leqq \sum_{\nu=1}^{k} 2^{\nu-1}|g(2^{-\nu}\rho)|\omega[2|g(2^{-\nu}\rho)|]. \end{gathered}$$

On the other hand

$$\begin{aligned} |g(\rho)| &\geqq \{1 - \tfrac{1}{2}\omega[2|g(2^{-1}\rho)|]\}2|g(2^{-1}\rho)| \geqq \cdots \\ &\geqq \prod_{j=1}^{\nu}\{1 - \tfrac{1}{2}\omega[2|g(2^{-j}\rho)|]\}2^{\nu}|g(2^{-\nu}\rho)|. \end{aligned} \tag{4.16}$$

It follows that

$$|R_k(\rho)| \leqq \tfrac{1}{2}|g(\rho)| \sum_{\nu=1}^{k} \prod_{j=1}^{\nu} \{1 - \tfrac{1}{2}\omega[2|g(2^{-j}\rho)|]\}^{-1}\omega[2|g(2^{-\nu}\rho)|].$$

Here the right-hand member tends to a finite limit when $k \to \infty$ if and only if

$$\sum_{\nu=1}^{\infty} \omega[2|g(2^{-\nu}\rho)|] < \sum_{\nu=1}^{\infty} \omega[(\tfrac{2}{3})^{\nu}\tau]$$

is convergent where the inequality is a trivial consequence of (4.16). But the second series converges by virtue of (4.15). It follows that $R_k(\rho)$ converges uniformly to a continuous limit for $0 \leqq \rho \leqq \rho_0$ when $k \to \infty$ and this implies that

$$\lim_{k\to\infty} 2^k g(2^{-k}\rho) \equiv l(\rho) \tag{4.17}$$

exists as a continuous function of ρ. Furthermore, one sees that if $l(\rho) = \rho m(\rho)$ then $m(2\rho) = m(\rho)$.

In exactly the same manner one proves that

$$\lim_{k\to\infty} 3^k g(3^{-k}\rho)$$

exists and a more detailed analysis shows that it also equals $l(\rho)$. This forces $m(\rho)$ to have the additional property $m(3\rho) = m(\rho)$, and log 2 and log 3 being incommensurable, this makes the continuous function $m(\rho)$ equal to a constant, a say. Since $g(\rho) \in \overline{E}_n^+$ by assumption, a will have the same property. From $l(\rho) = \rho a$, we conclude that (4.6) holds.

Condition (4.15) is not particularly restrictive and it is fully utilized in the proof of the theorem, but we have of course no evidence whatsoever that it is a necessary condition.

5. **The one-parameter sub-semi-groups.** In view of the discussion in §3, it is natural to replace A_1^3 by

A_3^3. *$T(a)$ is a strongly continuous function of a in $\overline{E}_n^+$.*

The basic theorem on sub-semi-groups reads

THEOREM 5.1. *Let the semi-group $\mathfrak{S} = \{T(a)\}$ satisfy assumptions A^1, A^2, A_3^3, A_1^4, A_4^4, and A_5^4. If a is any element of $\overline{E}_n^+$, then the sequence of operators $\{[T(2^{-j}a)]^{2^j}\}$ converges strongly to a limit $S(a) = T[f(a)]$ where $f(a)$ is defined by* (4.10). *The operators $\{S(\rho a)\}$, $0 < \rho < \infty$, form a sub-semi-group $\mathfrak{S}_a$ of $\mathfrak{S}$ and $\mathfrak{S}_{\alpha a} = \mathfrak{S}_a$ if $\alpha > 0$. If* (4.15) *also holds, then conversely if $\mathfrak{T} = \{S(\rho)\}$, $0 < \rho < \infty$, is a sub-semi-group of $\mathfrak{S}$, if*

$S(\rho) = T[g(\rho)]$ and $\lim_{\rho\to 0} g(\rho) = 0$, then there exists an $a \in E_n^+$ such that $\mathfrak{T} = \mathfrak{S}_a$.

The proof follows directly from Theorem 4.1 and 4.2. Referring back to A_1^4 and the definition of the power $(2^{-j}a;\, 2^j)$ one sees that

$$(5.1) \qquad [T(2^{-j}a)]^{2^j} = T[(2^{-j}a;\, 2^j)].$$

By (4.10) we know that $(2^{-j}a;\, 2^j) \to f(a)$ when $j \to \infty$ and by A_3^3

$$(5.2) \qquad T[(2^{-j}a;\, 2^j)]x \to T[f(a)]x \equiv S(a)x$$

for each x. That the operators $S(\rho a)$, $0 < \rho < \infty$, form a semi-group with the product law $S(\rho a)S(\sigma a) = S((\rho+\sigma)a)$ follows from the fact that $f(\rho a)$ satisfies equation (4.1).

If (4.15) holds, then we can apply Theorem 4.2. If then $\mathfrak{T} = \{S(\rho)\}$ is a one-parameter sub-semi-group of $\mathfrak{S}$ with the canonical product law, we can find a function $g(\rho)$ satisfying (4.1) such that $S(\rho) = T[g(\rho)]$. By assumption $g(\rho) \to 0$ with ρ so there is an a in E_n^+ such that $g(\rho) = f(\rho a)$ and, hence, $\mathfrak{T} = \mathfrak{S}_a$.

The equation

$$(5.3) \qquad p = f(\rho a), \qquad 0 < \rho < \infty$$

defines a path Γ_a in E_n^+, starting at the origin where it is tangent to the vector $p = \rho a$. The properties of $f(\rho a)$ listed in Theorem 4.1 show that Γ_a has a tangent almost everywhere and every arc of Γ_a corresponding to $0 < \rho < R$ is rectifiable. If $F(p,\, q)$ has continuous partial derivatives with respect to the components of q, we may show that $f(\rho a) = [\phi_1(\rho), \cdots, \phi_n(\rho)]$ is a solution of the system of first order differential equations

$$(5.4) \qquad \begin{cases} \dfrac{d\phi_j}{d\rho} = \displaystyle\sum_{k=1}^{n} \alpha_k F_{j,k}(\phi_1, \cdots, \phi_n;\, 0, \cdots, 0), \\ \phi_j(0) = 0. \end{cases} \qquad j = 1, \cdots, n,$$

Here $F_{j,k}$ is the derivative of the jth component of $F(p,\, q)$ with respect to the kth component of q and $a = (\alpha_1, \cdots, \alpha_n)$. This system, when available, usually offers more convenient determination of Γ_a than the functional equation (4.1) with the initial condition (4.6). It should be observed, however, that the latter define Γ_a uniquely in situations where the uniqueness theorems for differential equations do not apply and even the differential equations themselves may fail to exist.

Every path Γ_a is confined to E_n^+ and condition (4.15) may be used to show that Γ_a cannot return to the origin when ρ tends to a finite limit or to infinity. Under the same assumption, Γ_a is a simple arc. The general question of what happens to $f(\rho a)$ when $\rho \to \infty$ is very important. A particularly simple case is that in which $f(\rho a)$ tends to a finite limit p_0 for then $p_0 = F(p_0, p_0)$ and $T(p_0)$ is a *projection operator*.

A transformation semi-group, in contrast to a group, may contain projections and their parameters are determined by the equation

$$(5.5) \qquad F(p, p) = p$$

which defines a locus P in E_n^+. P always contains $p = 0$ and may reduce to this point. The origin is an isolated point of P since $|F(p, p) - p| > 0$ as long as $p \neq 0$ and $\omega(2|p|) < 1$ as we see from A_5^4. If a path Γ_a has an interior point $p_0 = f(\omega a)$ in common with P, then the operator $T[f(\rho a)]$ is periodic with period ω for $\rho > \omega$. This cannot happen, however, if condition (4.15) holds and if Int $\Gamma_a \epsilon E_n$. A path Γ_a may very well have its end point on P and P may be made up of such terminal points. This happens in the case of the projective semi-group on positive numbers for which all paths Γ_a are straight line segments joining the origin with the surface P which is a portion of a hyperbolic paraboloid in E_3.

The determination of all points p such that $p = f(a)$ for some a appears to be very difficult. It is clear that if $p = f(a)$, then we must be able to determine a sequence of points $\{p_k\}$ in E_n^+ such that $p_0 = p$ and

$$(5.6) \qquad F(p_k, p_k) = p_{k-1}, \qquad k = 1, 2, 3, \cdots .$$

Since $p_k = f(2^{-k}a)$ we must also have

$$(5.7) \qquad \lim_{k \to \infty} 2^k p_k = a.$$

It is possible to determine conditions under which this process may be carried through, but so far the results have been rather disappointing. In our theory it is much easier to determine the paths from the origin than to find the path, if any, which joins a given point p with the origin.

Let us observe that if $p = f(a)$ then

$$(5.8) \qquad T(p)x = \lim_{\delta \to 0} \exp \left\{\frac{1}{\delta} [T(f(\delta a)) - I]\right\} x, \qquad x \in \mathfrak{X},$$

in the sense of strong convergence. Cf. [6, p. 189].

6. **The infinitesimal generators.** With each one-parameter sub-semi-group $\mathfrak{S}_a$ defined above there is associated an *infinitesimal gen-*

erator $A(a)$ of $\mathfrak{S}$ which is defined by

$$\lim_{\delta\to 0}\frac{1}{\delta}[T(f(\delta a))-I]x=A(a)x \tag{6.1}$$

whenever the limit exists. The domain of $A(a)$ will be denoted by $\mathfrak{D}(a)$; it is clearly a linear subspace of $\mathfrak{X}$.

THEOREM 6.1. *Under the assumptions of Theorem* 5.1 *the set* $\mathfrak{D}(a)$ *is dense in* $\mathfrak{X}$ *for each a in* $\overline{E}_*^+$. *In particular,* $\mathfrak{D}(a)$ *contains all elements of the form*

$$\int_\alpha^\beta T[f(\rho a)]y\,d\rho,\qquad y\in\mathfrak{X},\ 0\leqq\alpha<\beta<\infty. \tag{6.2}$$

This is a well known result in the theory of one-parameter semigroups (N. Dunford [**2**], cf. [**6**, p. 185]). We also observe that for x in $\mathfrak{D}(a)$ we have

$$\frac{d}{d\rho}T[f(\rho a)]x=T[f(\rho a)]A(a)x=A(a)T[f(\rho a)]x. \tag{6.3}$$

The operator $A(a)$ which is closed is ordinarily unbounded on $\mathfrak{D}(a)$. Its resolvent is given by the Laplace transform

$$R[\lambda;A(a)]x=\int_0^\infty e^{-\lambda\rho}T[f(\rho a)]x\,d\rho,\qquad \Re(\lambda)>0. \tag{6.4}$$

The spectrum of $A(a)$ may very well fill the complementary half-plane $\Re(\lambda)\leqq 0$. $\mathfrak{D}(a)$ is the range of $R[\lambda; A(a)]$ for any fixed λ with $\Re(\lambda)>0$. If $R[\lambda; A(a)]$ is known, it determines $T[f(\rho a)]$ uniquely with the aid of the inversion formulas for the Laplace transform.

The mapping $a\to A(a)$ defines a correspondence between the vectors a of $\overline{E}_*^+$ and the infinitesimal generators $A(a)$ of $\mathfrak{S}$. Under suitable assumptions this correspondence is actually an isomorphism under the operations of addition and multiplication by positive numbers. Our previous postulates suffice for the scalar multiplication, however, since (6.1) shows that we have

THEOREM 6.2. *Under the assumptions of Theorem* 5.1 *we have* $\mathfrak{D}(\alpha a)=\mathfrak{D}(a)$ *for* $\alpha>0$ *and* $A(\alpha a)=\alpha A(a)$.

For the addition and for the fundamental theorems we need further restrictions and the following are convenient assumptions.

A_4^3. *There exists a positive* ρ_0 *such that* $|a|<\rho_0$, $|b|<\rho_0$, $a\neq b$ *implies that* $T(a)\neq T(b)$.

A_6^4. *$F(a, b)$ has continuous partial derivatives with respect to the components of a and b up to and including the third order.*

At every point of E_n^+ the function $F(a, b)$ may then be expanded in a Taylor series up to terms of the third order. Because of A_1^4 these expansions have a special form: at the origin we have in particular

$$\gamma_i = \alpha_i + \beta_i + \sum_j \sum_k \alpha_{jk}^i \alpha_j \beta_k + R_{3,i}, \qquad i = 1, 2, \cdots, n, \tag{6.5}$$

$$a = (\alpha_i), \quad b = (\beta_i), \quad c = (\gamma_i), \quad c = F(a, b).$$

The crux of the problem before us is to construct elements of $\mathfrak{X}$ dense in $\mathfrak{X}$ belonging to the domains of definitions of $A(a)$ and of $A(a)A(b)$ for every choice of a and b in E_n^+. For this purpose we shall use a modification of artifices due to N. Dunford [2], I. Gelfand [5], and L. Gårding [4]. The same device gives elements belonging to the domain of existence of the product of three or more infinitesimal generators, provided we assume the existence of enough derivatives.

Let $K(c)$ be a numerically-valued function of class $C^{(m)}$, $m \geqq 3$, defined in E_n^+. Let D be a bounded domain whose closure lies in E_n^+ and let $K(c)$ be integrable over D and vanish outside of D. If D_η is a homeomorphic image of D in E_n^+ such that no two corresponding points are at a distance of more than η apart, then we suppose that

$$\int |K(c)|\, dc = o(\eta) \qquad \text{when } \eta \to 0, \tag{6.6}$$

where the integration is extended over that part of D which is not in D_η. We require that the partials of order not greater than m have the same integrability properties. The class of all such kernels $K(c)$ will be denoted by $\mathfrak{K}$. Specifically we may choose D as the cube $0<\sigma<\gamma_j<\tau<\infty, j=1, \cdots, n$, and define $K(c)$ in the cube as

$$\begin{aligned} K(c) &= K(c; \sigma, \tau) \\ &= [C(\tau-\sigma)]^{-n} \exp\left\{-(\tau-\sigma)^2 \sum_{j=1}^{n} [(\tau-\gamma_j)(\gamma_j-\sigma)]^{-1}\right\} \end{aligned} \tag{6.7}$$

where $C = \int_0^1 \exp\{-[\gamma(1-\gamma)]^{-1}\}\, d\gamma$.

For any choice of $K(c)$ in $\mathfrak{K}$ we define

$$y = K[x] = \int_D K(c)T(c)x\, dc, \qquad x \in \mathfrak{X}. \tag{6.8}$$

For fixed $K(c)$ this is a linear bounded transformation of $\mathfrak{X}$ to $\mathfrak{X}$ the norm of which does not exceed the integral of $|K(c)|$ over D. If $K(c)=K(c;\sigma,\tau)$ the norm is at most one.

THEOREM 6.3. *Under the assumptions* A^1, A^2, A_3^3, *and* A_1^4 *to* A_6^4 *the set* $[\bigcap_a\mathfrak{D}(a)]\cap[\bigcap_{a,b}\mathfrak{D}[A(a)A(b)]]$ *is dense in* $\mathfrak{X}$ *and contains the set* $\mathfrak{K}[\mathfrak{X}]$ *of elements of the form* (6.8).

That $\mathfrak{K}[\mathfrak{X}]$ is dense in $\mathfrak{X}$ follows from

$$\lim_{\sigma\to 0}\lim_{\tau\to\sigma} K(c;\sigma,\tau)[x]=x \tag{6.9}$$

in the sense of strong convergence for every x.

We proceed to indicate briefly how one shows that $K[x]\in\mathfrak{D}(a)$ where, without restricting the generality, we may take $|a|=1$. We form

$$\begin{aligned}&\frac{1}{\delta}\{T[f(\delta a)]-I\}y\\ &\qquad=\frac{1}{\delta}\int_D K(c)\{T[F(f(\delta a),c)]-T(c)\}x\,dc.\end{aligned} \tag{6.10}$$

In order to find the limit of the right member, we have to study the mapping U_δ defined by $b=F[f(\delta a),c]$ which takes D into a set $D(\delta)$. Condition A_3^4 asserts that the correspondence is one-to-one for sufficiently small values of δ since $|f(\delta a)|\leqq B^{-1}[e^{B\delta}-1]<2\delta$ if $B\delta<1$. Further

$$|b-c|<(1+B|c|)|f(\delta a)|<M\delta$$

where M depends only upon B and D. The mapping is consequently a homeomorphism involving only a small distortion of D when δ is small. It follows that (6.6) holds with η replaced by δ and D_η by $D(\delta)$.

By A_3^4 we can solve the equation $b=F[f(\delta a),c]$ for c when δ is small obtaining the unique solution $c=\psi(b,f(\delta a))$. This solution is continuous by A_3^4; the added condition A_6^4 also makes it differentiable. It follows from the theorem on implicit functions that the solution $c=\psi(b,s)$ of the equation $F(s,c)=b$ has continuous partial derivatives with respect to the components of b and s of order not greater than 3 which is the limit for the existence of partials of $F(p,q)$ postulated in A_6^4. In this case $f(\delta a)$ has continuous derivatives with respect to δ of order not greater than 3 as is seen from equation (5.4). It follows that $\psi(b,f(\delta a))$ has continuous partials with respect to the components of b and with respect to δ of order not greater than 3. Further, the Jacobian

$$J(c; b) = J[\psi(b, f(\delta a)); b] \tag{6.11}$$

of the inverse transformation U_δ^{-1} from $D(\delta)$ to D is near to one uniformly in D when δ is small and has partial derivatives with respect to the components of b and with respect to δ of order not greater than 2.

From this we conclude that the right member of (6.10) may be written

$$\begin{aligned}
\delta^{-1}\int_{D(\delta)} K(c)T(b)xJ(c; b)db &- \delta^{-1}\int_D K(b)T(b)xdb \\
&= \int_{D_1} \delta^{-1}[K(c)J(c; b) - K(b)]T(b)xdb \\
&\quad + \delta^{-1}\int_{D_2} K(c)J(c; b)T(b)xdb - \delta^{-1}\int_{D_3} K(b)T(b)xdb \\
&= J_1 + J_2 + J_3,
\end{aligned}$$

where $D_1 = D \cap D(\delta)$, $D_2 = D(\delta) - D_1$, $D_3 = D - D_1$. Here the norms of J_2 and J_3 do not exceed $\|x\|$ times

$$\delta^{-1}\int_{D_2} |K(c)|J(c; b)db \quad \text{and} \quad \delta^{-1}\int_{D_3} |K(b)|db$$

respectively. Both of these expressions tend to zero with δ by (6.6). It follows that (6.10) tends to a limit so that $y \in \mathfrak{D}(a)$ and

$$\begin{aligned}
A(a)y &= \int_D K_1(b; a)T(b)xdb, \\
K_1(b; a) &= \frac{\partial}{\partial\delta}\{K[\psi(b, f(\delta a))]J[\psi(b, f(\delta a); b]\}_{\delta=0}.
\end{aligned} \tag{6.12}$$

For D near to the origin this becomes

$$\begin{aligned}
K_1(b; a) = &- K(b)\left[\sum_i \sum_j \alpha^i_{ji}\alpha_j + O(|b|)\right] \\
&- \sum_i K_i(b)\left[\alpha_i + \sum_j \sum_k \alpha^i_{jk}\alpha_j\beta_k + O(|b|^2)\right],
\end{aligned} \tag{6.13}$$

where $K_i(b)$ is the partial of $K(b)$ with respect to β_i. It should be observed that the remainder terms are independent of the kernel. We conclude that $A(a)K$ is a linear bounded transformation on $\mathfrak{X}$ to $\mathfrak{X}$ and

$$\|A(a)K\| \leqq \int_D |K_1(b; a)|\, db.$$

Moreover, it is not difficult to see that this bound is a bounded function of a for $|a| = 1$.

We have now to consider the existence of $A(a_1)A(a_2)y$. Here $A(a_2)y$ is given by (6.12) with a replaced by a_2. This integral is of the same type as (6.8) with a kernel $K_1(c; a_2)$ instead of $K(c)$. Here $K_1(c; a_2)$ has the same properties as $K(c)$ except for differentiability; however, $K_1(c; a_2) \in C^{(2)}$ at least and this more than suffices for our needs. The argument given above may consequently be used also to prove the existence of $A(a_1)A(a_2)y$. Further we see that $A(a_1)A(a_2)K$ is a bounded linear operator and the bound is a bounded function of a_1 and a_2 on the unit sphere in E_n^+. This completes the proof.

We have seen that the set $\mathfrak{K}[\mathfrak{X}]$ is dense in $\mathfrak{D}(a)$. Actually a stronger statement can be made and we can make assertions about the graphs of the operators in the relevant product spaces.

THEOREM 6.4. *The graph* $[y, A(a)y]$, $y \in \mathfrak{K}[\mathfrak{X}]$, *is dense in the graph* $[x, A(a)x]$, $x \in \mathfrak{D}(a)$, *in* $\mathfrak{X} \times \mathfrak{X}$. *More generally, the graph* $[y, A(a_1)y, \cdots, A(a_k)y]$, $y \in \mathfrak{K}[\mathfrak{X}]$, *is dense in the graph* $[x, A(a_1)x, \cdots, A(a_k)x]$, $x \in \bigcap_1^k \mathfrak{D}(a_j)$, *in* $\mathfrak{X} \times \cdots \times \mathfrak{X}$ (*$k+1$ factors*).

The proof is long and laborious so we shall merely sketch the argument for $k = 1$ and indicate briefly the extension to more dimensions. Since $A(a)$ is a closed linear operator the graph $\mathfrak{G}_1 = [x, A(a)x]$, $x \in \mathfrak{D}(a)$, with points g_1 may be made into a (B)-space under the norm $\|g_1\| = \|x\| + \|A(a)x\|$ with obvious definition of the algebraic operations. If the subset $\mathfrak{G}_{10} = [y, A(a)y]$, $y \in \mathfrak{K}[\mathfrak{X}]$, is non-dense in $\mathfrak{G}_1$, then there exists a linear bounded functional on $\mathfrak{G}_1$ which vanishes on $\mathfrak{G}_{10}$ without vanishing identically. Any bounded linear functional on $\mathfrak{G}_1$ is of the form

$$g_1^*(g_1) = x_1^*(x) + x_2^*[A(a)x]$$

where x_1^* and x_2^* are arbitrary linear bounded functionals on $\mathfrak{X}$. We have then for some special choice of x_1^* and x_2^* that

$$x_1^*(y) + x_2^*[A(a)y] = 0, \qquad y \in \mathfrak{K}[\mathfrak{X}], \tag{6.14}$$

and by assumption this does not hold for all x in $\mathfrak{D}(a)$ if we replace y by x. We now make a special choice of y in $\mathfrak{K}[\mathfrak{X}]$. We take as domain of integration the cube $C(\delta, \epsilon)$: $\delta < \gamma_i < \epsilon$, $i = 1, 2, \cdots, n$, and set $K(c) = 0$ outside the cube and equal to $\prod_1^n K_0(\gamma_i)$ in the cube where

$$K_0(\gamma) = \exp\left\{-\eta(\epsilon - \delta)[(\epsilon - \gamma)(\gamma - \delta)]^{-1}\right\}, \qquad \eta > 0.$$

The corresponding elements $y = y(x;\ \delta,\ \epsilon,\ \eta)$, $x \in \mathfrak{D}(a)$, are in $\mathfrak{R}[\mathfrak{X}]$. Substituting this value of y in (6.14) gives an identity in δ, ϵ, η, and x. By continuity we may let $\delta \to 0$ and afterwards $\eta \to 0$. Under these two operations $x_1^*(y)$ is carried into

$$\int_{C(0,\epsilon)} x_1^*[T(c)x]dc \tag{6.15}$$

since $K_0(\gamma) \to 1$ boundedly in $C(0, \epsilon)$. For the discussion of $x_2^*[A(a)y]$ we use (6.13) which gives the corresponding kernel. This expression involves two terms of which the first one leads to

$$\int_{C(0,\epsilon)} \left\{ \sum_i \sum_j \alpha_{ji}^i \alpha_j + O(|c|) \right\} x_2^*[T(c)x]dc. \tag{6.16}$$

We recall that the remainder term is independent of the kernel. The second term of (6.13) gives rise to n singular integrals of which the first one, $i = 1$, involves the integral

$$\eta \int_0^\epsilon K_0(\gamma_1)[(\gamma_1 - \epsilon)^{-2} - \gamma_1^{-2}]f_1(c)d\gamma_1$$

which is multiplied by $\prod_2^n K_0(\gamma_j)$ integrated with respect to the remaining $n-1$ variables from 0 to ϵ. Here

$$f_i(c) = \left[\alpha_i + \sum_j \sum_k \alpha_{jk}^i \alpha_j \gamma_k + O(|c|^2)\right] x_2^*[T(c)x].$$

Passing to the limit with η gives the result

$$f_1(\epsilon, \gamma_2, \cdots, \gamma_n) - f_1(0, \gamma_2, \cdots, \gamma_n)$$

as is easily seen. At the same time $\prod_2^n K_0(\gamma_j) \to 1$. Thus we see that the limit of $x_2^*[A(a)y]$ under the limit processes $\delta \to 0$, $\eta \to 0$ becomes the integral over the $(n-1)$-dimensional boundary of the cube $C(0, \epsilon)$ of the function which on the face $\gamma_i = \epsilon$ equals $f_i(c)$ and on the face $\gamma_i = 0$ equals $-f_i(c)$.

After performing these operations on $x_1^*(y) + x_2^*[A(a)y]$ we multiply by ϵ^{-n} and let $\epsilon \to 0$. The contributions from (6.15) and (6.16) add up to

$$x_1^*(x) - \left\{ \sum_i \sum_j \alpha_{ji}^i \alpha_j \right\} x_2^*(x). \tag{6.17}$$

The surface integral leads to two terms of which one arises from the partials of the factors $\alpha_{ij}+\sum_j\sum_k\alpha^i_{jk}\alpha_j\gamma_k+O(|c|^2)$ and gives a limit which cancels the second term of (6.17). The second term involves combinations of difference quotients of $x_2^*[T(c)x]$. Since everything else tends to a limit when $\epsilon\to 0$, this term must also tend to a limit which we denote by $X^*(x)$. We have then

$$x_1^*(x)+X^*(x)=0,\qquad x\in\mathfrak{D}(a). \tag{6.18}$$

Since $\mathfrak{D}(a)$ is dense in $\mathfrak{X}$, this must hold for all x so that $X^*=-x_1^*$. But for $x=y$ in $\mathfrak{R}[\mathfrak{X}]$ we have $X^*(y)=x_2^*[A(a)y]$ and $\mathfrak{R}[\mathfrak{X}]$ is dense in $\mathfrak{D}(a)$ so we must have $X^*(x)=x_2^*[A(a)x]$ for all x in $\mathfrak{D}(a)$. But this shows that (6.14) holds with y replaced by any x of $\mathfrak{D}(a)$ so that $g_1^*(g_1)=0$ for all g_1. Thus $\mathfrak{G}_{10}$ is dense in $\mathfrak{G}_1$.

The extension from 1 to k does not offer any new difficulties. The graph $\mathfrak{G}_k=[x, A(a_1)x, \cdots, A(a_k)x]$, $x\in\bigcap_1^k\mathfrak{D}(a_j)$, becomes a (B)-space under the norm $\|g_k\|=\|x\|+\sum_1^k\|A(a_j)x\|$ and the linear functionals on $\mathfrak{G}_k$ are of the form

$$g_k^*(g_k)=x_1^*(x)+x_2^*[A(a_1)x]+\cdots+x_{k+1}^*[A(a_k)x].$$

Proceeding as above with the same choice of $y=y(x;\,\delta,\,\epsilon,\,\eta)$, $x\in\bigcap_1^k\mathfrak{D}(a_j)$, and passing to the limit with the parameters, one obtains a relation of type (6.18) holding for all x under consideration. But for $x=y\in\mathfrak{R}[\mathfrak{X}]$ we have

$$X^*(y)=x_2^*[A(a_1)y]+\cdots+x_{k+1}^*[A(a_k)y]. \tag{6.19}$$

Here X^* has a unique extension when we pass from the dense set $\mathfrak{R}[\mathfrak{X}]$ to the set $\bigcap_1^k\mathfrak{D}(a_j)$ and the right-hand side of (6.19) has the obvious extension obtained by replacing y by x. It follows that (6.19) holds for all points of $\bigcap_1^k\mathfrak{D}(a_j)$ so that g_k^* is the zero functional and the subgraph is dense in $\mathfrak{G}_k$.

We now define an operator $U(a)$ by

$$U(a)x=\lim_{\delta\to 0}\delta^{-1}[T(\delta a)-I]x \tag{6.20}$$

whenever the limit exists.

THEOREM 6.5. *$U(a)y$ exists and equals $A(a)y$ if $y\in\mathfrak{R}[\mathfrak{X}]$.*

The existence of $U(a)y$ is proved by the argument used in the proof of Theorem 6.3. We have merely to replace $f(\delta a)$ by δa throughout. But $f(\delta a)-\delta a=O(\delta^2)$ whence it follows that

$$\psi(b, f(\delta a))-\psi(b, \delta a)=O(\delta^2),$$
$$J[\psi(b, f(\delta a)); b]-J[\psi(b, \delta a); b]=O(\delta^2),$$

so the right member of (6.13) is unchanged when we replace $f(\delta a)$ by δa. Hence $U(a)y=A(a)y$ for $y\in\mathfrak{R}[\mathfrak{X}]$. If it could be shown that $U(a)$ is a closed operator, then it would follow from Theorem 6.4 that the domain of $U(a)$ contains that of $A(a)$ and that equality between the operators holds in $\mathfrak{D}(a)$. At present we cannot decide this question and it is not vital for the following discussion.

THEOREM 6.6. *If* $y=K(c)[x]$ *and if* a_1, a_2, b *are points of* $\bar{E}_n^+$ *with* $|a_1|<\delta(D)$, $|a_2|<\delta(D)$, $|b|\leqq 1$, *then there exists a constant* C *depending only upon* $K(c)$ *such that*

$$|[T(a_1)-T(a_2)]y|\leqq C|a_1-a_2|\,\|x\| \tag{6.21}$$

and the same relation holds with y *replaced by* $A(b)y$.

This is proved by the method of Theorem 6.3. We omit the details.

We can now prove that the correspondence $a\to A(a)$ is a homomorphism under addition in the following sense.

THEOREM 6.7. $\mathfrak{D}(a_1+a_2)$ *contains* $\mathfrak{D}(a_1)\cap\mathfrak{D}(a_2)$ *and in the latter set*

$$A(a_1+a_2)x=A(a_1)x+A(a_2)x. \tag{6.22}$$

We start by proving (6.22) for $x=y\in\mathfrak{R}[\mathfrak{X}]$. In this set it is sufficient to prove the corresponding relation with A replaced by U and this is accomplished if we can show that

$$[T(\delta(a_1+a_2))-T(\delta a_1)-T(\delta a_2)+I]y=o(\delta)$$

when $\delta\to 0$. The left member equals

$$\{T(\delta(a_1+a_2))-T[F(\delta a_1,\delta a_2)]\}y+[T(\delta a_1)-I][T(\delta a_2)-I]y.$$

By Theorem 6.6 the norm of the first term does not exceed

$$C|\delta(a_1+a_2)-F(\delta a_1,\delta a_2)|\,\|x\|$$

which is $O(\delta^2)$ by A_5^4 since $\omega(\xi)=O(\xi)$. Since $y\in\mathfrak{D}(a_2)$, we have $[T(\delta a_2)-I]y=\delta A(a_2)y+o(\delta)$, so that

$$\begin{aligned}\|[T(\delta a_1)-I][T(\delta a_2)-I]y\| &\leqq \delta\|[T(\delta a_1)-I]A(a_2)y\|\\ &\quad+o(\delta)\|T(\delta a_1)-I\|\leqq C\delta^2|a_1|\,\|x\|+o(\delta)\end{aligned}$$

from which the assertion follows. In order to extend the validity of (6.22) from $\mathfrak{R}[\mathfrak{X}]$ to $\mathfrak{D}(a_1)\cap\mathfrak{D}(a_2)$ we argue as follows. By Theorem 6.4 the subgraph $[y, A(a_1)y, A(a_2)y]$, $y\in\mathfrak{R}[\mathfrak{X}]$, is dense in the graph $[x, A(a_1)x, A(a_2)x]$, $x\in\mathfrak{D}(a_1)\cap\mathfrak{D}(a_2)$. Hence for any x in the latter set we may find a sequence $y_n\in\mathfrak{R}[\mathfrak{X}]$ such that $y_n\to x$, $A(a_1)y_n\to A(a_1)x$, $A(a_2)y_n\to A(a_1)x$. It follows that

$$A(a_1 + a_2)y_n = A(a_1)y_n + A(a_2)y_n \to A(a_1)x + A(a_2)x.$$

Since $A(a_1+a_2)$ is also a closed operator, it follows that $A(a_1+a_2)x$ exists and that (6.22) holds.

We come now to the main theorem of this section.

THEOREM 6.8. *Let*

$$a = (\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$$

and set $A(e_k) = A_k$. *Then* $\mathfrak{D}(a)$ *contains the set* $\bigcap_1^n \mathfrak{D}(e_k) \equiv \mathfrak{D}_n$ *and for* x *in the latter set*

$$A(a)x = \alpha_1 A_1 x + \alpha_2 A_2 x + \cdots + \alpha_n A_n x. \tag{6.23}$$

The basic infinitesimal generators $A_1, A_2, \cdots, A_n$ *are linearly independent in* $\mathfrak{D}_n$.

The validity of (6.23) is an immediate consequence of the preceding theorem. A linear relation with constant coefficients between $A_1x, A_2x, \cdots, A_nx$ valid for all x in $\mathfrak{D}_n$ implies the existence of two distinct vectors a and b in E_n^+ such that $A(a)x = A(b)x$ in $\mathfrak{D}_n$. We know that the subgraph $[y, A(a)y]$, $y \in \mathfrak{R}[\mathfrak{X}]$, is dense in the graph of $A(a)$. A fortiori this is true for the graph $[x, A(a)x]$, $x \in \mathfrak{D}_n$. Now for $\Re(\lambda) > 0$ the set $[\lambda I - A(a)][\mathfrak{D}(a)]$ is dense in $\mathfrak{X}$ so the same must be true for $[\lambda I - A(a)][\mathfrak{D}_n] = [\lambda I - A(b)][\mathfrak{D}_n]$. From this we conclude that the linear bounded operators $R[\lambda; A(a)]$ and $R[\lambda; A(b)]$ coincide in a dense set and hence everywhere. By the inversion formulas for the Laplace transform, applied to formula (6.4), we conclude that the corresponding one-parameter semi-group operators $T[f(\rho a)]$ and $T[f(\rho b)]$ are identical for all values of ρ. But for all small values of ρ we have $f(\rho a) \neq f(\rho b)$ and $|f(\rho a)| < \rho_0$, $|f(\rho b)| < \rho_0$. Thus the result contradicts A_4^2 so we conclude that $A(a)$ and $A(b)$ are distinct operators on $\mathfrak{D}_n$ when $a \neq b$.

The infinitesimal generators $A(a)$ of $\mathfrak{S}$ form an n-dimensional system which is closed under addition and multiplication by positive numbers. On the other hand, multiplication by negative numbers is not allowed since a and $-a$ are not simultaneously in $\overline{E}_n$ if $a \neq 0$. In this respect there is a striking difference between the semi-module of infinitesimal generators of a semi-group and the Lie ring of generators of a group.

7. **The fundamental theorems.** We come now to the analogues of the three fundamental theorems of Lie.

THEOREM 7.1. *For* $y \in \mathfrak{R}[\mathfrak{X}]$ *and small values of* a *in* E_n^+

$$(7.1)\qquad \frac{\partial}{\partial \alpha_j} T(a)y = \sum_{k=1}^{n} \Gamma_{jk}(a)T(a)A_k y, \qquad j = 1, 2, \cdots, n,$$

where the matrix $(\Gamma_{jk}(a))$ *tends to the unit matrix when* $a \to 0$. *If the operator in the left member is closed, then its domain contains* $\mathfrak{D}_n$ *and* (7.1) *holds in* $\mathfrak{D}_n$.

The left side of (7.1) is the limit when $\delta \to 0$ of

$$(7.2)\qquad \delta^{-1}[T(a + \delta e_j) - T(a)]y.$$

Using A_3^4 we see that the equation $F(a, h) = a + \delta e_j$ has a unique solution in E_n^+, viz.

$$h = \psi(a + \delta e_j, a) = \chi_j(a, \delta) = c_j(a)\delta + O(\delta^2),$$

where the components $\Gamma_{jk}(a)$ of the vector $c_j(a)$ are determined by the linear system of equations

$$(7.3)\qquad \sum_{k=1}^{n} F_{ik}(a, 0)\Gamma_{jk}(a) = \delta_{ij}, \qquad i = 1, 2, \cdots, n.$$

Here F_{ik} as in formula (5.4) is the partial derivative of the ith component of $F(a, b)$ with respect to the kth component of b in which we set $b = 0$. Since $F_{ik}(a, 0) \to \delta_{ik}$ when $|a| \to 0$, the determinant of the system (7.3) is different from zero for small values of $|a|$ so that the $\Gamma_{jk}(a)$ may be determined. Further $\Gamma_{jk}(a) \to \delta_{jk}$ when $|a| \to 0$. Thus

$$(7.4)\qquad \begin{aligned} &\delta^{-1}[T(a + \delta e_j) - T(a)]y \\ &\qquad = T(a)\delta^{-1}\{T[\chi_j(a, \delta)] - I\}y \\ &\qquad = T(a)\delta^{-1}\{T[\delta c_j(a)] - I\}y \\ &\qquad\quad + T(a)\delta^{-1}\{T[\chi_j(a, \delta)] - T[\delta c_j(a)]\}y. \end{aligned}$$

The first term in the last member tends to $T(a)A[c_j(a)]y$ when $\delta \to 0$ by Theorem 6.5 while the second term tends to θ by Theorem 6.6. Hence the limit exists and by Theorem 6.8 it equals

$$T(a)A[c_j(a)]y = \sum_{k=1}^{n} \Gamma_{jk}(a)A_k y.$$

The left side of this equation involves a closed operator, but it is not a priori obvious that the partials of $T(a)$ are closed operators; if they are, then Theorem 6.4 shows immediately that (7.1) holds in $\mathfrak{D}_n$.

Theorem 7.1 also gives us

$$(7.5)\qquad T(a)A_i y = \sum_{k=1}^{n} \Delta_{ik}(a) \frac{\partial}{\partial \alpha_k} T(a)y$$

where $(\Delta_{ik}(a))$ is the inverse of the matrix $(\Gamma_{jk}(a))$. It follows from (7.3) that

$$\Delta_{ik}(a) = F_{ki}(a, 0). \tag{7.6}$$

The second and third fundamental theorems involve the *structural constants* of the semi-group defined by

$$\gamma^{i}_{jk} = \alpha^{i}_{jk} - \alpha^{i}_{kj}, \qquad \alpha^{i}_{jk} = \left(\frac{\partial^2 F_i}{\partial\alpha_j\partial\beta_k}\right)_{0,0} \tag{7.7}$$

in the notation of (6.5). We note that

$$\frac{\partial}{\partial\alpha_j}\Delta_{ik}(a) \to \alpha^{k}_{ji}, \qquad |a| \to 0. \tag{7.8}$$

THEOREM 7.2. *If* $y \in \mathfrak{R}[\mathfrak{X}]$

$$[A_i, A_j]y \equiv (A_iA_j - A_jA_i)y = \sum_{m=1}^{n} \gamma^{m}_{ij} A_m y. \tag{7.9}$$

If the operator in the first member is closed, then the relation holds in $\mathfrak{D}_n$.

For the proof we use (7.5) twice forming

$$\begin{aligned} T(a)A_iA_jy &= \sum_{k=1}^{n} \Delta_{ik}(a)\frac{\partial}{\partial\alpha_k}[T(a)A_jy] \\ &= \sum_{k=1}^{n} \Delta_{ik}(a)\frac{\partial}{\partial\alpha_k}\left\{\sum_{m=1}^{n} \Delta_{jm}(a)\frac{\partial}{\partial\alpha_m}T(a)y\right\}. \end{aligned}$$

Interchanging i and j, subtracting and simplifying we get

$$T(a)[A_i, A_j]y = \sum_{m=1}^{n} \Delta^{m}_{ij}(a)\frac{\partial}{\partial\alpha_m}T(a)y$$

where

$$\Delta^{m}_{ij}(a) = \sum_{k=1}^{n}\left\{\Delta_{ik}(a)\frac{\partial}{\partial\alpha_k}\Delta_{jm}(a) - \Delta_{jk}(a)\frac{\partial}{\partial\alpha_k}\Delta_{im}(a)\right\},$$

which tends to γ^{m}_{ij} when $|a| \to 0$ by (7.8) since $\Delta_{ik}(a) \to \delta_{ik}$. Formula (7.1) shows that the partial of $T(a)y$ with respect to α_m tends to $A_m y$. It should be observed that the second order partials of $T(a)y$ which arise in the process, but cancel in the subtraction, actually exist when $y \in \mathfrak{R}[\mathfrak{X}]$. This follows from the fact that in (7.4) we may replace y by $A_j y$ and still carry through the limit process. The second partials may consequently be found by formal differentiation of formula (7.1). If $[A_i, A_j]$ is a closed operator, then Theorem 6.4 shows that (7.9) holds in $\mathfrak{D}_n$.

The second fundamental theorem asserts that each *commutator* $[A_i, A_j]$ is a linear combination of basic infinitesimal generators, but in the semi-group case it does not follow that $[A_i, A_j]$ is also an infinitesimal generator, that is, we cannot always find a vector c in E_n^+ such that $[A_i, A_j]y = A(c)y$. In particular, we note that if $[A_i, A_j]$ is an infinitesimal generator, $[A_j, A_i]$ cannot be one.

THEOREM 7.3. *The structural constants satisfy*

$$\gamma^{i}_{jk} = -\gamma^{i}_{kj}, \tag{7.10}$$

$$\sum_{m=1}^{n} \left[\gamma^{p}_{im}\gamma^{m}_{jk} + \gamma^{p}_{jm}\gamma^{m}_{ki} + \gamma^{p}_{km}\gamma^{m}_{ij}\right] = 0. \tag{7.11}$$

Here (7.10) follows from (7.7) while (7.11) follows from the relation

$$[A_i, [A_j, A_k]] + [A_j, [A_k, A_i]] + [A_k, [A_i, A_j]] = \Theta \tag{7.12}$$

which holds when the operator on the left acts on the subspace $\mathfrak{K}[\mathfrak{X}]$.

8. **Conclusions.** The preceding theory raises perhaps more questions than it answers. Let us list some directions in which further research is desirable.

(1) Determine the set of points c such that $c = f(a)$ for some a. In particular, are all finite boundary points of this set "accessible" and found by solving the equation $F(p, p) = p$?

(2) Extend the investigation to other parameter sets.

(3) Prove that the partials of $T(a)$ and the commutators are closed operators.

(4) Formulate and prove converses of the fundamental theorems.

(5) Is it possible to embed the given semi-group $\mathfrak{S}$ in a group of, in general, unbounded operators, the group being generated by the set $\sum \alpha_k A_k$ with real α's not necessarily positive?

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RAMIFICATIONS, OLD AND NEW, OF THE EIGENVALUE PROBLEM

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Since this is a lecture dedicated to the memory of Josiah Willard Gibbs let me start with that purely mathematical discovery which Gibbs contributed to the theory of Fourier series. Fourier series have to do with the eigenvalues and eigenfunctions of the oldest, simplest, and most important of all spectrum problems, that of the vibrating string. In preparing this lecture, the speaker has assumed that he is expected to talk on a subject in which he had some first-hand experience through his own work. And glancing back over the years he found that the one topic to which he has returned again and again is the problem of eigenvalues and eigenfunctions in its various ramifications. It so happens that right at the beginning of my mathematical career I wrote two papers on what we now call the Gibbs phenomenon.

1. **Gibbs phenomenon.** Take a simple periodic function with a discontinuity, for example, the function $1^0(x)$ of period 2π which equals 0 for $-\pi < x < 0$ and 1 for $0 < x < \pi$. In a letter to the editor of Nature published on April 27, 1899, Gibbs, correcting a statement in a previous letter, pointed out that the limit of the graphs of the partial sums $y = 1^0_n(x)$ of the Fourier series of $1^0(x)$ includes not only the vertical ascent from the level 0 to the level 1 at $x = 0$, but extends vertically beyond it by a specific amount. A. A. Michelson had started the discussion in Nature by criticizing the way in which the mathematicians are wont to describe the limit of the sequence of those partial sums; he had pleaded for adding to the two horizontal levels the vertical precipice. Today we find in the notion of uniform convergence the most adequate analysis of the phenomenon. Introduce the sinus integral

$$\mathrm{Si}\,(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{\sin \xi}{\xi}\, d\xi$$

and consider a closed interval I, say $-\pi/2 \leqq x \leqq \pi/2$, containing only the one discontinuity at $x = 0$. It is, of course, not true that the difference between $1^0(x)$ and the nth partial sum $1^0_n(x)$ converges

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uniformly to zero in I, but it is true that $1_n^0(x) - \mathrm{Si}\ (nx)$ does so. Thus the graph of $1_n^0(x)$ for large n is essentially that of the undulating function Si (x) compressed at the ratio $1:n$ in the x-direction.

Instead of direct summation one can apply to infinite series, in particular to Fourier series, other methods of summation. Let me mention only one here: in the Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ add the factor e^{-n^2t} to the nth term and then let the positive parameter t = time in the resulting sum

$$\sum_n a_n e^{-n^2t} e^{inx}$$

converge to zero. Since that sum is a solution of the equation of heat conduction

$$\partial^2 u/\partial x^2 - \partial u/\partial t = 0,$$

I call this the heat conduction summation. The Gibbs phenomenon for this summation is ruled by the function

$$\mathrm{Er}\ (x) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{x} e^{-\xi^2} d\xi$$

in the same sense as it is ruled by Si (x) for the direct summation. This is an immediate consequence of the fact that Er $(x/2t^{1/2})$ is actually a solution—though not a periodic one—of the heat equation.

In the two papers just mentioned [**1**][1] I considered the Gibbs phenomenon for a certain general type of summation methods. My chief concern was with the simplest two-dimensional case of eigenfunctions, namely Laplace's expansion of functions on a sphere in terms of spherical harmonics. Nothing new occurs if the function has a discontinuity along a smooth line with a continuous tangent. But quite an impressive mountain landscape develops in the neighborhood of a point where this line makes an angle. There was one specific one-dimensional problem which attracted my attention. It deals with a circular metal ring consisting of two halves of different conductivities α and β. Assume the normalization $\alpha+\beta=1$. If one of the halves has a temperature of 100° C. at the time $t=0$, the other of 0°, how will the temperature level off in the progress of time? The solution can be easily expressed by means of the function Er, and this gives the Gibbs phenomenon for the heat conduction summation of the corresponding eigenfunction expansion. The Gibbs phenomenon for direct summation is a more subtle question because the distribution of the eigenvalues λ depends on the arithmetical character of the numbers α and β. Only if they are rational, the problem may be settled by fairly direct computation (A). The result would carry over to an arbitrary irrational α if it were true that α can be approximated by a sequence of fractions a_n/c_n ($n=1, 2, \cdots$) such that

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[1] The letters (A), (B), · · · refer to a number of notes printed at the end of the paper (and not included in the actual lecture as delivered at the Columbus meeting). The bold face numerals [**1**], · · · refer to the bibliography.

$$n(\alpha - a_n/c_n) \to 0, \qquad c_n/n \to 0 \qquad \text{for } n \to \infty.$$

It is simple enough to show that this can be done, and thus to determine the Gibbs phenomenon for arbitrary conductivities α, β. Our lemma on Diophantine approximations proves at the same time the equidistribution mod. 1 of the multiples $n\alpha$ of an irrational number α.

When, not so long after, I learned through Felix Bernstein about the problem of mean motion in Lagrange's linear theory of perturbation for the planetary system, a problem P. Bohl had connected with that of equidistribution mod. 1, I remembered this investigation and tried to settle the question of equidistribution in a more general form [2]. This is an example of how experience in one field of mathematics may give one the lead in an entirely different field. It is chiefly for this lesson that I have mentioned here this early work of mine, by which, in a very modest way, I paid homage to the genius of Gibbs about forty years ago.—Incidentally, Fourier series provided the basis for the analytic method which I brought to bear on the general problem of deciding whether a given sequence of real numbers $\xi_1, \xi_2, \cdots$ is equidistributed mod. 1. Such equidistribution can be formulated as a statement concerning the mean value of any Riemann integrable function $f(x)$ of period 1 for the argument values $x=\xi_1, \xi_2, \cdots$; and the gist of the method lies in the observation that verification of the statement for the special periodic functions $e^{2\pi n i x}$ ($n=0, \pm 1, \pm 2, \cdots$) is sufficient.

2. **Limit circle and limit point.** As one knows, Fourier series were generalized by Sturm and Liouville so as to cover the eigenvalues λ and eigenfunctions $\phi(s)$ of the self-adjoint differential equation

$$L_\lambda(\phi) \equiv \left\{\frac{d}{ds}\left(p(s)\frac{d\phi}{ds}\right) - q(s)\phi(s)\right\} + \lambda\phi(s) = 0, \tag{1}$$

subject to a real linear boundary condition at either end of the interval $0 \leqq s \leqq l$. The coefficients $p(s) > 0$ and $q(s)$ are given real continuous functions in this interval. Let the abbreviation ϕ' be used for $p(s)d\phi/ds$. A real linear boundary condition for $s=l$ is of the form

$$\phi'(l) - h\cdot\phi(l) = 0 \tag{2}$$

with a real constant h (not excluding $h=\infty$). For a solution ϕ of $L_\lambda(\phi)=0$ and a solution ϕ_* of $L_{\lambda_*}(\phi_*)=0$ we have the simple Green's formula

$$[\phi\phi_*' - \phi_*\phi']_0^l = (\lambda - \lambda_*)\int_0^l \phi\phi_* ds. \tag{3}$$

It shows that eigenfunctions ϕ, ϕ_* belonging to two distinct eigenvalues λ, λ_* are necessarily orthogonal, $\int_0^l \phi\phi_*\cdot ds=0$. On taking $\lambda_*=\bar{\lambda}$, $\phi_*=\bar{\phi}$ one finds that for a non-real λ the function $\phi(s)$ cannot satisfy a real linear boundary condition at both ends without vanishing identically; for under these circumstances our equation would give $\int_0^l \phi\bar{\phi}ds=0$. The positive-definite character of the integrand $\phi\bar{\phi}$ is decisive here.

The spectrum of the eigenvalues is discrete provided the differential equation is regular at both ends, that is, provided $p(s)$, $q(s)$ are continuous and $p(s)$ actually positive throughout the closed interval $0\leqq s\leqq l$. If we make this assumption only for the right-open interval $0\leqq s<l$, as we shall now do, then the end $s=0$ stays regular, but the end $s=l$ is (possibly) singular. Let us throw the singular end into $s=+\infty$. Under these circumstances one must expect that a continuous spectrum will appear side by side with the point spectrum. Moreover it seems that sometimes a boundary condition is required at the singular end, just as it would be for a regular one, but sometimes not.

The very first result by which I added my mite to our stock of mathematical knowledge had to do with the clarification of this issue [3]. Since one cannot vouch that the spectrum will not cover the entire real λ-axis, I had the simple idea (not as trivial at that time as it has now become) to determine Green's function $G(s, t)$ neither for $\lambda=0$ nor for any real λ, but for a λ in the upper half-plane, $\mathfrak{J}\lambda>0$, for example for $\lambda=i$. Let $\eta(s)$, $\theta(s)$ designate the two solutions $\phi(s)=\phi(s;\lambda)$ of (1) determined by the initial conditions

$$\eta(0) = 1,\ \eta'(0) = 0 \quad\text{and}\quad \theta(0) = 0,\ \theta'(0) = 1$$

respectively, and then consider the solution

$$\phi(s) = w\cdot\eta(s) - \theta(s) \tag{4}$$

which combines them by means of an arbitrary constant w. The question naturally arises for which values of w this $\phi(s)$ satisfies a real linear boundary condition (2) at $s=l$. The answer is: for those w

that lie on a certain circle C_l in the complex w-plane. Indeed the condition (2) gives

$$(w\eta' - \theta') - h(w\eta - \theta) = 0 \qquad \text{for } s = l$$

or

$$w = \frac{\theta'(l) - h\theta(l)}{\eta'(l) - h\eta(l)},$$

and this fractional linear or Möbius transformation $h \rightarrow w$ maps the real h-axis upon a circle in the w-plane. Here λ is a given value in the upper half-plane. We now compare two such values λ, λ_0. Whereas $\phi(s) = \phi(s; \lambda)$, η, θ, and so on refer to λ, let ϕ_0, η_0, θ_0, and so on refer to λ_0. By picking a point w^0 on $C_l^0 = C_l(\lambda_0)$ one fixes the coefficient h of the real boundary condition (2). Clearly the point $w = w(\lambda)$ on $C_l = C_l(\lambda)$ for which (4) satisfies the same boundary condition as $\phi_0 = w^0 \cdot \eta_0 - \theta_0$ at $s = l$ proceeds from w^0 by a certain Möbius transformation $w^0 \rightarrow w$. Points w^0 and w on C_l^0 and C_l thus related may be called homologous points.

We now face the task of transferring these obvious answers to the limit $l \rightarrow \infty$. For that purpose the definitions of the circle C_l and of the homology between the two circles $C_l(\lambda_0)$ and $C_l(\lambda)$ must first be given a new form, one that looks more complicated but is in fact more instructive. Put $\lambda_* = \bar{\lambda}$, $\phi_* = \bar{\phi}$ in (3):

$$[\phi\bar{\phi}' - \bar{\phi}\phi']_0^l = (\lambda - \bar{\lambda}) \cdot \int_0^l \phi\bar{\phi}\,ds.$$

On account of this identity the requirement that ϕ satisfies a real linear boundary condition at $s = l$, $\phi\bar{\phi}' - \bar{\phi}\phi' = 0$ for $s = l$, is equivalent to the relation

$$\mathfrak{J}\lambda \int_0^l \phi\bar{\phi}\,ds = \mathfrak{J}w \qquad \text{for } \phi = w\eta - \theta.$$

This is indeed the equation of a circle C_l in the upper half w-plane. The points of the circular disk (C_l) bounded by C_l are characterized by the inequality

$$\int_0^l \phi\bar{\phi}\,ds \leqq \mathfrak{J}w/\mathfrak{J}\lambda.$$

This shows at once that $C_{l'}$ lies inside C_l if $l' > l$. Hence with l tending to infinity, C_l shrinks either to a limit circle or a limit point $C = C(\lambda)$. This alternative, limit circle or limit point, is clearly the correct formulation of the question whether or not to impose a boundary condition at $s=\infty$.

Next we have to put the description of the homology mapping $w^0\rightarrow w$ of the circle C_l^0 upon C_l into a form suitable for passage to the limit $l\rightarrow\infty$. Pick a point w^0 on $C_l^0=C_l(\lambda_0)$ and form

$$\phi_0(s) = \phi(s;\lambda_0) = w^0\cdot\eta(s;\lambda_0) - \theta(s;\lambda_0). \tag{5}$$

The solution

$$\phi(s) = \phi(s;\lambda) = w\cdot\eta(s;\lambda) - \theta(s;\lambda), \tag{6}$$

satisfying at the end $s=l$ the same real linear boundary condition as $\phi_0(s)$, is obtained from $\phi_0(s)$ by solving the linear integral equation

$$\phi_0(s) = \phi(s) - (\lambda-\lambda_0)\int_0^l G^0(s,t)\cdot\phi(t)dt$$

the kernel $G^0(s, t)$ of which is Green's function for λ_0:

$$G^0(s,t) = \begin{cases}\phi_0(s)\cdot\eta_0(t) & (t\leqq s),\\ \eta_0(s)\cdot\phi_0(t) & (s\leqq t).\end{cases}$$

From the solution $\phi(s)$, (4), one gets $w=w(\lambda)$ as its initial value $\phi(0)$. This prescription at once carries over to the limit $l\rightarrow\infty$; one has simply to replace integration from 0 to l in our integral equation by one extending from 0 to ∞. Of course in (5) the factor w^0 is now supposed to be a point on the circle C^0 (whether that is a real circle or degenerates into a point). Indeed, on trying to solve in the simplest way the integral equation thus resulting, namely by the Neumann series, one finds by direct estimates that the series converges within the circle around λ_0 in the λ-plane that touches the real axis. Hence analytic continuation encounters no obstacle, and $\phi(0;\lambda)=w(\lambda)$ is a regular analytic function in the entire upper half λ-plane. The method works in both the limit-circle and the limit-point cases. In the latter w^0 is *the* limit point for λ_0, and the construction gives *the* limit point $w=w(\lambda)$ for λ. Whether the singular end $s=\infty$ is of the limit circle or limit point type does not depend on the value of λ as long as λ is restricted to the upper half-plane. In the limit circle case it follows easily that the homology mapping $w^0\rightarrow w=w(\lambda)$ of C^0 onto C is a Möbius transformation with coefficients depending analytically on λ (for $\Im\lambda>0$).

If one replaces the differential equation (1) by the corresponding difference equation, one arrives at a neat formulation for the theory of Stieltjes' continued fractions and his moment problem. It was treated by E. Hellinger in a manner analogous to the one outlined here for the differential equation [4]. The moment problem is a very special limiting case of an interpolation problem in the theory of analytic functions $w=w(\lambda)$ of a complex variable λ first studied by G. Pick and R. Nevanlinna [5]. It concerns analytic functions $w(\lambda)$ defined in the upper half λ-plane, $\mathfrak{J}\lambda>0$, the values of which have themselves positive imaginary parts. For the moment let us call them positive functions. How far is such a function determined if its values $w(\alpha_n)$ are prescribed for a sequence of points $\lambda=\alpha_1, \alpha_2, \cdots$ in the upper half-plane? The differential problem corresponding to this interpolation or difference problem can be put in the form of a system of two linear differential equations of the first order for two unknowns ϕ, ϕ' containing the spectral parameter λ in broken linear fashion,

$$(\mathrm{S}_\lambda) \qquad \begin{cases} \dfrac{d\phi}{ds} = \dfrac{\lambda a_1(s) - b_1(s)}{\lambda a(s) - b(s)}\cdot\phi'(s), \\[2ex] \dfrac{d\phi'}{ds} = \dfrac{\lambda a_2(s) - b_2(s)}{\lambda a(s) - b(s)}\cdot\phi(s) \end{cases}$$

with real coefficients which satisfy the inequalities

$$k'(s) = a_1(s)b(s) - b_1(s)a(s) > 0, \quad k(s) = a(s)b_2(s) - b(s)a_2(s) > 0.$$

This general system now replaces our former system (1) or

$$(\mathrm{L}_\lambda) \qquad \frac{d\phi}{ds} = \frac{1}{p(s)}\cdot\phi'(s), \qquad \frac{d\phi'}{ds} = (q(s) - \lambda)\cdot\phi(s).$$

The theory of limit circles and of the homology mapping of the limit circles for different values of λ carries over practically without alteration to this more general problem [6]. The decisive point is the positive definite character of the integrand

$$\frac{k\phi\bar{\phi} + k'\phi'\bar{\phi}'}{(\lambda a - b)(\bar{\lambda} a - b)},$$

that appears in Green's formula and takes over the role played by $\phi\bar{\phi}$ in the problem (L_λ). By direct constructive solutions of integral equations one thus proves the fundamental facts about the Nevanlinna interpolation problem, which Nevanlinna himself had derived with the aid of some of the strong "existential" methods characteristic for the theory of analytic functions, such as the Vitali theorem (B).

3. Expansion theorem for ordinary self-adjoint linear differential

equations of second order with singular end. For the classical problem (L_λ) the investigation of the singular end $s=\infty$ is merely a preliminary to the study of expansions by eigenfunctions. What one has to expect can be predicted when one first replaces the singular end ∞ by the regular end l. But one has to obliterate the feature of a discrete spectrum by writing the sum over the eigenvalues in the expansion formula as a Stieltjes integral involving a non-decreasing step function. Rather than attempt to carry out the passage to the limit $l\to\infty$ one seeks to verify the formula thus obtained directly for the interval $0\leqq s<\infty$ with the singular end ∞. For simplicity's sake let us prescribe the boundary condition $\phi'(0)=0$ at the regular end $s=0$. Then we know a priori that for any eigenvalue $\lambda=\lambda_n$ the function $\eta(s;\lambda_n)$ must be the eigenfunction. The eigenvalues are those real values of λ for which $\eta(s;\lambda)$ satisfies a given real linear boundary condition $\eta'(l;\lambda)-h\cdot\eta(l;\lambda)=0$ at the end $s=l$.

Choose a definite λ_0 with positive imaginary part, for example $\lambda_0=i$. Fixing the coefficient h amounts to fixing a definite point w^0 on the circle $C_l(\lambda_0)=C^0$. The expansion of the arbitrary function $f(s)$ is then given by

$$f(s)\sim\sum_n a_n\cdot\eta(s;\lambda_n)\quad\text{where}\quad a_n=\int_0^l\eta(s;\lambda_n)r_l(\lambda_n)f(s)ds$$

and

$$r_l(\lambda_n)=1\Big/\int_0^l(\eta(s;\lambda_n))^2ds.$$

Let $\Delta=(\lambda_1,\lambda_2)$ be any interval on the real λ-axis, and, $x(\lambda)$ being any function of the real λ, let Δx stand for the difference $x(\lambda_2)-x(\lambda_1)$. We may now define a nondecreasing step function $\rho_l(\lambda)$ by the equation

$$\Delta\rho_l=\sum_{\lambda_n\in\Delta}r_l(\lambda_n). \tag{7}$$

(If one of the ends λ_1,λ_2 of the interval is an eigenvalue, the summand $r_l(\lambda_1)$ or $r_l(\lambda_2)$ in (7) should be counted with the weight 1/2 only.) After forming

$$\Delta P(s)=\int_\Delta\eta(s;\lambda)d\rho(\lambda),\qquad\Delta\alpha=\int_0^l\Delta P(s)\cdot f(s)ds \tag{8}$$

our expansion appears as the following Stieltjes integral extending over the real λ-axis:

$$f(s) \sim \int_{-\infty}^{+\infty} \eta(s;\lambda)\,d\alpha(\lambda). \tag{9}$$

One has good reason to hope that in this form the expansion theorem will carry over to the interval $(0, \infty)$ with the singular end ∞, and in anticipation of this result we have dropped the subscript l in (8). The whole problem boils down to determining the nondecreasing function $\rho(\lambda)$.

For a finite interval $0 \leqq s \leqq l$ and a non-real λ one easily proves (C) the following expansion, which is uniformly convergent with respect to s:

$$\mathfrak{J}\phi(s;\lambda) = \int_{-\infty}^{+\infty} \mathfrak{J}\,\frac{1}{\mu - \lambda}\cdot \eta(s;\mu)\,d\rho(\mu) \tag{10}$$

in particular $(s=0)$

$$\mathfrak{J}w(\lambda) = \int_{-\infty}^{+\infty} \mathfrak{J}\,\frac{1}{\mu - \lambda}\cdot d\rho(\mu). \tag{11}$$

We expect these equations to hold even for the infinite interval—although $\rho(\lambda)$ may then cease to be a step function—if $w(\lambda)$ and $\phi(s;\lambda)$ are constructed from a (or the) point w^0 on the limit circle C^0 according to the prescription given before. Denote by Δ_ϵ the segment Δ after it has been raised by the positive amount ϵ in the direction of the imaginary axis. One computes $\Delta\rho$ and $\Delta\mathrm{P}(s) = \int_\Delta \eta(s;\ \lambda)\,d\rho(\lambda)$ from (11) and (10) as the limits

$$\Delta\rho = \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{\Delta_\epsilon} \mathfrak{J}w(\lambda)\cdot d\lambda, \tag{12}$$

$$\Delta\mathrm{P}(s) = \lim_{\epsilon\to 0} \frac{1}{\pi} \int_{\Delta_\epsilon} \mathfrak{J}\phi(s;\lambda)\cdot d\lambda.$$

The first formula determines ρ.

When turning these heuristic arguments into an actual proof one should first endeavor to prove the existence of the limit (12) and then to establish the expansion (9), (8), with the density differential $d\rho$ thus constructed. Mean convergence of the expansion is to be expected for any square integrable $f(s)$; certain slight restrictions imposed upon $f(s)$ will insure ordinary uniform convergence.

Many authors have written on our subject. My own first approach was based on Hilbert's general theory of spectral decomposition of a bounded symmetric linear operator and specialized it by taking advantage of the particular circumstances prevailing for the differential problem in question, above all, of the fact that for any eigenvalue λ the eigenfunction, $\eta(s; \lambda)$, is known a priori [7]. M. H. Stone's procedure in his book on *Linear transformations in Hilbert space* [8] is of the same character, but he is able to utilize the machinery of general concepts developed in the twenty intervening years for axiomatized Hilbert space. Earlier, the Stieltjes method, on which Hellinger had founded Hilbert's general theory [9], had been directly applied to the special differential problem by E. Hilb [10]; but he did not carry it so far as to obtain the explicit construction of the differential $d\rho$. Recently E. C. Titchmarsh in several papers and in his book on *Eigenfunction expansions* [11] resumed this direct approach. The basic equation (12) is due to him. Yet his construction of $w(\lambda)$ and of $d\rho$ is not as direct as I should wish them. Also a number of contributions made by A. Wintner and P. Hartman during the last two years ought to be mentioned [12]. The formula (12) was rediscovered by Kunihiko Kodaira (who of course had been cut off from our Western mathematical literature since the end of 1941); his construction of ρ and his proofs for (12) and the expansion formula (9), still unpublished, seem to clinch the issue. It is remarkable that forty years had to pass before such a thoroughly satisfactory direct treatment emerged; the fact is a reflection on the degree to which mathematicians during this period got absorbed in abstract generalizations and lost sight of their task of finishing up some of the more concrete problems of undeniable importance.

4. **Inequalities and asymptotic laws for eigenvalues.** But let us drop this matter now and turn to another subject, that of the asymptotic distribution of the eigen-frequencies for the two- or more-dimensional membrane and for other oscillating continua. H. A. Lorentz had impressed upon the mathematicians the urgency for physics of a settlement of this question. For a pupil of Hilbert around 1910 it was natural to visualize the question as one concerning integral equations. By means of a real symmetric kernel $K(s, t)$ one introduces the linear operator $u \to Ku$, more explicitly $u(s) \to \int_0^1 K(s, t)u(t)dt$, in the vector space of all real-valued continuous functions $u = u(s)$ defined over the interval $0 \leqq s \leqq 1$. If the integral $\int_0^1 u(s)v(s)ds$ is taken as the scalar product (u, v) of any two vectors u, v in this space, then the quadratic integral form

$$K\langle u\rangle = \int_0^1 \int_0^1 K(s, t)u(s)u(t) \cdot dsdt$$

is the scalar product of u and Ku. The reciprocal eigenvalues κ and corresponding eigenvectors ϕ are the solutions of the equation $K\phi = \kappa\cdot\phi$. Let the positive κ's be arranged in descending order, $\kappa_1 \geqq \kappa_2 \geqq \cdots$, and the corresponding eigenfunctions ϕ_n so chosen as to form an orthonormal system, $(\phi_m, \phi_n) = \delta_{mn}$. Then κ_n is the maximum of the form K under the auxiliary conditions

$$\|u\|^2 = (u, u) \leqq 1, \qquad (u, \phi_i) = 0 \qquad (i = 1, \cdots, n-1).$$

Let $\omega_1(s), \cdots, \omega_{n-1}(s)$ be any $n-1$ functions. The fundamental lemma which made my investigation possible states that there exists a vector u of length $\|u\| = 1$ which is orthogonal to $\omega_1, \cdots, \omega_{n-1}$,

$$(u, \omega_1) = 0, \cdots, (u, \omega_{n-1}) = 0,$$

such that $K\langle u\rangle \geqq \kappa_n$. This characterizes κ_n independently of the preceding eigenvalues and eigenfunctions as the "minimum of a maximum." The construction of such a u as the lemma requires is easy enough: a suitable linear combination $c_1\phi_1(s) + \cdots + c_n\phi_n(s)$ of the first n eigenvectors will do the trick. I used this lemma (in a slightly different form) for the purpose of carrying over to all κ_n statements that are evident for the first reciprocal eigenvalue κ_1 [13].

Here is an example. Suppose you add to a kernel K a positive-definite one k, that is, one for which $k\langle u\rangle \geqq 0$. It is clear that the first reciprocal positive eigenvalue κ_1^* of $K^* = K + k$ is greater than or equal to κ_1; for κ_1 is the maximum of $K\langle u\rangle$ and κ_1^* the maximum of $K\langle u\rangle + k\langle u\rangle$ under the condition $\|u\|^2 \leqq 1$. Our lemma carries the inequality over to all κ's arranged in descending order: $\kappa_n \leqq \kappa_n^*$.

This result is of immediate application to the two-dimensional membrane problem. Let the membrane cover a region S of (Jordan) area V. With the argument P ranging over the points of S the eigenvalues λ and eigenfunctions $\phi(P)$ satisfy the differential equation $\Delta\phi + \lambda\phi = 0$ in S and the condition $\phi = 0$ along the boundary S' of S. The differential equation together with the boundary condition is equivalent to the integral equation

$$\phi(P) - \lambda \int_S G(P, Q)\cdot\phi(Q)\cdot dQ = 0,$$

the symmetric kernel of which is the Green's function $G(P, Q)$. Divide S by a line l into two parts S_1, S_2 and let G_1, G_2 be their Green's functions (setting $G_i(P, Q) = 0$ if one of the argument points P, Q or both are outside S_i). It can easily be shown that the kernel $G - (G_1 + G_2)$ is positive-definite. Hence the nth eigenvalue λ_n of G is less than or equal to the nth eigenvalue λ_n' of $G_1 + G_2$. The eigenvalues of $G_1 + G_2$ are the eigenvalues of a membrane covering S which is kept fixed not only along the boundary S', but also along the line l. By combining this result with the known asymptotic distribution of the membrane eigenvalues for a square it could readily be deduced that the fixing of the membrane along l does not alter the asymptotic distribution,

$$\lambda_n/\lambda'_n \to 1 \qquad \text{for} \quad n \to \infty,$$

and moreover to establish the law of asymptotic distribution, according to which the number $N(\lambda)$ of eigenvalues less than λ equals asymptotically $(V/4\pi)\cdot\lambda$.

After the first world war Courant resumed this sort of problems [14]. If I see correctly, his essential contribution is not the minimum-maximum principle formulated in our fundamental lemma, but its application to a fairly general typical situation. The maximum of a quadratic form $K\langle u\rangle$ depending on a vector u of length not greater than 1 in a vector space $\mathfrak{H}$ is lowered if additional restrictions are imposed upon u, for example, if u is restricted to a linear subspace $\mathfrak{H}'$ of $\mathfrak{H}$. If the quadratic form is what Hilbert calls completely continuous, this obvious statement may be put into the inequality $\kappa'_1 \leqq \kappa_1$. The quadratic form $K\langle u\rangle$ in $\mathfrak{H}'$ is $(u, K'u)$ where K' is the operator K followed by perpendicular projection upon $\mathfrak{H}'$. While κ_1 is the first reciprocal eigenvalue of the operator K in $\mathfrak{H}$, κ'_1 has the same significance for the operator K' in $\mathfrak{H}'$. The lemma carries the inequality $\kappa'_1 \leqq \kappa_1$ over to all reciprocal (positive) eigenvalues, $\kappa'_n \leqq \kappa_n$. It is clear that the fixing of a membrane along the line l introduces a new restriction, and hence Courant's observation at once gives rise to the inequality $\lambda_n \leqq \lambda'_n$, derived before in another way, for the eigenvalues λ_n, λ'_n of the undivided and divided membrane. Anyone familiar with the abstract concept of Hilbert space who ponders a little more closely upon the situation to which the principle is applied here, will describe it as follows. All continuous functions $u(P)$ in the closed region S with continuous first derivatives in the interior, such that $u=0$ at the boundary and the Dirichlet integral $D\langle u\rangle = \int_S (\text{grad } u)^2 dP$ is finite, form a functional space $\mathfrak{H}$. Define the square $\|u\|^2$ of the length of a vector u in this space by $D\langle u\rangle$, and not by $I\langle u\rangle = \int_S u^2 dP$. The closure of $\mathfrak{H}$ with respect to this metric is a Hilbert space $\overline{\mathfrak{H}}$ and $I\langle u\rangle$ is a completely continuous quadratic form in $\overline{\mathfrak{H}}$. Without altering the other conditions for the functions $u \in \mathfrak{H}$ add the restriction $u=0$ along the line l. The closure of the subspace $\mathfrak{H}'$ thus obtained is a closed subspace $\overline{\mathfrak{H}}'$ of $\overline{\mathfrak{H}}$, and the relation $\kappa'_n \leqq \kappa_n$ holds for the reciprocal eigenvalues of the form $I\langle u\rangle$ in $\overline{\mathfrak{H}}$ and in $\overline{\mathfrak{H}}'$ respectively. (Forming the closure is essential since the eigenfunctions of the divided membrane are in $\overline{\mathfrak{H}}'$ but in general not in $\mathfrak{H}'$, because of the jump of their normal derivative along l. To be sure, by going back to the proof of the fundamental lemma, one can avoid this whole abstract set-up. But for the moment we are interested in the general formulation.)

Let then $K\langle u\rangle$ again be a completely continuous quadratic form in a Hilbert space $\mathfrak{G}$, and $\mathfrak{H}$ be a closed subspace of $\mathfrak{G}$. For simplicity's sake we assume $K\langle u\rangle$ to be positive-definite and denote the eigenvalues of $K\langle u\rangle$ in $\mathfrak{G}$ and in $\mathfrak{H}$ by λ_n and μ_n respectively. We have seen that

$$\lambda_n \leqq \mu_n. \tag{13}$$

Splitting $\mathfrak{G}$ into $\mathfrak{H}$ and its perpendicular subspace and choosing a basis $p_1, p_2, \cdots$ for the latter (preferably an orthonormal basis), we can pass from $\mathfrak{H}$ to $\mathfrak{G}$ by a sequence of intermediary subspaces $\mathfrak{H}\subset\mathfrak{H}'\subset\mathfrak{H}''\subset\cdots$, $\mathfrak{H}^{(\nu)}\rightarrow\mathfrak{G}$ with $\nu\rightarrow\infty$, by adding one vector of this basis after the other, $\mathfrak{H}^{(\nu)}=\mathfrak{H}+\{p_1, \cdots, p_\nu\}$. Or we can submit the vectors u of $\mathfrak{G}$ to one after the other of the conditions $(u, p_1)=0$, $(u, p_2)=0, \cdots$, and thus obtain a descending sequence $\mathfrak{G}\supset\mathfrak{G}'\supset\mathfrak{G}''\supset\cdots$ with $\mathfrak{G}^{(\nu)}\rightarrow\mathfrak{H}$ for $\nu\rightarrow\infty$. If λ_n is known, the inequality (13) gives a lower bound for μ_n, and the second of our sequences gives rise to an increasing sequence of such lower bounds,

$$\lambda_n \leqq \lambda_n' \leqq \lambda_n'' \leqq \cdots, \quad \lim_{\nu\rightarrow\infty} \lambda_n^{(\nu)} = \mu_n.$$

If μ_n is known, the same inequality gives an upper bound for λ_n, which by the first sequence of subspaces may be extended into a whole sequence of decreasing upper bounds, $\mu_n\geqq\mu_n'\geqq\mu_n''\geqq\cdots$, $\lim_{\nu\rightarrow\infty}\mu_n^{(\nu)}=\lambda_n$. With some right we may call the first procedure the Rayleigh-Ritz method and with more right ascribe the second to A. Weinstein [15].

The basic situation encountered here is that of a Hilbert space $\mathfrak{H}$ that splits into a subspace $\mathfrak{H}'$ and a perpendicular ν-dimensional space $\{p_1, \cdots, p_\nu\}$ spanned by the vector basis $p_1, \cdots, p_\nu$. N. Aronszajn recently developed two neat formulas concerning this situation [16]. I shall here mention but the one that corresponds to the Weinstein process, the descent from $\mathfrak{H}$ to $\mathfrak{H}'$, and omit the other which refers to the inverse Rayleigh-Ritz process. Let $R(\zeta)$ be the resolvent of the operator K in $\mathfrak{H}$, so that $u=R(\zeta)v$ is the solution of the equation $u-\zeta\cdot Ku=v$. Moreover let λ', u' be an eigenvalue and corresponding eigenvector of K' in $\mathfrak{H}'$. This fact is expressed by the relations

$$u' \in \mathfrak{H}', \qquad u' - \lambda' \cdot Ku' = \beta_1 p_1 + \cdots + \beta_\nu p_\nu.$$

If we suppose that λ' coincides with no eigenvalue λ_n of K then the constants β_i cannot all be zero. By means of the resolvent $R(\zeta)$ the second relation takes on the form $u' = \sum_{j=1}^{\nu} \beta_j \cdot R(\lambda')p_j$. The first relation requires that the vector u' is perpendicular to all the p_i,

$$\sum_j \beta_j(p_i, R(\lambda')p_j) = 0 \qquad (i, j = 1, \cdots, \nu).$$

These ν equations for the ν unknowns β_j have a nontrivial solution only if the determinant

$$\det (p_i, R(\zeta)p_j) = W_\nu(\zeta)$$

vanishes for $\zeta = \lambda'$. Divide $W_\nu(\zeta)$ by Gram's determinant $G_\nu = \det (p_i, p_j)$. The quotient

$$W(\zeta) = W_\nu(\zeta)/G_\nu$$

is clearly independent of the choice of the basis $p_1, \cdots, p_\nu$. From the theory of the resolvent one knows that $W(\zeta)$ is a meromorphic function with simple poles at the eigenvalues λ_n of K. On the other hand, we have seen that $W(\zeta)$ vanishes for a value $\zeta = \lambda'$ different from all the λ_n if and only if λ' is an eigenvalue of K' in $\mathfrak{H}'$. Hence we shall not find the following equation of Aronszajn too surprising:

$$W(\zeta) = \left(\frac{-1}{\zeta}\right)^{\nu} \cdot \prod_n \frac{\zeta - \lambda_n'}{\zeta - \lambda_n}.$$

I regret that shortness of time prevents me from illustrating these general developments by their applications to the classical problems of elastic and electromagnetic oscillations (**D**).

5. **Zeta-function of the membrane and asymptotic laws for its eigenfunctions.** The physicist will not be satisfied with a knowledge of the asymptotic behavior of the eigenvalues alone; also that of the eigenfunctions should be investigated. Carleman was the first to attack this more difficult problem by a new powerful method [**17**]. Last year we had the good fortune to have with us at the Institute Dr. Åke Pleijel, who had extended Carleman's investigations [**18**], and Dr. Minakshisundaram, who independently of Carleman had just found a modification of Carleman's method shedding new light on the whole problem. Let us again envisage the two-dimensional membrane. Carleman made use of the Green's function of the "meson" equation $\Delta u - k^2 u = 0$ with the positive parameter k^2 (for

the boundary condition $u=0$). Minakshisundaram [19] used instead Green's function $G(P, Q; t)$ of the heat equation

$$\Delta u - \partial u/\partial t = 0 \tag{14}$$

with the positive time parameter t.

$$u(P; t) = \int_S G(P, Q; t) \cdot f(Q) \cdot dQ$$

is the temperature of the disk S at the point P and at the moment t if $f(P)$ describes the initial distribution of temperature and the boundary of S is kept constantly on the temperature zero. For the infinite plane $G(P, Q; t)$ is

$$G_0(P, Q; t) = \frac{1}{4\pi t} \cdot \exp\left(-\frac{r_{PQ}^2}{4t}\right)$$

where r_{PQ} denotes the distance of the two points P, Q. For an arbitrary domain S we write

$$G(P, Q; t) = G_0(P, Q; t) - g(P, Q; t).$$

For a fixed P in the interior of S, the compensating term $g(P, Q; t)$ is a solution of (14) which vanishes for $t \to 0$ and on the surface S' of S has the same boundary values as the principal part G_0.

The maximum which the principal term G_0 assumes when P and t are given, but Q varies over the boundary S', is

$$H_0(P; t) = \frac{1}{4\pi t} \exp\left(-\frac{l_P^2}{4t}\right),$$

l_P denoting the shortest distance of P from the boundary. As a function of t this H_0 is on the increase from $t=0$ to $t=T=4/l_P^2$ and then decreases. A simple argument shows that, as long as $0<t\leqq T$, the compensating function $g(P, Q; t)$ of Q is positive and reaches its maximum at the boundary, or

$$0 < g(P, Q; t) \leqq H_0(P; t) \tag{15}$$

for all Q in S and $0<t\leqq T$.

In terms of the orthonormal system of eigenfunctions $\phi_n(P)$ and their eigenvalues λ_n Green's function is expressible as the sum

$$\sum_n e^{-\lambda_n t}\phi_n(P)\phi_n(Q). \tag{16}$$

The series

$$\zeta(P, Q; s) = \sum_{n} \frac{\phi_n(P)\phi_n(Q)}{\lambda_n^s} \tag{17}$$

may be called the ζ-function and accordingly (16) the θ-function of the membrane problem. We know that (17) converges uniformly in P and Q for all complex s the real part σ of which is greater than or equal to 2. For $s=1, 2, 3, \cdots$ the ζ-function gives Green's function $G(P, Q)$ of the membrane and its successive iterations.

One of Riemann's methods for deriving the properties of the ordinary ζ-function was based on a connection between the θ-function and the ζ-function, which at once carries over to the general functions here considered as follows:

$$\Gamma(s)\cdot\zeta(P, Q; s) = \int_0^\infty G(P, Q; t)\cdot t^{s-1}\cdot dt. \tag{18}$$

On the basis of this relation Minakshisundaram proves that $\Gamma(s) \cdot\zeta(P, Q; s)$ is a regular function of s in the entire s-plane if P, Q are two distinct inner points of S, but that it is regular except for a simple pole at $s=1$ with residue $1/4\pi$ if Q and P coincide. Still following Riemann, Minakshisundaram splits the integral $\int_0^\infty$ on the right of (18) into $\int_0^T+\int_T^\infty$. Because of the uniform convergence of $\zeta(P, Q; 2)$, the θ-function (16) falls off exponentially with $t\rightarrow\infty$, and hence the integral

$$\int_T^\infty G(P, Q; t)\cdot t^{s-1}dt$$

is a regular-analytic function of s in the whole s-plane. If $P\neq Q$ the formula

$$G(P, Q; t) = \frac{1}{4\pi t}\exp\left(-\frac{r_{PQ}^2}{4t}\right) - g(P, Q; t)$$

together with the estimate (15) proves that $G(P, Q; t)$ also goes down exponentially to zero with $1/t\rightarrow\infty$, and hence the integral $\int_0^T$ is likewise regular-analytic in s. This proves the result for $P\neq Q$. However if $Q=P$ then

$$G(P, P; t) = \frac{1}{4\pi t} - g(P, P; t).$$

The second part is positive and does not exceed $H_0(P; t)$; therefore

$\int_0^T g(P, P; t) \cdot t^{s-1} dt$ is regular in s, however

$$\int_0^T \frac{1}{4\pi t} t^{s-1} dt = \frac{1}{4\pi} \cdot \frac{T^{s-1}}{s-1}$$

has a pole at $s=1$ with the residue $1/4\pi$. That completes the proof. $\zeta(P, Q; s)$ itself has zeros for the same values for which the regular function $1/\Gamma(s)$ has zeros, namely for $s=0, -1, -2, \cdots$.

Integration over P, one might think, would give for

$$\zeta(s) = \int_S \zeta(P, P; s) \cdot dP = \sum_n \lambda_n^{-s}$$

the following result

$$\Gamma(s) \cdot \zeta(s) = \frac{V}{4\pi} \cdot \frac{1}{s-1} + \text{a regular function } R(s). \tag{19}$$

Unfortunately this conclusion is too hasty: if P is near the boundary, $H_0(P; t)$ is not so small. By carrying out the integration over P in the relation

$$G(P, P; t) = \frac{1}{4\pi t} - g(P, P; t), \qquad 0 \leqq g(P, P; t) \leqq H_0(P; t),$$

one finds that the remainder $R(s)$ in (19) is regular at least for $\sigma > 1/2$; it seems difficult to go beyond the vertical $\sigma = 1/2$.

Standard devices familiar from the theory of Riemann's ζ-function permit one to deduce from this behavior of the ζ-function of the membrane Carleman's asymptotic formulas

$$\sum_{\lambda_n \leqq \lambda} (\phi_n(P))^2 \sim \lambda/4\pi, \qquad N(\lambda) = \sum_{\lambda_n \leqq \lambda} 1 \sim V\lambda/4\pi$$

and also the "incoherence relation"

$$\sum_{\lambda_n \leqq \lambda} \phi_n(P)\phi_n(Q) = o(\lambda) \qquad \text{for } P \neq Q.$$

I feel that these informations about the proper oscillations of a membrane, valuable as they are, are still very incomplete. I have certain conjectures on what a complete analysis of their asymptotic behavior should aim at; but since for more than 35 years I have made no serious attempt to prove them, I think I had better keep them to myself.

In general, it can not be expected that our ζ-function satisfies a functional equation of the Riemann type; one may guess that this feature depends on the homogeneity of the domain of integration. Such a domain is the circumference of the unit circle. Functions on it are functions $f(x)$ of period 2π. The periodic eigenfunctions ϕ and corresponding eigenvalues λ of $d^2\phi/dx^2+\lambda\phi=0$ are

$$\phi(x) = e^{inx}, \qquad \lambda_n = n^2 \qquad\qquad (n = 0, \pm 1, \pm 2, \cdots).$$

This leads straight to the Riemann ζ-function $\sum_{n=1}^{\infty} n^{-2s}$ usually denoted not by $\zeta(s)$ but by $\zeta(2s)$. It was therefore natural that Minakshisundaram should investigate the spherical harmonics on a k-dimensional sphere (in $k+1$-dimensional space). Here he found indeed a sort of Riemann functional equation, the structure of which is, however, essentially more complicated than in the classical Riemann case $k=1$.

The two-sphere is homogeneous because it permits a compact transitive Lie-group σ of transformations s into itself, namely the group of rotations. The spherical harmonics of order l form a $(2l+1)$-dimensional linear manifold that is invariant with respect to the group of rotations and has the property of irreducibility in this regard. Consider arbitrary (complex-valued) continuous functions on the sphere and define the scalar product of two such functions f and g by the integral $\int \bar{g}(P)f(P)\cdot d\omega_P$ formed by means of the invariant area element $d\omega_P$. It is obvious how to generalize this situation to any homogeneous manifold S of points P, that is, any manifold that permits a compact transitive Lie-group σ of transformations s, $P\to sP$. The existence of an invariant volume element on such a manifold (which itself is of necessity compact) follows easily from the fact that a compact Lie-group has an invariant volume element ds. We normalize the unit for measuring volumes on the group so that the total volume $\int ds$ of the group becomes 1. The integrals with respect to s are then in truth mean values. The transform sf of a function $f=f(P)$ on S is defined by $sf(sP)=f(P)$ or $sf(P)=f(s^{-1}P)$. A set $\phi_1(P), \cdots, \phi_h(P)$ of functions on S, or the manifold of their linear combinations $\phi(P)=x_1\cdot\phi_1(P)+ \cdots +x_h\cdot\phi_h(P)$, is invariant if each $s\phi_i(P)$ is a linear combination $\sum_j \omega_{ji}(s)\cdot\phi_i(P)$ of the ϕ_i themselves. Then $s\to\|\omega_{ij}(s)\|$ is a representation of degree h of the group σ. Inequivalent irreducible invariant sets are orthogonal to each other. Besides orthogonality there is the completeness relation, to which we shall presently return. Thus we are in possession of the "eigenfunctions" of the homogeneous manifold, the sequence of which is subdivided into irreducible invariant sets of finite length. Theorems about summability of expansions in terms of these eigenfunctions have been proved by S. Bochner [20]. But so far they are eigenfunctions without eigenvalues. It was the young Dutch physicist H. B. G. Casimir who, prompted by the applications of group theory to quantum mechanics, found the eigenvalues. Indeed he constructed an invariant self-adjoint differential operator Δ working on arbitrary functions $f(P)$ which is the analogue of the Laplace operator on a sphere, and he was able to show that the functions of a given irreducible invariant set satisfy an equation $\Delta\phi+\lambda\phi=0$ with a constant λ characteristic for the entire set [21]. Having the eigenfunctions, one can, following a suggestion by Bochner, form the ζ-function

$$\zeta(P, Q; z) = \sum_n \frac{\phi_n(P)\phi_n(Q)}{\lambda_n^z}$$

and might expect that this function, in addition to having the properties quite generally established by Minakshisundaram, will satisfy a functional equation of Riemann's type. But this is a question that remains to be investigated.

6. Integral equations and the group-theoretic completeness relation. The proof of the completeness relation for invariant sets on a homogeneous manifold S is one of the most surprising applications of the eigenvalue theory of integral equations. If the manifold S is the compact Lie group itself under the influence of its left translations, then this theorem states the completeness of the totality of all irreducible representations of the group. But the method for its proof, developed in 1927 by F. Peter and the speaker [22], not only carries over to the homogeneous manifolds, but applies to a far more general situation, that is best described in axiomatic terms [23]. We replace the functions on the homogeneous manifold by vectors f in a vector space Σ and suppose that Σ bears a Hermitian metric defined by a scalar product (g, f) with the usual properties including the positive character of $(f,f)=\|f\|^2$. An abstract compact Lie group σ is given and a representation of its elements s by linear transformations $f\rightarrow sf$ in our vector space. The invariance of the metric is assumed, $(sg, sf) = (g,f)$. Let f be a given vector. All vectors that will occur in our construction are prepared from f by forming linear combinations of its transforms sf. Besides Σ we envisage the "vector space" Ξ of all continuous functions $\xi=\xi(s)$ on the group manifold σ and define a linear mapping $\xi\rightarrow g$ of Ξ onto Σ by

$$g = \mathfrak{f}\xi = \int \xi(s)\cdot sf\cdot ds,$$

and its Hermitian conjugate, a mapping $g\rightarrow\xi$ of Σ onto Ξ, by

$$\xi = \mathfrak{f}^* g = (sf, g).$$

The mapping $\mathfrak{f}^*\mathfrak{f} = \mathfrak{F}$ of Ξ into itself has the positive-definite Hermitian kernel $H(s, t) = (sf, tf)$. By Erhard Schmidt's method we construct its largest reciprocal eigenvalue γ and an orthonormal set of eigenfunctions $\phi_1(s), \cdots, \phi_h(s)$ for it. Repetition of the construction gives the reciprocal eigenvalues in descending order, $\gamma > \gamma' > \cdots$, and the sought-for completeness relation results from the well known fact that the trace of H equals the sum of the reciprocal eigenvalues,

$$\|f\|^2 = (f, f) = h\gamma + h'\gamma' + \cdots .$$

Indeed $\int H(s, t)\cdot\phi_i(t)\cdot dt = \gamma\cdot\phi_i$ may be written in the form

$$\mathfrak{f}\phi_i = \gamma^{1/2}\cdot g_i, \qquad \mathfrak{f}^* g_i = \gamma^{1/2}\cdot\phi_i$$

where the first equation is to be taken as the definition of the vector g_i. The g_i then form an orthonormal invariant set, and if the Fourier coefficients $(g_i, f) = \alpha_i$ are introduced one finds that

$$h\gamma = |\alpha_1|^2 + \cdots + |\alpha_h|^2.$$

Thus the completeness relation follows, stating that the orthonormal sequence $g_1, g_2, \cdots, (g_m, g_n) = \delta_{mn}$, resulting from our construction and consisting of sections of finite length, each of which is an invariant set, makes the absolute square sum $|\alpha_1|^2 + |\alpha_2|^2 + \cdots$ of the Fourier coefficients $\alpha_i = (g_i, f)$ not only $\leq \|f\|^2$, as is trivial (Bessel's inequality), but actually $= \|f\|^2$. The construction picks out those invariant sets that contribute to f and $\|f\|^2$.

This feature is quite essential when, with Harald Bohr and J. von Neumann [24], all restrictions concerning the group σ are abandoned. The construction still works, provided one supposes f to be "almost periodic." But in general there are under these circumstances more than denumerably many inequivalent irreducible invariant sets of vectors; but f itself picks out those among them that matter for f.—The assumption of almost periodicity is highly restrictive. One may instead impose some slight restriction on the group, e.g., local compactness, and at the same time admit a far wider class of vectors f. In that case nothing resembling completeness is to be expected unless one includes also representations of infinite degree. This step has recently been taken by D. Rykov and I. Gelfand in Russia, by V. Bargmann and I. Segal in this country [25].

I think it is time for me to stop here. I have not even touched on the extension of Hilbert's theory of bounded to non-bounded linear operators, which came about under the pressure of quantum mechanics, nor to the connection between spectral decomposition and ergodic theory. Other mathematicians at other times have spoken or will speak on these subjects with more competence than I could. I hope you have taken this lecture for what it was meant to be: a *Plauderei*, the chat of a man who has reached the age where it is more pleasant to remember the past than to look forward into the future. Even so, it gives him a little satisfaction to see that the issues to which the efforts of his youth were dedicated have kept alive over the years and are still in the process of unfolding their implications.

NOTES

(A) There are two classes of eigenfunctions. But since one of them does not contribute to the expansion of the discontinuous function $1^0(x)$, only the eigenvalues λ^2 belonging to the other class have to be taken into account; they are determined by the transcendental equation

$$\beta \cdot \tan \frac{\alpha\lambda\pi}{2} + \alpha \cdot \tan \frac{\beta\lambda\pi}{2} = 0.$$

It is easily seen that for every integer n this equation has exactly one root λ_n of the form $\lambda_n = 2n + \theta_n$, $-1 < \theta_n < 1$. If α and β are rational,

$$\alpha = a/c, \quad \beta = b/c, \quad a + b = c; \quad a, b, c \text{ integers}, \quad c > 0,$$

then θ_n has the period c, $\theta_{n+c} = \theta_n$, and this circumstance makes a fairly explicit evaluation of the nth partial sum $1_n^0(x)$ possible.

(B) The main fact is as follows: The given values $\beta_1 = w(\alpha_1)$, $\beta_2 = w(\alpha_2)$, $\cdots$ have to satisfy a sequence of inequalities of which the first, $\mathfrak{I}\beta_1 > 0$, involves only β_1, the second β_1 and β_2, and so on. If these inequalities are fulfilled then there are two possibilities, which are distinguishable by a convergence criterion. In the first, the limit point case, the problem has a unique solution; in the second, the limit circle case, the manifold of all solutions $w(\lambda)$ is obtained from that of all positive functions $z(\lambda)$ by a certain Möbius transformation

$$w(\lambda) = \frac{A(\lambda) \cdot z(\lambda) + B(\lambda)}{C(\lambda) \cdot z(\lambda) + D(\lambda)}$$

with coefficients A, B, C, D that are regular analytic functions of λ in the upper half λ-plane.

(C) Indeed replacing λ_0, λ by λ, $\bar{\lambda}$ in (6) one finds

$$\mathfrak{I}\phi(s) = \mathfrak{I}\lambda \cdot \int_0^l G(s, t) \cdot \bar{\phi}(t) \cdot dt,$$

and therefore $\mathfrak{I}\phi(s)$ has a uniformly convergent expansion in terms of the eigenfunctions $\eta(s;\lambda_n)$. For the integral $\int_0^1\phi(s)\eta(s;\lambda_n)\cdot ds$ one obtains from (3) the value $1/(\lambda_n-\lambda)$.

(D) The eigenvalues $\lambda=\mu_n$ of $\Delta u+\lambda u=0$ corresponding to the boundary condition $\partial u/\partial n=0$ (normal derivative of u equal to zero; "acoustic eigenvalues") are the reciprocals of the successive maxima of $I\langle u\rangle$ under the restriction $D\langle u\rangle=1$, or the successive minima of $D\langle u\rangle$ under the restriction $I\langle u\rangle=1$. The boundary condition $\partial u/\partial n=0$ gets lost, as it were, in the process of closure under the metric defined by $D\langle u\rangle$. Hence $\mu_n\leqq\lambda_n$, where λ_n, as before, are the membrane eigenvalues corresponding to the boundary condition $u=0$.

The equation for the oscillations of a plate,

$$\Delta\Delta u - \lambda^2\cdot u = 0 \qquad \text{in } S,$$

arises from minimizing

$$\int_S (\Delta u)^2\cdot dP \tag{20}$$

under the auxiliary condition $I\langle u\rangle=1$. Let μ_n^2 be the eigenvalues of the clamped plate,

$$\text{boundary conditions } u = 0, \quad \frac{\partial u}{\partial n} = 0,$$

and λ_n^2 those for the "half-free" plate,

$$\text{boundary conditions } u = 0, \quad \Delta u = 0. \tag{21}$$

Again the boundary condition $\Delta u=0$ gets lost in the process of closure under the metric defined by (20). Hence $\lambda_n^2\leqq\mu_n^2$; one can further expect that λ_n and μ_n follow the same asymptotic law. Weinstein observed that the eigenvalues of the half-free plate (as their notation indicates) are simply the squares of the membrane eigenvalues. Indeed if $\Delta\Delta u=\lambda^2 u$ $(\lambda>0)$ set $\Delta u=-\lambda v$, so that $\Delta v+\lambda u=0$. The boundary conditions (21) give $u=0$, $v=0$ along S', hence $(u+v)/2$ is a membrane eigenfunction with the eigenvalue λ and $(u-v)/2$ for $-\lambda$. But the membrane has no negative eigenvalues; consequently $u-v=0$ and $(u+v)/2=u$.

It was by a somewhat similar remark that I had previously reduced the elastic oscillations of a three-dimensional body asymptotically to the three-dimensional membrane problem [26]. The vector field $\mathfrak{v}(P)$ describing a proper oscillation of the elastic body satisfies an equation

$$(\mathrm{E}) \qquad a\cdot \operatorname{grad}\operatorname{div}\mathfrak{v} - b\cdot \operatorname{rot}\operatorname{rot}\mathfrak{v} + \lambda\mathfrak{v} = 0 \qquad \text{in } S$$

with two elastic constants a, b. Impose the boundary conditions

$$(\mathrm{E}') \qquad \mathfrak{v} \text{ normal,} \qquad \operatorname{div}\mathfrak{v} = 0 \quad \text{on the surface } S' \text{ of } S.$$

For $\phi = \operatorname{div}\mathfrak{v}$ one finds $a\cdot\Delta\phi + \lambda\phi = 0$ in S and the boundary condition $\phi = 0$; hence if ϕ is an eigenfunction of the membrane problem with the eigenvalue λ/a then $\mathfrak{v} = \operatorname{grad}\phi$ satisfies (E), (E′). If, however, $\operatorname{div}\mathfrak{v} = 0$ throughout S we have

$$b\cdot\Delta\mathfrak{v} + \lambda\mathfrak{v} = 0, \qquad \operatorname{div}\mathfrak{v} = 0 \text{ in } S; \qquad \mathfrak{v} \text{ normal on } S'$$

(since $\Delta\mathfrak{v} = \operatorname{grad}\operatorname{div}\mathfrak{v} - \operatorname{rot}\operatorname{rot}\mathfrak{v}$), or λ/b is an eigenvalue of the problem of radiation in a Hohlraum S whose wall S' is a perfect mirror. Denoting the numbers of eigenvalues $\leqq\lambda$ of the membrane, the radiation, and the elastic problem (E) & (E′) by $N_m(\lambda)$, $N_r(\lambda)$ and $N_e(a,b;\lambda)$ respectively, we thus find the relation

$$N_e(a, b; \lambda) = N_m(\lambda/a) + N_r(\lambda/b).$$

For $a = b = 1$ the left side of (E) turns into $\Delta\mathfrak{v} + \lambda\mathfrak{v}$. Since asymptotically the boundary conditions are of no influence we must have the asymptotic relation

$$N_e(1, 1; \lambda) \sim 3N_m(\lambda), \quad \text{that is,} \quad N_m(\lambda) + N_r(\lambda) \sim 3N_m(\lambda)$$

or $N_r(\lambda) \sim 2N_m(\lambda)$ and thus

$$N_e(a, b; \lambda) \sim N_m(\lambda/a) + 2N_m(\lambda/b).$$

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THE ANNUAL MEETING OF THE SOCIETY

The fifty-sixth Annual Meeting of the American Mathematical Society was held at Columbia University, New York City, Tuesday to Thursday, December 27–29, 1949, in conjunction with the annual meetings of the Mathematical Association of America and the American Association for the Advancement of Science. Registration headquarters for the Society and the Association were in the Lobby of Earl Hall, Columbia University. Residence headquarters were at the Hotel Governor Clinton. Over seven hundred persons registered, including the following six hundred sixty nine members of the Society (a record attendance for an Annual meeting):

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Prim, Hans Rademacher, Tibor Rado, H. V. Rådstrom, L. R. Raines, Gordon Raisbeck, K. G. Ramanathan, O. J. Ramler, J. F. Randolph, G. N. Raney, H. E. Rauch, L. T. Ratner, G. E. Raynor, M. J. Rees, M. S. Rees, C. F. Rehberg, Irving Reiner, I. M. Reiner, Eric Reissner, G. W. Reitwiesner, Russell Remage, R. W. Rempfer, Daniel Resch, Helene Reschovsky, C. N. Reynolds, D. P. Richardson, Moses Richardson, C. E. Rickart, P. R. Rider, R. F. Rinehart, J. F. Ritt, R. K. Ritt, E. K. Ritter, H. E. Robbins, R. E. Roberson, J. H. Roberts, R. A. Roberts, J. E. Robinson, S. L. Robinson, V. N. Robinson, C. A. Rogers, Saul Rosen, Ira Rosenbaum, Murray Rosenblatt, P. C. Rosenbloom, J. H. Rosenbloom, M. A. Rosenlicht, Arthur Rosenthal, J. B. Rosser, M. F. Rosskopf, H. L. Royden, Herman Rubin, Herbert Ruderfer, Walter Rudin, C. W. Saalfrank, Bernard Sachs, Charles Salkind, Charles Saltzer, J. E. Sammet, R. G. Sanger, Arthur Sard, S. S. Saslaw, F. E. Satterthwaite, A. C. Schaeffer, A. T. Schafer, R. D. Schafer, Robert Schatten, Samuel Schecter, Henry Scheffé, Eugene Schenkman, Albert Schild, E. R. Schneckenburger, J. P. Scholz, K. C. Schraut, Pincus Schub, A. L. Schurrer, Abraham Schwartz, B. L. Schwartz, G. E. Schweigert, C. H. W. Sedgewick, I. E. Segal, Wladimir Seidel, George Seifert, D. B. Shaffer, H. S. Shapiro, I. M. Sheffer, L. W. Sheridan, Seymour Sherman, L. P. Siceloff, K. M. Siegel, J. A. Silva, L. L. Silverman, Annette Sinclair, James Singer, M. H. Slud, L. L. Smail, P. A. Smith, R. E. Smith, W. M. Smith, Ernst Snapper, A. K. Snyder, W. S. Snyder, Andrew Sobczyk, E. S. Sokolnikoff, J. J. Sopka, D. E. South, E. H. Spanier, D. E. Spencer, V. E. Spencer, M. R. Spiegel, A. H. Sprague, E. R. Stabler, E. P. Starke, J. R. K. Stauffer, M. P. Steele, N. E. Steenrod, I. A. Stegun, S. K. B. Stein, Fritz Steinhardt, Wolfgang Sternberg, R. W. Stokes, R. R. Stoll, R. C. Strodt, Walter Strodt, D. J. Struik, D. M. Studley, E. A. Sturley, Gabor Szego, T. T. Tanimoto, Olga Taussky-Todd, J. S. Taylor, M. E. Taylor, A. J. Terzuoli, Feodor Theilheimer, J. M. Thomas, D. L. Thomsen, W. J. Thron, Michael Tikson, G. L. Tiller, John Todd, L. F. Tolle, C. B. Tompkins, M. M. Torrey, J. I. Tracey, W. R. Transue, H. M. Trent, W. J. Trjitzinsky, C. A. Truesdell, A. W. Tucker, J. W. Tukey, Annita Tuller, J. L. Ullman, F. E. Ulrich, Eugene Usdin, E. P. Vance, J. L. Vanderslice, Henry Van Engen, H. E. Vansant, A. H. Van Tuyl, S. I. Vrooman, M. C. Waddell, D. H. Wagner, G. L. Walker, S. E. Walkley, J. L. Walsh, R. M. Walter, W. G. Warnock, D. R. Waterman, G. C. Webber, J. V. Wehausen, Alexander Weinstein, Louis Weisner, M. J. Weiss, F. P. Welch, David Wellinger, J. G. Wendel, E. H. Wetherell, F. J. Weyl, George Whaples, M. E. White, D. E. Whitford, P. M. Whitman, A. M. Whitney, D. R. Whitney, Hassler Whitney, G. T. Whyburn, L. S. Whyburn, W. M. Whyburn, D. V. Widder, V. A. Widder, N. A. Wiegmann, Norbert Wiener, Albert Wilansky, E. S. Wolk, Y. K. Wong, M. Y. Woodbridge, M. A. Woodbury, Arthur Wouk, F. M. Wright, M. A. Wurster, Bertram Yood, D. M. Young, J. W. Young, J. W. T. Youngs, N. J. Zabb, L. A. Zadeh, Oscar Zariski, Arthur Zeichner, Daniel Zelinsky, P. W. Zettler-Seidel, A. D. Ziebur, J. A. Zilber, H. J. Zimmerberg, Leo Zippin, Antoni Zygmund.

The twenty-third Josiah Willard Gibbs Lecture, entitled *The problem of sensory prosthesis*, was delivered by Professor Norbert Wiener of the Massachusetts Institute of Technology at 4:30 P.M., Wednesday, December 28. Professor J. L. Walsh, President of the American Mathematical Society, was the presiding officer.

At a joint session of Section A and the Society at 2:00 P.M., Tuesday, December 27, Professor G. T. Whyburn of the University of

Virginia gave an address, *The open mapping medium in topological analysis*, as a retiring Vice President of the American Association for the Advancement of Science. Professor E. J. McShane, Chairman of Section A, presided.

The Committee to Select Hour Speakers for Annual and Summer Meetings invited two speakers. On Wednesday, December 28, Professor Sarvadaman Chowla of the University of Kansas gave an address on *The Riemann zeta and allied functions* at 9:30 A.M. Vice President W. T. Martin presided. At 2:00 P.M. Professor L. V. Ahlfors of Harvard University gave an address on *The classification of open Riemann surfaces*. Vice President Hassler Whitney presided.

The annual Business Meeting and Election of Officers was held at 10:00 A.M., Thursday, December 29, with President J. L. Walsh in the chair. Details of proceedings are reported in the sequel. The Cole Prize in Algebra was awarded to Professor Richard Brauer of the University of Michigan for his work in group theory and in particular for the following paper: *On Artin's L-series with general group characters*, Ann. of Math. (2) vol. 48 (1947) pp. 502–514. Professor Brauer gave a brief address on his prize-winning paper.

A tea for members of the mathematical organizations and their guests was given by Columbia University at 4 P.M. Tuesday at the Faculty House.

On Wednesday evening at the Casa Italiana, there was a musicale by Mr. and Mrs. Everett Anderson and Mr. Harold Triggs of the Columbia University Department of Music.

There was a conducted tour of the Museum of Modern Art on Thursday afternoon.

The dinner Thursday evening in the John Jay Hall Dining Room was attended by 240 members and guests. The toastmaster, Professor P. A. Smith, introduced Dr. George B. Pegram, Vice President of Columbia University who gave a brief address of welcome. He was followed by Professor J. L. Walsh, President of the American Mathematical Society; Dr. H. M. MacNeille, Executive Director of the American Mathematical Society; Professor Jerzy Neyman, President of the Institute of Mathematical Statistics; and Professor R. E. Langer, President of the Mathematical Association of America. The toastmaster then introduced Professor M. R. Hestenes who offered resolutions of thanks and appreciation to Columbia University, the Committee on Arrangements, and all who had helped make the meetings successful and enjoyable. This resolution was adopted by a rising vote.

The Council met at 6:30 P.M. on December 28, 1949.

The Secretary announced the election of the following one hundred and thirty-one persons to ordinary membership in the Society:

Mr. John Pruyn van Alstyne, Hamilton College, Clinton, N.Y.;
Professor Henry Adalbert Antosiewicz, Montana State College;
Mr. Elihue Barden, Alcorn College, Alcorn, Miss.;
Mr. James Earl Barney, II, University of Kansas;
Dr. Howard Gordon Bergmann, City College, New York, N.Y.;
Miss Irma Berkowitz, University of Alabama;
Mr. Alvin Kilian Bettinger, Creighton University, Omaha, Neb.;
Miss Laura Blakeley, Armstrong College, Savannah, Ga.;
Dr. Felix Earl Browder, Massachusetts Institute of Technology;
Mr. Beverley Merl Brown, Tusculum College, Greeneville, Tenn.;
Mr. Howard Harry Brown, Franklin Institute, Philadelphia, Pa.;
Mr. Robert William Bryant, Miami University, Oxford, Ohio;
Dr. J. Richard Buchi, Ripon College, Ripon, Wis.;
Mr. Alberto Pedro Calderón, University of Chicago;
Mr. Jean Mitchener Calloway, University of Pennsylvania;
Mrs. Dorothy Wolking Campbell, University of Wisconsin;
Mr. Richard Crawford Campbell, University of Colorado;
Mr. George Casseb, Saint Mary's University of Texas, San Antonio, Tex.;
Mr. Bruce Bampton Clark, Grinnell College, Grinnell, Iowa;
Professor Willie E. Clark, Agricultural, Mechanical and Normal College, Pine Bluff, Ark.;
Mr. Daniel Edmund Coffey, Montana State University;
Mr. Warren Alan Couch, Washington University;
Mr. John Norris Crawford, Texas Southmost College, Brownsville, Tex.;
Professor Dorothy Ada Curry, Wilberforce University, Wilberforce, Ohio;
Professor Rene Joseph De Vogelaere, Laval University, Quebec, Que., Canada;
Mr. Robert Benjamin Davis, Massachusetts Institute of Technology;
Dean Louis Aspell Deesz, Youngstown College, Youngstown, Ohio;
Mr. Leroy John Derr, Tulane University of Louisiana;
Mr. Raymond Allen Dibrell, Jr., Southwest Texas State College, San Marcos, Tex.;
Mr. Robert John Dickson, Jr., California Institute of Technology;
Professor Israel Edward Drabkin, City College, New York, N.Y.;
Mr. Georges Roland Dubé, Yale University;
Mr. James Eells, Jr., Amherst College, Amherst, Mass.;
Professor Gustav Elfving, University of Helsingfors, Helsingfors, Finland;
Mr. John W. Forman, University of Kansas;
Mr. Harold Forstat, Purdue University;
Mr. Colonel Douglas Gardner, Okolona College, Okolona, Miss.;
Mr. Eugene Joseph Germino, Saint John's University, Brooklyn, N.Y.;
Mr. Walter Adelbert Glass, Drew University, Madison, N.J.;
Mr. William Goldstein, State Teachers College, Trenton, N.J.;
Mr. Jacob Festus Golightly, Jacksonville Junior College, Jacksonville, Fla.;
Mr. Edward Gordon Goman, College of Puget Sound, Tacoma, Wash.;
Miss Lillian Gough, University of Buffalo;
Mr. Richard Briggs Grekila, University of Maryland;
Mr. Vincent Harold Haag, Franklin and Marshall College, Lancaster, Pa.;
Mr. Herman David Hartstein, School of Business, Washington University;
Mr. Arthur Hendler, Rensselaer Polytechnic Institute, Troy, N.Y.;

Professor Boyd Herbert Henry, Parsons College, Fairfield, Iowa;
Mr. John Stephenson Hicks, Socony-Vacuum Laboratories, Paulsboro, N.J.;
Mr. Harley A. Hill, Portland, Ore.;
Miss Jean E. Hirsh, Queens College, Flushing, N.Y.;
Mr. Cullen Squaere Hodge, Fisk University, Nashville, Tenn.;
Mr. Robert James Howerton, Regis College, Denver, Colo.;
Dr. Sze Tsen Hu, Tulane University of Louisiana;
Professor Gilbert Agnew Hunt, Cornell University;
Mr. Edward Ormond Hynard, Manhattan College, New York, N.Y.;
Professor Wendell Gilbert Johnson, Phillips University, Enid, Okla.;
Mr. John Ernest Kelley, University of Miami, Coral Gables, Fla.;
Mrs. Rachel Geck Kirkendall, Highland Park School, Highland Park, Mich.;
Mr. John Sharpless Klein, Williams College, Williamstown, Mass.;
Mr. Karl John Klein, Loras College, Dubuque, Iowa;
Mr. Micha Klein, Mekoroth Water Co., Ltd., Tel-Aviv, Israel;
Dr. Paul Koditschek, Long Island University, Brooklyn, N.Y.;
Mr. Allan Theodore Kovitz, Polytechnic Institute of Brooklyn;
Mr. Andrew V. Kozak, Concord College, Athens, W. Va.;
Professor Anthony Edward Labarre, Jr., University of Idaho, Moscow, Idaho;
Mr. James Richard Larkin, University of Kansas, Lawrence, Kan.;
Mr. Sim Lasher, Chicago, Ill.;
Mr. Eric Liban, New York University;
Mr. Johnathan Wattam Lindsay, Texas Technological College, Lubbock, Tex.;
Professor Wilhelm Ljunggren, University of Bergen, Bergen, Norway;
Mr. Manice de Forest Lockwood, III, New York, N.Y.;
Dr. Henry Francis Joseph Lówig, University of Tasmania, Tasmania, Australia;
Dr. George Lorentz, University of Toronto;
Professor Fred Wilbur Lott, Jr., Iowa State Teachers College, Cedar Falls, Iowa;
Mr. Cyrus Ray McAllister, University of Idaho, Moscow, Idaho;
Mr. Charles Wilson McArthur, Tulane University of Louisiana;
Miss June Marie McArtney, University of Buffalo;
Dean (Henry) Stirling McCall, Norman College, Norman Park, Ga.;
Mr. (James) Edward McGaughy, Lawrence College, Appleton, Wis.;
Mr. Kenneth M. McMillin, University of Minnesota;
Mr. Nathaniel Macon, University of North Carolina;
Professor David Middleton, Physics Department, Harvard University;
Mr. Henri Monik, Detroit Institute of Technology;
Miss Mary Prudence Moseley, Southern Methodist University, Dallas, Tex.;
Miss Martha Rheuba Mouser, Wilson Teachers College, Washington, D.C.;
Mr. Howard Warren Moyer, Lost City, W. Va.;
Mr. Robert George Needels, Loyola University of Los Angeles;
Mr. Edward Peter Neuburg, University of Vermont and State Agricultural College;
Miss Virginia Anne Ohlson, Purdue University, Lafayette, Ind.;
Mr. Clellie Curtis Oursler, Gary Center, Indiana University;
Dr. Ting Kwan Pan, University of California, Berkeley, Calif.;
Mr. Kenneth Stephen Phelan, Saint John's University, Brooklyn, N.Y.;
Professor Willis Lloyd Pickard, State University of New York, New Paltz, N.Y.;
Mr. John F. Purdy, Goodyear Tire and Rubber Co., Akron, Ohio;
Mr. Edward Roy Rang, University of Wisconsin;
Mr. Roy Franklin Reeves, Iowa State College of Agriculture and Mechanic Arts;
Professor Thomas Robert Richards, Wilkes College, Wilkes-Barre, Pa.;

Professor Charles Lathan Riggs, Kent State University, Kent, Ohio;
Dr. Julien Ashton Ripley, Jr., Physics Dept., Montgomery Junior College, Bethesda Md.;
Miss Oral Maye Robbins, State Teachers College, Milwaukee, Wis.;
Mr. David Rosen, University of Pennsylvania;
Professor Jules Perry Russell, Polytechnic Institute of Brooklyn;
Miss Cecile Blanche Salwen, Hunter College;
Mr. Leonard Schieber, Jr., Lees-McRae College, Banner Elk, N.C.;
Mr. Robert Charles Seber, Rockford College, Rockford, Ill.;
Mr. Bernard David Seckler, Long Island University, Brooklyn, N.Y.;
Mr. Nathan Thomas Seeley, Jr., Agricultural, Mechanical and Normal College, Pine Bluff, Ark.;
Dr. Esther Seiden, University of California, Berkeley, Calif.;
Mr. Sharad Shankar Shrikhande, College of Science, Nagpur, India;
Mr. Luiz Oswaldo Teixeira de Silve, Polytechnic School, Catholic University, Rio de Janeiro, Brazil;
Professor Ralph Axel Staal, University of New Brunswick;
Mr. George Henry Stanley, Jr., Colby College, Waterville, Me.;
Mr. David Sternberg, Physics Department, Columbia University;
Professor Joseph Talacko, Marquette University, Milwaukee, Wis.;
Mr. Nien Yee Tang, University of Washington;
Professor Abdelnour Simon Thomas, Boston College, Chestnut Hill, Mass.;
Mr. Morris Edward Tittle, Agricultural and Mechanical College of Texas, College Station, Tex.;
Mr. John Edward Vollmer, University of Maryland, College Park, Md.;
Professor Enrico Giovanni Volterra, Illinois Institute of Technology;
Mr. Frank Joseph Wagner, Creighton University, Omaha, Neb.;
Mr. Charles Kenneth Wasserman, University of Idaho, Moscow, Idaho;
Miss Betty Rose Weber, University of South Carolina;
Mr. Edwin Weiss, Massachusetts Institute of Technology;
Mr. William Joseph Wells, State Teachers College, Mankato, Minn.;
Mr. George Nichols White, Jr., Brown University;
Professor Mary Elizabeth Williams, Skidmore College, Saratoga Springs, N.Y.;
Mr. Robert Joel Wisner, University of Washington;
Mr. George E. Witter, University of Idaho, Moscow, Idaho;
Professor Gerhard Norval Wollan, North Georgia College, Dahlonega, Ga.;
Sister Mary Rupert Zeiser, Mount Saint Clare Academy, Clinton, Iowa.

It was reported that the following one hundred forty-one persons had been elected to membership on nomination of institutional members as indicated:

University of Alabama: Professor James Clifton Eaves, Mr. Judson James Hart;
University of British Columbia: Mr. James Lewin McGregor;
Brooklyn College: Mr. Leonard Geller;
Brown University: Dr. Jacques Heyman, Mr. John Allen Lewis;
University of California, Berkeley: Messrs. Hewitt Kenyon and Maurice Sion;
University of California, Los Angeles: Messrs. James Richard Jackson and Elmer Edwin Osborne;
California Institute of Technology: Messrs. Benjamin Bernholtz, Robert Yost Dean, William Thomas Guy, Robert Hunter Owens, Carleton Bernard Solloway, and Richard Bennett Talmadge;

University of Chicago: Messrs. Leonard David Berkovitz, Murray Gerstenhaber, Oscar Irving Litoff, Ernest Arthur Michael, and I. Alex Rosenberg;

University of Cincinnati: Professor Emilio Baiada;

City College, New York: Messrs. Robert John Aumann, Leon Ehrenpreis, Carl Engelman, and Leopold Flatto;

Columbia University: Messrs. Maurice Auslander and Alfred Lorenzo Duquette, Miss Alice Krikorian, Mr. Burt Jules Morse;

Cornell University: Mr. Jerome Blackman, Miss Joanne Elliott, Messrs. Carl Samuel Herz and Ralph Gordon Selfridge;

Harvard University: Messrs. Henry Nelson Browne, Jr., Robert Lawrence Graves, Alan Francis Kay, Lawrence Friedman Markus, Robert Osserman, Eoin Laird Whitney, and Norman Zachary Wolfsohn;

Illinois Institute of Technology: Messrs. Robert Alonzo Eubanks and Nathaniel Roy Goodman, Miss Elizabeth Catherine Kleinhans, Messrs. Howard S. Levin, Jack Milton Miller, Seymour Victor Parter, Stewart Schlesinger, Leo Andrew Schmidt, and William Barry Snarr;

University of Illinois: Mr. Martin Nixon Chase, Miss Marjorie Baldwin Djorup, Messrs. Hans Carl Flesch, Donald Gordon Higman, Joseph Hope Hornback, Mitsuru Kadoyama, Jerome Henry Manheim, Merle Wallace Milligan, William Rolen Orton, Jr., Donald Lawrence Sullivan, and Robert John Weeks;

Institute for Advanced Study: Drs. Raoul Bott and Raouf Doss, Professors Olof Hanner and Kunihiko Kodairo;

Iowa State College of Agriculture and Mechanic Arts: Mr. Robert J. Lambert;

State University of Iowa: Mr. Robert Vincent Hogg;

Johns Hopkins University: Messrs. Edward Daniel Carey, Ralph Elwood Keirstead, Jr., William Edward Moore, Norman L. Rabinovitch, Alexander Cormac Smith;

University of Kansas: Mr. Robert Charles Fisher;

Kenyon College: Mr. Daniel Rightor Mason;

Lehigh University: Messrs. Benjamin Collins Kenny and Hubert Howard Snyder;

University of Maryland: Professors George Gerald O'Brien and John Aloysius Tierney;

Massachusetts Institute of Technology: Mr. John Weber Carr, III, Mrs. Violet Bushwick Haas, Messrs. Flavio Botelho Reis and William Lucas Root;

Michigan State College: Mr. Henry Arthur Hanson;

University of Michigan: Messrs. Fred Brafman and Joshua Chover, Miss Margaret Louise Comstock, Mr. Arthur Covington Downing, Jr., Rev. Lester Joseph Heider, Messrs. Robert Maxwell Lauer, John Francis Riordan, and Shih-Hua Tsao;

University of Missouri: Mr. H. C. Griffith;

New York University: Messrs. Ignace I. Kolodner and Chankey Nathaniel Touart;

Ohio State University: Mr. Edley Wainright Martin, Jr., Miss Vera Marie Tozzer;

Oklahoma Agricultural and Mechanical College: Mr. George B. Pedrick;

University of Oregon: Mr. Luther Peh-Hsiun Cheo;

Pennsylvania State College: Mr. Joseph D. E. Konhauser;

University of Pennsylvania: Mr. Athanasios Papoulis;

Purdue University: Mr. Obert Benjamin Moan;

College of St. Thomas: Mr. Richard Anthony Zemlin;

Smith College: Miss Elinor Constance Martin;

Stanford University: Messrs. Kenneth Lloyd Cooke, William James Firey, Howard Ashley Osborn, and Edgar C. Smith, Jr.;

Swarthmore College: Mr. Albert Ross Eckler;

Syracuse University: Mr. Frank Pao-ming Pu;

University of Tennessee: Mr. Eldon Eugene Posey;

Department of Pure Mathematics, University of Texas: Miss Lida Baker Kittrell, Messrs. Herbert Allen Morris, Ernest Tilden Parker, and Spurgeon Eugene Smith;

University of Toronto: Messrs. Geoffrey Eric Norman Fox, Harold Stanley Heaps, Steve Alexander Kushneriuk, and Adrian Eugen Scheidegger;

University of Virginia: Messrs. Frank Thomas Dresser, David Barrington Lowdenslager, Leonard Robert Schlauch, and Robert Fones Williams;

University of Washington: Messrs. Chi-Yuan Lee and Fred H. Ringey;

Wellesley College: Miss Joyce B. Friedman;

University of Wisconsin: Messrs. William Francis Ames, Paul Axt, Robert Joseph Buehler, William F. Donoghue, Jr., Wendell Helms Fleming, Melvin Henriksen, Erwin Kleinfeld, and Paul William Knaplund, Miss Joan K. Ross, Messrs. Garth Hollis Markle Thomas and Arnold Wendt;

Yale University: Messrs. John Daniel Baum, Thaddeus Bankson Curtz, Leon William Green, Henry George Jacob, Jr., and Sol Schwartzman.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematiker Vereinigung: Dr. Wilhelm R. Fischer, University of Pennsylvania (nominee); Professor Alwin Walther, Technische Hochschule, Darmstadt, Germany; London Mathematical Society: Professor Patrick Du Val, University of Georgia; Dr. H. G. Eggleston, Swansea University College, Wales; Mr. David Emrys Evans, University College of North Wales; Mr. Marcus Campbell Goodall, Institute for Advanced Study (nominee); Mr. Walter Kurt Hayman, Brown University (nominee); Miss Marjorie Littleton, H. H. Wills Physical Laboratory, University of Bristol; Dr. Claude Ambrose Rogers, University College, London, and member, Institute for Advanced Study (nominee); Dr. Hsien-Chung Wang, Louisiana State University; Polish Mathematical Society: Professor Adam Feliks Bielecki, University of Lublin; Professor Mieczysław Kwiryn Biernacki, University of Marie Curie-Skłodowska; Mrs. Helena Alina Raś, University of Warsaw; Dr. Roman Sikorski, University of Warsaw; Société Mathématique de France: Dr. Shmuel Agmon, Rice Institute; Mr. Jean Arbault, University of Poitiers; Professor Claude Gilbert Chabauty, University of Strasbourg; Mr. Robert Louis Croisot, University of Poitiers; Professor Leonce Lesieur, University of Poitiers; Dr. Jacques Riquet, Institut Henri Poincaré; Professor Laurent Schwartz, University of Nancy; Wiskundig Genootschap te Amsterdam: Dr. Jacob Korevaar, Purdue University.

The following universities were elected as institutional members of the Society: University of Massachusetts; Princeton University; University of Wichita.

The Secretary is pleased to report at this time that the ordinary membership of the Society is now 4064, including 339 nominees of institutional members and 49 life members. There are also 107 institutional members. The total attendance at all meetings in 1949 was 2470; the number of papers read was 580; there were 13 hour addresses, 5 addresses and 16 papers at the Applied Mathematics Symposium, 1 Gibbs Lecture, and 4 Colloquium Lectures; the number of members attending at least one meeting was 1470.

It was reported that the Council had voted by mail to approve the list of Honorary Presidents and Honorary Chairmen for the 1950 International Congress, as recommended by the Organizing Committee. The Council had also voted by mail to accept for publication in Mathematical Surveys a manuscript by Claude Chevalley entitled *Introduction to the theory of algebraic functions of one variable.*

The following appointments by President J. L. Walsh of representatives of the Society were reported: Professor H. A. Rademacher on the Editorial Board of the American Year Book for a term of three years beginning January 1, 1950; Professor T. R. Hollcroft at the inauguration of the Very Reverend Juvenal Lalor as President at St. Bonaventure College on September 22, 1949; Professor R. H. Marquis at the inauguration of William Bay Irvine as President of Marietta College on October 15, 1949; Professor E. G. Olds at the inauguration of Will W. Orr as President of Westminster College on October 14, 1949; Professor D. S. Morse at the One Hundred Twenty-fifth Anniversary of the founding of Rensselaer Polytechnic Institute on October 13–15, 1949; Professor W. M. Whyburn at the inauguration of Arthur Hollis Edens as President of Duke University on October 22, 1949; Professor G. E. Raynor at the inauguration of David E. Weinland as President of Moravian Seminary and College for Women on October 22, 1949; Professor David Blackwell at the inauguration of Alonzo G. Moŕon as President of Hampton Institute on October 29, 1949; Professor B. W. Jones at the inauguration of Albert Charles Jacobs as Chancellor of the University of Denver on November 19, 1949; Professor Gweneth Humphreys at the inauguration of George Tyler Miller as President of Madison College on December 10, 1949.

The following additional appointments by the President were reported: Professors R. E. Langer (Chairman), E. F. Beckenbach, and S. S. Wilks as a committee to nominate Society representatives on Policy Committee; Dr. A. S. Householder (Chairman), Professors J.

A. Cooley, G. A. Garrett, W. S. Snyder, J. M. Thomas, and W. M. Whyburn as Committee on Arrangements for meeting at Oak Ridge, Tennessee, on April 21–22, 1949; Professors W. M. Whyburn (Chairman), Tomlinson Fort (term to expire December 31, 1951), and J. M. Thomas (term to expire December 31, 1950) as Committee to Select Hour Speakers for Southeastern Sectional Meetings; Professor B. P. Gill as Chairman of the committee to consider non-editorial problems of the Memoirs (to replace Professor A. W. Tucker); Professor Einar Hille as Chairman of the Committee on the Role of the Society in Mathematical Publication (to replace Professor R. L. Wilder); Miss Grace Bolton and Professor Arthur Sard as tellers for the 1949 annual election; Professors C. G. Phipps (Chairman), R. G. Blake, W. A. Gager, H. M. Gehman, W. R. Hutcherson, F. W. Kokomoor, H. A. Meyer, and W. M. Whyburn as a Committee on Arrangements for 1950 Annual Meeting; Professor A. A. Albert as a member of the Committee to Select Hour Speakers for Summer and Annual Meetings for the period 1950–1951 (committee now consists of Professors J. R. Kline, Chairman, A. A. Albert, and G. A. Hedlund); Professor E. G. Begle as member of the Committee to Select Hour Speakers for Eastern Sectional Meetings for the period 1950–1951 (committee now consists of Professors T. R. Hollcroft, Chairman, E. G. Begle, and D. V. Widder); Professor Walter Leighton as member of the Committee to Select Hour Speakers for Western Sectional Meetings for the period 1950–1951 (committee now consists of Professors J. W. T. Youngs, Chairman, Ralph Hull, and Walter Leighton); Professor D. C. Spencer as a member of the Committee to Select Hour Speakers for Far Western Sectional Meetings for the period 1950–1951 (committee now consists of Professors J. W. Green, Chairman, E. F. Beckenbach, and D. C. Spencer); Professor H. S. Wall as a member of Committee on Visiting Lectureships for the period 1950–1952 (committee now consists of Professors Hassler Whitney, Chairman, H. S. Wall, and Norbert Wiener); Professor J. B. Rosser as liaison officer for Quarterly of Applied Mathematics for the period 1950–1952; Professors K. O. Friedrichs and John von Neumann as members of Committee on Applied Mathematics for the period 1950–1952; Professors Garrett Birkhoff and G. C. Evans as members of Committee on Role of the Society in Mathematical Publication for the period 1950–1952; Professor R. A. Johnson as a member of Committee on Printing Contracts for the period 1950–1952.

It was reported to the Council that: Professor G. E. Uhlenbeck has accepted the invitation to deliver the Gibbs Lecture at the 1950

Annual Meeting; Professor Einar Hille has been appointed a member of the Organizing Committee of the International Congress to replace the late Dean R. G. D. Richardson; Professor Einar Hille was substituted for Professor M. H. Stone as a nominee for the Mathematical Reviews Editorial Committee on the ballot for the annual election; the usual biennial membership campaign was held in October, 1949, at which time invitations to join the Society were sent to 4,203 teachers of collegiate mathematics (including 485 members of the Mathematical Association of America); during 1950 Professor G. T. Whyburn will act as Managing Editor of the Transactions and Memoirs, Professor C. C. MacDuffee will act as Chairman of the Colloquium Editorial Committee, Professor A. W. Tucker will act as Chairman of the Mathematical Surveys Editorial Committee, Dean E. B. Stouffer will act as Chairman of the Bulletin Editorial Committee, and Professor Nathan Jacobson will act as Chairman of the Proceedings Editorial Committee.

At the annual election which closed on December 29, and at which 744 votes were cast, the following officers were elected:

President Elect, Professor John von Neumann.

Vice Presidents, Professors E. J. McShane and R. L. Wilder.

Associate Secretaries, Professors J. W. Green, W. M. Whyburn, and J. W. T. Youngs.

Members of Editorial Committee of the Bulletin, Professors G. B. Price and A. C. Schaeffer.

Members of Editorial Committee of the Transactions, Professors L. V. Ahlfors and J. L. Doob.

Member of the Editorial Committee of the Colloquium Publications, Professor R. L. Wilder.

Member of the Editorial Committee of Mathematical Reviews, Professor Einar Hille.

Member of the Editorial Committee of Mathematical Surveys, Professor W. T. Martin.

Representative on the Editorial Board of the American Journal of Mathematics, Professor Samuel Eilenberg.

Members at large of the Council, Professors Warren Ambrose, P. R. Halmos, Mark Kac, S. B. Myers, and D. C. Spencer.

The Council voted to approve April 28, 1950, as an additional date for the spring meeting in the far west. Dates of other meetings in 1950 were set as follows: October 28 in New York City; November 24–25 in Chicago; November 25 in the far west; December 27–29 at the University of Florida. The Council also accepted an invitation to hold the October, 1951, meeting at the National Bureau of Standards in

Washington, D. C.

In an appendix to this report are excerpts from the report of the Treasurer for the fiscal year 1949 as verified by the auditors. A copy of the complete report will be sent, on request, to any member of the Society.

The American Journal of Mathematics, which is a joint enterprise of the American Mathematical Society and The Johns Hopkins University, reported that it had published 975 pages in 1949.

The Librarian reported the following additions to the Library: 309 bound volumes of periodicals, 124 books, and 139 pamphlets (including 41 dissertations).

Certain invitations to give addresses in 1950 were announced; Professor C. L. Siegel for the February meeting in New York City; Professors J. S. Frame, Sam Perlis, and M. F. Smiley for the February meeting in East Lansing, Michigan; Professors Atle Selberg and R. P. Boas for the April meeting in Washington, D. C.

The Bulletin Editorial Committee reported that the backlog of the journal was such that a period of approximately eighteen months must now elapse between acceptance and publication of a particular paper. A recommendation that a total of 1438 pages be authorized for the 1950 Bulletin and Proceedings was accepted by the Council and referred to the Trustees. This represents an increase of 249 pages over the 1949 volume of the Bulletin. Professors C. B. Allendoerfer, G. Hochschild, and J. B. Diaz were reported as new Assistant Editors for the Proceedings.

The Transactions Editorial Committee reported that the transfer of several long papers to the Memoirs had greatly relieved the backlog of papers awaiting publication in the Transactions. Additional funds for the publication of the Memoirs were requested and approved, subject to a vote of the Trustees. The Council voted to authorize the Committee on Printing Contracts, in consultation with the editors, to study the possibility of economizing in the cost of the Transactions by reducing the margins.

The Mathematical Reviews Editorial Committee reported that 766 pages of reviews had been published, as against 638 pages for 1948; the increase was due partly to the policy of encouraging longer reviews for Russian papers but also to the increase in mathematical production in most countries. During 1949 the Société Mathématique de France and the Unione Matematica Italiana were added to the list of sponsors. The subscription list as of November, 1949, was 2,126.

The Council authorized the Executive Committee, with the

Trustees, to negotiate a contract with Columbia University in which the interests of the Society with respect to housing would be protected and in which the library of the Society would be used as a proper bargaining asset.

A recommendation in connection with the Executive Editorship of Mathematical Reviews was accepted and referred to the Trustees for approval.

The Council voted to act as a sponsor of a symposium on the *Eigenvalue problem* to be held about the middle of June, 1950, at Oklahoma Agricultural and Mechanical College.

The Council voted to accept invitations to hold the 1951 Annual Meeting at Brown University and the 1953 Summer Meeting at Laval University; the latter meeting will be in conjunction with the meeting of the Canadian Mathematical Congress and the 100th anniversary of the charter of Laval University.

A report on activities of the Policy Committee was presented, including the following items: (1) Plans for a Union Conference to be held at Columbia University for three days immediately preceding the International Congress, to be attended by delegates of National Committees which have been discussing statutes and by-laws for a new International Mathematical Union; (2) Discussion of the possible establishment of a Mathematics Foundation, to receive and administer funds contributed for the general support of mathematics, by a subcommittee of the Policy Committee; (3) Study by a subcommittee of the needs of mathematics which might be met by a National Science Foundation.

The Organizing Committee of the International Congress reported that satisfactory progress had been made toward a solution of the difficult problem of visas. All indications point to a successful Congress of a truly international character at Cambridge in the summer of 1950.

On recommendation of the Committee on the Role of the Society in Mathematical Publication, the Council voted to recommend to the Trustees that for a period of three years the Society contribute an annual subsidy not to exceed $1500 to the proposed Pacific Journal of Mathematics.

The Council voted to omit the 1950 Symposium in Applied Mathematics.

It was voted to advance by forty-eight hours the deadline for papers to appear in the programs of Society meetings.

The President was authorized to appoint a committee to study the relation of the Bulletin and Proceedings to individual membership dues and to reconsider the matter of individual dues.

Abstracts of the papers read follow. Presiding officers at the sessions for contributed papers were Professors W. M. Whyburn, Arnold Dresden, C. A. Truesdell, Wallace Givens, R. H. Bruck, B. O. Koopman, F. W. Perkins, G. L. Walker, Dr. R. S. Burington, Professors A. A. Bennett and M. R. Hestenes.

Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 99 was presented by Mr. Macon, 131 by Professor Goffman, 135 by Mr. Rogers, 142 by Professor Rosenbloom, 147 by Professor Brunk, 154 by Professor Rado, 170 by Professor Lin, 180 by Professor Sard, 183 by Mr. O'Brien, 195 by Professor Kasner, 210 by Professor Cohen.

Professor Samuelson was introduced by Professor W. T. Martin and Professor Baiada by Professor C. N. Moore.

ALGEBRA AND THEORY OF NUMBERS

99. A. T. Brauer and Nathaniel Macon: *On the approximation of irrational numbers by the convergents of their continued fractions.* II.

Let ξ be any positive irrational number and A_n/B_n the convergents of its expansion as a regular continued fraction. If λ_n is defined by $|\xi - A_n/B_n| = \lambda_n^{-1}B_n^{-2}$, then the best possible lower bounds for the sums of any k consecutive λ_n are obtained. Hurwitz [Math. Ann. vol. 39 (1891) pp. 279–284] proved that the limit of the arithmetic means of the first n numbers λ_i equals $5^{1/2}$ for $(1+5^{1/2})/2$ and its equivalent numbers. It is proved here that the lower limit of these arithmetic means is not less than $5^{1/2}$ for every irrational number. (Received November 14, 1949.)

100. R. H. Bruck: *A numerical invariant for finite nets.*

The author initiates a study of "imbedding problems" for nets. A (finite) net N of degree k, order n is a system of "points" and k classes of "lines" such that: (i) each point is on one line of each class; (ii) two lines of distinct classes have a common point; (iii) a line has n points. (Finite affine planes, sets of orthogonal latin squares, finite loops, can be interpreted as nets.) An integer i is "represented" by N provided there exists an integer-valued point-function f which sums to i over each line of N. The invariant $\phi(N)$ is the l.p.i. represented by N; $\pi(N)$, the l.p.i. represented with a non-negative f. A n.a.s.c. that there exist n points of N, no two collinear, is that $\pi(N)=1$; a necessary condition, that $\phi(N)=1$. The invariant ϕ is explicitly calculated for affine planes and loops. Relations are obtained between the ϕ's of a net and its homomorphic image; sharper relations, for direct products. As an application: if n is a product of coprime prime-powers, the smallest of which is $k-1$, there exists a set of $k-2$ mutually orthogonal $n \times n$ latin squares to which no square can be added. (Received December 12, 1949.)

101. K. T. Chen: *Integration in free groups and commutator calculus.*

To each word in a free group F with n generators a path in a Euclidean n-space is assigned. The integral of any polynomial along this path depends only on the element of F represented by the word. The totality of elements of F for which the integral of every polynomial of degree less than d vanishes forms a subgroup of F which is proved to be the product of the dth lower central commutator group ${}^{d}F$ with the second upper central commutator group $F^{(2)}$. Using this it is found that ${}^{d-1}F\cdot F^{(2)}/{}^{d}F\cdot F^{(2)}$ is a free abelian group with $(d-1)C_{d+n-2,d}$ generators ($n\geqq 2$). Further the theory can be applied to the computation of the above factor group for any group G given by a finite number of generators and relations, thus yielding numerical invariants of G. (Received November 10, 1949.)

102*t*. Sarvadaman Chowla and Paul Erdös: *A theorem on the distribution of the values of L-functions.*

The authors prove the following result: Let (d/n) denote Kronecker's symbol defined for $d\equiv 0$, 1 (mod 4) and d not equal to a perfect square. Define for $s>0$, $L_d(s) = \sum_1^\infty (d/n)n^{-s}$. Denote by $g_s(a, x)$ the number of integers $d\leqq x$ such that (i) $d\equiv 0, 1(4)$, (ii) $d\neq t^2$, (iii) $L_d(s)<a$. Then if $s>3/4$, $\lim_{x\to\infty} g_s(a, x)/2^{-1}x=g_s(a)$ exists; furthermore $g(-\infty)=0$, $g(\infty)=+1$, and $g_s(a)$, the distribution function, is a continuous and steadily increasing function of a. (Received November 14, 1949.)

103*t*. Sarvadaman Chowla: *On the nonvanishing of a certain infinite series.*

The author proves the following result: For every integer n let $f(n)$ be an integer such that (i) $f(n)=f(n')$ whenever $n'\equiv n$ (mod p), (ii) $f(n)=-f(-n)$. Then $\sum_1^\infty f(n)n^{-1}\neq 0$, unless all $f(n)$ are zero. Here p is a prime such that p and $(p-1)/2$ are both primes. The paper also contains a proof of an extension of this theorem due to Siegel. (Received November 14, 1949.)

104*t*. Harvey Cohn: *Finiteness conditions for a convex body in hyperspace.*

If a convex body in d-dimensional space contains a hypersphere of radius R about the origin, then unless it contains an integral lattice point within the hypersphere of radius C_d/R^{d-1}, it will lie entirely within the latter hypersphere. There is a best constant C_d as $R\to 0$, but it is more difficult to determine. Tentative bounds for C_d are found by letting the Dirichlet boxing in principle oppose the Perron transferal principle. (Received November 11, 1949.)

105*t*. Harold Davenport: *Euclid's algorithm in certain quartic fields.*

It is proved in this paper that Euclid's algorithm can hold in only a finite number of quartic fields of the totally complex type. The method is a development of that already applied by the writer to real quadratic fields and to cubic fields of negative discriminant, in a series of papers in course of publication. An additional complication, as compared with those cases, arises from the fact that the inequalities which naturally present themselves are not quite precise enough to be of service. (Received December 27, 1949.)

106. H. C. Davis: *Closure lattices.* Preliminary report.

Corresponding to a Boolean algebra B there is defined (uniquely) a lattice $L(B)$ with the following properties: (1) It contains the partition lattice over B as a sublattice; (2) it is pseudo-complemented; (3) if B is finite, $L(B)$ is semi-modular. There is a natural dual homomorphism of $L(B)$ into the lattice of closure algebras over B—the latter lattice being defined in immediate generalization of the case where B is the field of all subsets of some set. The extension to the case where B is replaced by a more general lattice is discussed. (Received December 14, 1949.)

107. D. T. Finkbeiner: *A general dependence relation for lattices.*

Let M be an arbitrary set, and let D be a relation between the elements and subsets of M. D is called a dependence relation if D satisfies (1) $mD(S\vee m)$ for arbitrary $S\subseteq M$, and (2) mDS and SDT imply mDT. A subset S is called closed if mDS implies $m\in S$, and the closed subsets form a complete lattice L. MacLane (Duke Math. J. vol. 4 (1938) pp. 455–468) has characterized dependence relations for which L is a semi-modular point lattice. In this paper M is considered to be partially ordered, and a corresponding characterization is given for dependence relations which preserve the ordering of M, in that L is a complete lattice whose set of completely join irreducible elements is isomorphic to M. Furthermore L satisfies a point-free exchange axiom which defines semi-modularity for lattices in which coverings may not exist. If M is not ordered, this characterization reduces to the classical result for semi-modular point lattices. (Received November 14, 1949.)

108t. A. L. Foster: *On n-ality theories in rings and their logical algebras; tri-ality principle in three-valued logics.*

The K-ality theory, as launched in a series of previous publications, represents a simple abstract framework which numbers among its realizations—and thus serves to unify—disciplines as widely dissimilar as, for example, the classical Boolean theory together with many new extensions thereof on the one extreme, and the traditional invariant-transformation theories (for example, tensors, groups, forms, geometries, and so on) on the other. The broad program involves the study of arbitrary disciplines (for example, rings, and so on) "in" different "coordinate systems" belonging to a group K of admissible "coordinate transformations" in the discipline. Thus, specialized to rings R and with K chosen as C, the "simple complementation" group (of order 2) generated by $x^* = 1 - x$, one obtains the C-ality (or "simple" duality) theory of rings, which further reduces to the classical Boolean duality when R is Boolean. The present communication, while still confined to rings, is not restricted to the C-level. The *K-ality theorems*, successors of the simple duality notions, are given, and various problems elevated to the general K-level. Among these is the concept *ring-logic* (mod K), generalizing the classical Boolean ring-Boolean algebra (logic) equivalence. (Received November 25, 1949.)

109t. A. L. Foster: *p-rings and their Boolean-vector representation.*

In connection with the K-ality theory (see previous abstract), the general structure of p-rings is studied. Using certain results previously obtained, it is found that all p-rings are isomorphic with a certain (generalized) hypercomplex "vector p-ring" over a suitable Boolean algebra—namely the idempotent Boolean sub-algebra of the p-ring. (In the vector representation, the vectors are componentially determined, but are not in general additive—hence the expression "generalized.") This leads to a generalized hypercomplex representation of p-rings which is a generalization of the familiar Stone representation for the special Boolean ($p=2$) case. Earlier results on p-rings of McCoy and Montgomery are subsumed and generalized. (Received November 9, 1949.)

110*t*. A. L. Foster: *Ring-algebras (logics) and p-rings.*

The ring-logic (mod N) status of p-rings for $p>3$ was left unsolved in a previous paper (see first abstract above). Applying the results on the structure (decomposition and p-vector synthesis) of p-rings obtained later (see preceding abstract), the general theorem is proved: each p-ring is a ring-logic (mod N). (Received November 9, 1949.)

111. Emil Grosswald: *On some algebraic properties of the Bessel polynomials.*

For Bessel polynomials (see paper by H. L. Krall and Orrin Frink, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 100–115), two asymptotic formulas are established and upper bounds for the errors terms are found. Using recursion formulas, it is shown that all the roots of Bessel polynomials are simple and non-real, except a single root, x_n, of the polynomials of odd degree n, which is real and satisfies $-1\leqq x_n\leqq -1/n$. Let p be an odd prime, m a positive integer, q an integer less than p^m; then, using mainly Dumas' theorem (Journal de Mathématiques (6) vol. 2 (1906) pp. 191–258), it follows that Bessel polynomials are algebraically irreducible in the field of rational numbers, at least for $n\leqq 323$, and also for $n>323$, if n is of one of the forms: p^m, p^m+1, $q\cdot p$, $q\cdot p-1$, p^m-1, 2^m-1, $q\cdot p^m$ (in the last 3 cases, with some restrictions); if not irreducible, any Bessel polynomial contains an irreducible factor of degree not less than $A\cdot(n-1)$, where $A>16/17$. Using theorems of I. Schur (Preuss. Akad. Sci. Sitzungsber. (1929) pp. 443–449) and Dedekind, it is found that all irreducible Bessel polynomials (except, perhaps, for $n=9, 11, 12$) lead to equations without affect. (Received November 7, 1949.)

112*t*, Franklin Haimo: *A representation for Boolean algebras.*

For a direct sum G of additive Abelian G_α, elements $x=\{x_\alpha\}$ and $y=\{y_\alpha\}$ are said to be in the relation $x\perp y$ if $x_\alpha\neq 0$ implies $y_\alpha=0$, and likewise with the x's and y's interchanged. Abelian groups which have a partial ordering like that of a direct sum, where $x\geqq y$ means that $y\perp x-y$, are called vector ordered groups (quasi vector ordered if the generalization is based on weak rather than on strong summation). It is proved that every complete Boolean algebra can be represented faithfully by the suitably ordered set of all endomorphisms bounded above by the identity endomorphism on a vector ordered group. Each of these endomorphisms is splitting, and their image spaces contain elements maximal therein. Conversely, such endomorphisms on a vector ordered group form a complete Boolean algebra. The not necessarily complete Boolean algebras are linked, in a somewhat similar fashion, to the quasi vector ordered groups. (Received November 14, 1949.)

113*t*. Franklin Haimo: *Decomposition of complete Boolean algebras.*

Direct decompositions of vector ordered groups (see the preceding abstract) are connected with join decompositions of the corresponding complete Boolean algebra. To each infinite cardinal corresponds a pair of decompositions, one for the group, the other for the algebra. One of the components of the group decomposition turns out to be a strong direct sum of subgroups, each of which is the image space of a splitting endomorphism. Correspondingly, every complete field of sets can be generated by a set of disjoint sets in the field. Each of these generators is small in that each contains, set-theoretically, only a limited number of such generators. This subsumes a result due to Tarski, Fund. Math. vol. 24 (1935) p. 197, Theorems 6. (Received November 14, 1949.)

114. I. N. Herstein: *On a conjecture on simple groups.*

If G is a finite group, let Γ_p be the group ring G over a field of characteristic p, and N_p the radical of Γ_p. Consider the following two assertions: (A) A simple group of odd order is of prime order. (B) If the order, n, of G is odd, then for some prime p, $p|n$, there exists a $g\in G$, $g\neq 1$, with $g-1\in N_p$. The result found in the paper is that (A) and (B) are equivalent; that (B) implies (A) follows easily by showing that $\{g\in\sigma | g-1\in N_p\}$ is a normal p-subgroup of G. To show (A) implies (B), use is made of Clifford's theorem on the representations induced in normal subgroups. (Received November 14, 1949.)

115. B. W. Jones: *An extension of Meyer's theorem on indefinite ternary quadratic forms.*

Let f be a properly or improperly primitive indefinite ternary quadratic form of determinant d such that its matrix has integer elements, Ω is the g.c.d. of its two-rowed minor determinants and $d=\Omega^2\Delta$ determines Δ. The principal result of this paper is: there is one class in the genus of f if neither Ω nor Δ is divisible by 4, they are not both even and if, for every prime factor p common to Ω and Δ, the following conditions hold: 1. p^2 divides neither Ω nor Δ; 2. There exists a q, depending on p, which is either a prime or double a prime, does not divide $2d$, and such that every solution of $x^2-qy^2\equiv 1 \pmod{p}$ is congruent (mod p) to a solution of $x^2-qy^2=1$. If $(p\pm 1)/2$ is a prime for proper choice of the ambiguous sign, condition 2 holds. Hence it seems likely that there is an infinite number of primes p satisfying this condition. (Received November 14, 1949.)

116. W. J. LeVeque: *On the equation $a^x-b^y=1$.*

It is shown that the Diophantine equation $a^x-b^y=1$ has at most one solution unless $a=3$, $b=2$, when it has just the two (1, 1) and (2, 3). If a is even and b is odd, the only possible solution is with $x=\beta/\alpha$, where 2^α is the highest power of 2 dividing a and 2^β is the highest power of 2 dividing $b+1$. If a is odd and b is even and $(a, b)\neq(3, 2)$, then the only possible solution is with $y=t$, where t is half the exponent to which b belongs (mod a). As an application, it is shown that the identity $\sum_{k=1}^{n} k^3=(\sum_{k=1}^{n} k)^2$ is the only one of its kind; that is, that the only solution (r, s, t) of $\sum_{k=1}^{n} k^s=(\sum_{k=1}^{n} k^t)^r$ is (2, 3, 1). (Received November 14, 1949.)

117*t*. R. C. Lyndon: *Groups with one defining relation.*

If $R=Q^q$ for Q a word in a free group F and q maximal, then an identity $\prod T_i R_i^{e_i} T_i^{-1}=1$ (where T_i in F, $e_i=\pm 1$) implies that the indices fall into pairs (i, j) such that $e_i=-e_j$ and that, for some p_i, $T_i\equiv T_jQ^{p_i}$ modulo R. A slightly more general theorem is proved, by a method derived from Magnus (J. Reine Angew. Math. (1930)). From incidence matrices (with operators) one calculates the Eilenberg-MacLane groups $H^n(G, K)$ for G defined by a single relation, and any coefficients K. For $n>2$ these groups are the exact analogues of the special case G cyclic of order q (infinite cyclic if $q=1$), as given by Eilenberg and MacLane (see Bull. Amer. Math. Soc. vol. 55 (1949) p. 22). (Received November 14, 1949.)

118*t*. F. I. Mautner: *Unitary representation of locally compact groups. II.*

This is a continuation of *Unitary representations*. I. Whereas part I dealt to a large extent with discrete groups, the problems of the present part II are mainly relevant in the case of non-discrete groups. Indeed the main problem treated is the structure of the operator algebra W generated by any given unitary representation of a semi-simple Lie group. And after various preliminary sections dealing with general properties of unitary representations and generalized direct sums of Hilbert spaces (cf. von Neumann, *Reduction theory*, Ann. of Math. (1949)) it is shown that if G is the 2×2-unimodular group (or the Lorentz group), then the ring W generated by any continuous unitary representation of G gives rise to factors of type I only in the sense of von Neumann (loc. cit.). An easy consequence of this is a proof of the (known) Fourier-inversion formula for G, which uses relatively few special properties of G. The analogous problem for arbitrary semi-simple Lie groups is reduced to a problem about the behavior of invariants of the Lie algebra (such as the fundamental bilinear form) under unitary group-representations. (Received November 10, 1949.)

119. A. J. Penico: *The Wedderburn principal theorem for Jordan algebras.*

The title of this paper is the same as that of a paper by Albert (Ann. of Math. vol. 48 (1947) pp. 1–7). He considered the "special" Jordan algebras obtained from associative algebras by the quasi-multiplication $ab=(a\cdot b+b\cdot a)/2$. The algebras considered here are the "general" Jordan algebras defined by the identities: $ab=ba$, $a(a^2b)=a^2(ab)$. The following theorem is proved: If A is a Jordan algebra over a field of characteristic zero, and if N is the radical of A, then there exists a subalgebra S of A such that A is the direct sum $A=S+N$, and $S\cong A-N$. For $N^2\neq(0)$, and A with unity element, it is proved that AN^2 is an ideal of A properly contained in N, and, by the usual inductive proof, the theorem is reduced to the case $N^2=(0)$. This latter case is then established for $A-N$ any one of the five types of split Jordan algebras. (Received November 14, 1949.)

120*t*. P. A. Samuelson: *Solving linear equations by continuous substitution.*

To solve the n linear equations $ax=b$, define rectangular matrices $[{}_0A_{ij}]=[a, b]$ and $[{}_kA_{ij}]=[{}_{k-1}A_{ij}(1-\delta_{ik})+({}_{k-1}A_{kk})^{-1}\,{}_{k-1}A_{kj}\{-{}_{k-1}A_{ik}(1-\delta_{ik})+\delta_{ik}\}]$ where $k=1, 2, \cdots, n$ and $[\delta_{ij}]=I$, the identity matrix. Then the writer's solution is $[{}_nA]=[I, a^{-1}b]$, whatever the number of columns of b. This invariant ritual, easily mechanized, avoids the usual "back-solution" but at the cost of about $n^3/6$ extra multiplications beyond the $n^3/3$ involved in the closely related conventional Gauss-Chio-Doolittle-Crout-Aitken solution of asymmetrical equations. The validity of the present variant of one of Aitken's methods follows from the elementary fact that the typical successive substitution process can be used on *all* n equations, not just on the dwindling number of interdependent equations involved in the reduction process. It may be useful to note that if ${}_0A$ is bordered below by $[c, d]$, then ${}_nA$ will be bordered by $[0, d-ca^{-1}b]$; this tells how to interpret all parts of ${}_kA$ for any k. (Received December 23, 1949.)

121. Ernst Snapper: *Completely primary rings.* III.

Let R and S be completely primary rings, that is, commutative rings with unit elements whose radicals $N(R)$ and $N(S)$ are maximal ideals. Let $R\subset S$ and let $N(S)$ be nilpotent. Assume that the field extension $\bar{R}=R/N(R)\subset S/N(S)=\bar{S}$ either has characteristic zero or preserves p-independence. For any ideal $\mathfrak{a}\subset R$, the smallest ideal of S which contains $\mathfrak{a}$ is denoted by $\mathfrak{a}^*$. If $(N(R))^* = N(S)$, S is a *principal extension* of R. If furthermore $\mathfrak{a} = \mathfrak{a}^* \cap R$ for each $\mathfrak{a} \subset R$, S is a *canonical extension* of R. The theory of algebraic and transcendental ring extensions, reported at the October meeting of the Society in New York, is used to prove that two canonical extensions S and S' are equivalent ring extensions of R if and only if $\bar{S}$ and $\bar{S}'$ are equivalent field extensions of $\bar{R}$. The proof uses the theorem that, whether or not S is a canonical extension, there exists a principal, completely primary ring extension T of R, where $R \subset T \subset S$ and $T/N(T) = \bar{S}$; this fact is closely connected with a theorem of I. S. Cohen on local rings. (Received November 10, 1949.)

122t. M. A. Woodbury: *On games whose matrices are the sums of the matrices of other games.*

Let $G^{(k)}$ $(k=1, 2, \cdots, K)$ be the finite two-person zero-sum game with the matrix $(G_{ij}^{(k)}$ $(i=1, 2, \cdots, m; j=1, 2, \cdots, n)$. Let $v(G)$ be the value of the game G and by $X(G)$ and $Y(G)$ respectively denote the (convex) sets of good strategies for the first and second players for G. Define the sum of two games by the addition of their matrices. If $\bigcap_{k=1}^{K} X(G^{(k)}) \neq \phi$ then $(^*) v(\sum_{k=1}^{K} G^{(k)}) \geqq \sum_{k=1}^{K} v(G^{(k)})$, since there exists a strategy x such that for all $k=1, 2, \cdots, K$, $xG^{(k)}y \geqq v(G^{(k)})$. If also $\bigcap_{k=1}^{K} Y(G^{(k)}) \neq \phi$ then the equality sign holds. Clearly $X(\sum_{k=1}^{K} G^{(k)}) \supset \bigcap_{k=1}^{K} X(G^{(k)})$ but the inclusion may be proper as is shown by the case $G^{(1)} = -G^{(2)} =$ a matrix whose first row is $1, -1$ and whose second row is $-1, 1$. Another example, $G_1 =$ a matrix whose first row is $1, -1$ and whose second row is $-1, 1$, $G_2 =$ a matrix whose first row is $2, -1$ and whose second row is $-2, 1$, due to D. Gale shows that the equality sign may hold in $(^*)$ even if $\bigcap_{k=1}^{K} Y(G^{(k)}) = \phi$. A result of A. W. Tucker follows as an easy consequence of this result $(^*)$, namely: Let the matrix of G be divided into blocks by subdividing the strategies of the first and second players into M and N groups respectively, denote by G_{ij} the game defined by the corresponding block and by v_{ij} the value of this game. Further let (v_{ij}) be the matrix of a new game V. Then the relation $v(V) \leqq v(G)$ holds provided that $\bigcap_{j=1}^{N} X(G_{ij}) \neq 0$ for $i=1, 2, \cdots, M$. (Received November 14, 1949.)

123. Daniel Zelinsky: *Rings with ideal nuclei.*

Decomposition theorems like those of Kaplansky and of van Dantzig are proved for certain complete topological rings with a neighborhood system at zero consisting of ideals. Such a ring decomposes as a (complete, usually infinite) direct sum exactly when its discrete homomorphs decompose concordantly. Such a concordant decomposition is at hand if the ring has a restricted minimum condition on open ideals and is either semisimple with first countability axiom or is commutative. The summands are respectively discrete classical simple rings, or primary rings like $\mathfrak{p}$-adic completions of the original ring. A corollary on Dedekind rings implies that an algebraic number field topologized so as to have a bounded open additive subgroup has as completion a local direct sum of $\mathfrak{p}$-adic fields. (Received November 15, 1949.)

ANALYSIS

124. T. M. Apostol: *Identities involving the coefficients of certain Dirichlet series.*

The Dirichlet series $\phi(s) = \sum a(n)n^{-s}$ under consideration are of signature $(\lambda, \kappa, \gamma)$ in Hecke's terminology, converge absolutely for $\sigma > \kappa > 0$, have at worst a first order pole at $s=\kappa$, and have functional equations of the form $(\lambda/2\pi)^s\Gamma(s)\phi(s) =(\lambda/2\pi)^{\kappa-s}\Gamma(\kappa-s)\phi(\kappa-s)$. The author derives the identity $\sum_{n=0}^{\infty} a(n)(x-n)^q/q! =\rho x^{\kappa+q}\Gamma(\kappa)/\Gamma(\kappa+q+1)+\gamma(\lambda/2\pi)^q x^{(\kappa+q)/2} \sum_{n=1}^{\infty} a(n)n^{-(\kappa+q)/2}J_{\kappa+q}(4\pi\lambda^{-1}(nx)^{1/2})$ where J_ν is the ordinary Bessel function of order ν and ρ is the residue of $\phi(s)$ at $s=\kappa$. The series on the right converges absolutely for positive $q>\kappa-1/2$. This includes as special cases many known identities in analytic number theory. In particular, when $a(n)$ is the number of representations of n as the sum of two squares this gives the integrated form of Hardy's identity in the famous problem concerning lattice points in a circle. The proof is based on a certain Mellin integral for $J_\nu(z)$. The method used also applies to identities of a different type involving the functions $F_p(z)=\cos(\pi p/2)J_p(z) -\sin(\pi p/2)(Y_p(z)-2K_p(z)/\pi)$ and $V_p(z)=(F_{1-p}(z)-F_{1+p}(z))/p$ where Y_p and K_p are the usual Bessel functions. Mellin integrals for these functions are also obtained in the course of the analysis. (Received October 26, 1949.)

125*t*. Helmut Aulbach: *Some geometrical inequalities for sets in Hilbert space.*

Certain constants $\rho(S)$ and $\sigma(S)$ are introduced for any bounded set S in Hilbert space. $\rho(S)$ was first implicitly introduced by K. Loewner in *Grundzuge einer Inhaltstheorie im Hilbertschen Raume*, Ann. of Math. vol. 40 (1939), and $\sigma(S)=\lim_{n\to\infty} s_n^*$, where s_n^* is the least upper bound of the sides of n-dimensional simplices whose vertices are in S. By the side of an n-dimensional simplex T is meant the length of the edge of that regular n-dimensional simplex whose volume equals the volume of T. A lemma is then proved which states that the side of an n-dimensional simplex T is not greater than the geometric mean of the sides of the $(n-1)$-dimensional simplices which constitute the faces of T. Using this lemma, it is shown that $\rho(S)\leqq\sigma(S)/2^{1/2}$ and a set is exhibited for which the equality holds. Furthermore, a necessary and sufficient condition for a closed set to be compact is that it be bounded and $\sigma=0$. (Received November 18, 1949.)

126. Emilio Baiada: *An isoperimetric problem for ship-bodies.*

Let $y=\phi(x)$, $z=0$, $0\leqq x\leqq 1$, $\phi(0)=\phi(1)=0$, be an A.C. positive function (profile), and $x=f(z)$, $y=0$ (bow); $x=g(z)$, $y=0$ (stern) two arbitrary A.C. functions such that $g(0)=1$, $f(0)=0$. A suitable law being given (similitude in the simplest case), out of the profile and for any $0<z_1\leqq m$, we obtain curves $y=\Phi(x, z_1)$, $z=z_1$, similar to the profile and with first and second end points on bow and stern lines respectively. The solid [ship or wing-body] enclosed by the surface built up and planes $z=0$, $y=0$, $z=m$, has an "external" area A and volume V. Two problems are solved: I. Minimum of A, V being given. The author obtains (1) Existence theorem, (2) Euler equations, (3) When A is minimum, bow and stern are C_1 and convex, and $\phi(x)$ being convex, they meet in a point with tangent parallel to the x-axis. II. Same problem, but profile being given in shape but not in size; (1), (2), (3) are obtained and (4) length of minimum body is found, (5) Euler equation reduces to the first order. (Received November 14, 1949.)

127*t*. Stefan Bergman: *On solutions with algebraic character of linear partial differential equations. I.*

The author associates with every harmonic function $H(X)$, $X=(x, y, z)$, a function $\chi(Z, Z^*)=\chi_1(Z, Z^*)+(ZZ^*)^{1/2}\chi_2(Z, Z^*)$, $x=2(ZZ^*)^{1/2}$, $y=-i(Z+Z^*)$, $z=(Z-Z^*)$, of two complex variables. χ_k are analytic at $Z=Z^*=0$. $P_1(\chi)=2\int_{T=0}^{1} u^{1/2}\{d[u^{1/2}$

$\cdot(u\zeta^{-1}T^2, u\zeta(1-T^2))]/du\}dT$, $u=x+\zeta Z+\zeta^{-1}Z^*$ transforms χ (the C_3-associate of H) into a function $f(u, \zeta)$ (the B-associate of H); $p_2(f)=(2\pi i)^{-1}\int_{|\zeta|=1}f(u, \zeta)d\zeta/\zeta$ transforms f into $H(X)$. The $\{\chi\}$ and $\{f\}$ form algebras, while $\{H\}$ forms only a linear space. The author studies functions H and harmonic vectors $H=H_1i+H_2j+H_3k$, div $H=0$, curl $H=0$, generated by P_2 from rational $f(u, \zeta)=p(u, \zeta)/q(u, \zeta)$, $p(u, \zeta) =\sum_{\nu=0}^{n} u^\nu(\sum_{s=-(N-\nu)}^{N-\nu} b_{\nu s}\zeta^s)$, $q(u, \zeta)=\sum_{\nu=1}^{m} u^\nu(\sum_{s=-(M-\nu)}^{M-\nu} a_{\nu s}\zeta^s))$. He obtains algebraic harmonic functions $H(X)=R(\zeta,X)=\sum_{\mu=+N-M-1}^{N+M-1} G_{\mu-N+M+1}(\zeta^\mu/(\partial Q/\partial\zeta))_{\zeta=\zeta^{(\nu)}}$ if $M<N$, where $G_s=\sum_\kappa b_{j,-M+j+s-\kappa}\Gamma_{j\kappa}$, $\Gamma_{\gamma,\lambda}=\sum_{j=[\lambda/2]}^{\min(\lambda,\nu)} 2^{-\nu}C_{\nu,j}C_{j,\lambda-j}(iy-z)^{\nu-j}(2x)^{2j-\lambda}(iy+z)^{\lambda-j}$, $\lambda\leqq 2\nu$, and the $\zeta^{(\nu)}(X)$ satisfy the equation $Q(\zeta; X)=\sum_{\nu=0}^{2N} A_\nu(X)\zeta^\nu=0$. Fixing q and letting p vary, one obtains a family of algebraic functions which are single-valued on the R-manifold (an analogue of a Riemann surface) $Q(\zeta; X)=0$. Canonical representation for $R(\zeta, X)$ is derived. An analogue of the Riemann-Roch theorem is established. (Received December 19, 1949.)

128*t*. Lipman Bers: *Abelian minimal surfaces*. Preliminary report.

Let $\phi(x, y)$ be a solution of the equation of minimal surfaces. If $\phi(x, y)$ can be continued analytically along any path in the (x, y)-plane which avoids a finite number of points, and if this continuation leads to only a finite number of determinations of $\phi_x-i\phi_y$, the minimal surface $z=\phi(x, y)$ is called Abelian. Relations are established between the number of critical points (points where $\phi_x=\phi_y=0$) of the surface and the number and character of the singularities. A parametric representation of all such surfaces is obtained in terms of Abelian integrals. This representation establishes a one-to-one correspondence between Abelian minimal surfaces and a class of real algebraic plane curves, and associates infinitely many Abelian minimal surfaces to any closed orthosymmetric Riemann surface of positive genus. Two Abelian minimal surfaces belong to the same (abstract) Riemann surface if and only if the corresponding algebraic curves are connected by a birational transformation which takes real points into real points. (Received November 14, 1949.)

129. Lipman Bers: *Partial differential equations and generalized analytic functions*.

The author considers functions $f=u(x, y)+iv(x, y)$ satisfying $u_x=\sigma(x, y)v_y$, $u_y=-\sigma(x, y)v_x$, σ being a given positive function defined over the whole Riemann sphere and possessing Hölder-continuous partial derivatives of first order. He proves the existence of such "pseudo-analytic" functions which behave like $a(z-z_0)^r$ (r arbitrary rational number). He also establishes theorems generalizing the theorems of classical function theory on the characterization of analytic functions by a differentiability requirement, on the existence of derivatives of all orders, on Taylor and Laurent expansions, on the classification of singularities, on representation of rational functions by partial fractions, on zeros of polynomials, on the Cauchy integral, on the logarithmic function, on approximation by polynomials, and on the uniformization of algebraic functions. (The case $\sigma=a(x)b(y)$ was treated by Gelbart and the author, cf. Trans. Amer. Math. Soc. (1944), by different methods.) (Received December 23, 1949.)

130. P. W. Carruth: *Maximal functions of ordinals*.

Let $\{a_\alpha\}$, $0\leqq\alpha<\beta$, be an ordinal sequence of ordinals. Every one-to-one map $\alpha(\alpha')$ of the ordinals less than β onto themselves reindexes the system $\{a_\alpha\}$ as $\{a_{\alpha(\alpha')}\}$, and defines a sum $\sum_{0\leqq\alpha'<\beta}a_{\alpha(\alpha')}$ and a product $\prod_{0\leqq\alpha'<\beta}a_{\alpha(\alpha')}$. Let S

denote the class of sums and P the class of products thus obtained. B. Dushnik, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 240–247, has given a sufficient condition that S contain a maximum. Let $\{c_\alpha\}$, $0 \leqq \alpha < \tau$, be the ordinal sequence such that $c_\alpha \leqq c_\delta$ whenever $0 \leqq \alpha < \delta < \tau$, and such that any ordinal in $\{c_\alpha\}$ is "repeated" either k times or ω_σ times according as it is "repeated" in $\{a_\alpha\}$ the finite cardinal k times or $\aleph_\sigma$ times. Conditions that S contains a maximum are determined, these conditions being both necessary and sufficient whenever τ is such that the power of the set of ordinals less than τ is equal to that of the set of ordinals less than the smallest transfinite remainder of τ. Results analogous to most of these, concerning the class P, also are obtained. (Received November 14, 1949.)

131. L. W. Cohen and Casper Goffman: *On completeness in the sense of Archimedes.*

It is shown that an ordered abelian group G is complete in the sense of Archimedes if and only if, for every proper isolated subgroup $I \subset G$, G/I is non-discrete and topologically complete. (Received August 15, 1949.)

132. V. F. Cowling: *On some representations of functions defined by Taylor and Newton series.*

Let $f(z) = \sum_{n=K}^{\infty} a_n z^n$ (K integral and not less than 0) with finite nonzero radius of convergence R. Suppose there exists a single-valued function $a(t)$ analytic in some right half-plane $\text{Re}(z) \geqq L$ and there satisfying conditions (1) $a(n) = a_n$, $n = [L]+1, \cdots$, (2) $a(t) = O(t^{-k})$, $k \geqq 1$. Then choosing L non-integral and greater than K (if necessary) one can write $f(z) = a_K z^K + \cdots + a_{[L]} z^{[L]} + \int_0^\infty (\phi(u)(ze^{-u})^{[L]+1}/(1-ze^{-u}))\,du$ where $\phi(u) = (1/2\pi i)\int_{L-i\infty}^{L+i\infty} e^{ut}a(t)dt$, for all z not on that portion of the positive real axis from 1 to ∞. Similar results are obtained for Newton series. Examples are given. (Received November 14, 1949.)

133. J. M. Danskin: *On the existence of minimizing surfaces in parametric problems in the calculus of variations.*

The problem solved is that of minimizing a certain functional $I(S)$ among all Fréchet surfaces S bounded by a given Jordan curve g in space which bounds some surface of finite area. $I(S)$ is defined for all surfaces of finite Lebesgue area by an integral due to Cesari which reduces for sufficiently regular representations to the form $I(S) = \int\int f(x^1, x^2, x^3, X^1, X^2, X^3)\,dudv$, where f is even, positively homogeneous in the generalized Jacobians X^i, positively regular, and positive definite. The lower semicontinuity of this functional with respect to uniform convergence is known. It thus suffices to obtain a uniformly convergent minimizing sequence. From an initial, arbitrary minimizing sequence results a minimizing sequence of polyhedra represented conformally on the unit circle. A subsequence of the representing sequence of vector functions converges uniformly on the boundary to a representation of g, and also along each of a countable everywhere dense set of parallels to the axes, and weakly in a Morrey space $\mathfrak{P}_2$ to a continuous limit function. Each surface of a further subsequence is modified by a geometrical process, so that the modified sequence is also minimizing and converges uniformly to the limit function obtained above. (Received November 21, 1949.)

134*t*. H. C. Davis: *Symmetry in Banach spaces.*

The following theorem is proved. A Banach space is a Hilbert space if and only if there exists, for any vector g and in any 2-space containing g, a nonzero f such that $\|f+\alpha g\| = \|f-\alpha g\|$ for all scalars α. The proof is by straightforward construction of an inner product. Two other symmetry characterizations of Hilbert space (one of them known previously) are corollaries. [It has been pointed out that this theorem was proved by R. C. James, Duke Math. J. vol. 12 (1945) pp. 291–302, Theorem 4.7 and Corollary 4.7.] (Received November 15, 1949.)

135. Aryeh Dvoretzky and C. A. Rogers: *Absolute and unconditional convergence in normed linear spaces.*

The well known conjecture that the equivalence of absolute and unconditional convergence of series in a Banach space implies the finite dimensionality of the space is established. This is done by exhibiting in every infinitely dimensional Banach space a series which is unconditionally but not absolutely convergent. The problem of constructing such a series is reduced to proving the following result: There exist positive constants $c_1, c_2, \cdots$ with $\liminf_{n\to\infty} c_n/n = 0$, having the property that, if K is any bounded n-dimensional convex body with the origin as centre, there are n points $X_1, \cdots, X_n$ on the boundary of K such that all 2^n points $(\pm X_1 \pm X_2 \pm \cdots \pm X_n)/c_n$ are in K. The points of contact of K with an inscribed ellipsoid of maximum volume are considered, and the existence of such constants with $c_n = O(n^{3/4})$ is established. (Received November 14, 1949.)

136. W. F. Eberlein: *A note on ergodic means.*

Connections between Abel and Cesaro means on one-parameter semi-groups of operators have been obtained by an extension to Banach spaces of classical Tauberian theory. It is shown that such relations follow in a direct and much more transparent fashion from the author's abstract ergodic theorem. (Received November 14, 1949.)

137. D. B. Goodner: *Normed linear spaces which have property* P_1.

A normed linear space X has property P_s, $s \geqq 1$, if and only if for every normed linear space Y containing X there is a projection T, $\|T\| \leqq s$, of Y onto X. Theorem 1. If the normed linear space X has property P_1 and the unit sphere C of X has an extreme point, then C is the closed convex hull of its extreme points. Theorem 2. A finite-dimensional normed linear space X has property P_1 if and only if the unit sphere C of X is a "parallelepiped." Theorem 3. A normed linear space X does not have property P_1 if there exists in X a hyperplane $H = \{x \mid f(x) = 1 = \|f\|\}$ such that $H \cap C$ (C the unit sphere of X) is an n-dimensional simplex, $3 \leqq n$. (Received November 14, 1949.)

138. R. G. Helsel: *Remarks on the isoperimetric inequality.*

Radó [Trans. Amer. Math. Soc. vol. 61 (1947) pp. 530–555] has established the spatial isoperimetric inequality for a general closed surface in terms of its Lebesgue area and an enclosed volume defined as the integral of the absolute value of the topological index with respect to the surface. In the present paper it is shown that a stronger inequality is obtained if the enclosed volume is defined as follows. Let P be a polyhedron and $V_R(P)$ its enclosed volume as defined by Radó. Then for a general surface S, let the enclosed volume be $V^*(S) = \inf \liminf V_R(P_n)$, where the limit inferior is with respect to a sequence of polyhedra which converges in area to S and the infimum is taken for all such sequences. It follows from a paper of the author [Duke Math. J. vol. 16 (1944) pp. 111–118] that $V^*(S) = V_R(S)$ if S is a surface which occupies a point set of zero three-dimensional Lebesgue measure; however, the present paper includes an example to show that $V^*(S)$ can exceed $V_R(S)$ for surfaces occupying a point set with positive measure. (Received November 7, 1949.)

139. M. R. Hestenes: *Gradients of integrals in the calculus of variations.*

The purpose of the author is to study the concept of the gradient of a function $F(x)$ in a Hilbert space and to apply the results obtained to problems in the calculus of variations. In particular it is shown, under certain conditions, that if y_n is the gradient of F at x_n and α is a suitably chosen positive constant, the sequence $x_{n+1}=x_n-\alpha y_n$ will converge to a minimum point x_0. It is shown further that if the norm is properly chosen this sequence obtained is identical with one obtained by an application of Newton's method of differential corrections. The conditions imposed are too stringent to be applicable to the general integral in the calculus of variations, but is applicable to cases when the Euler equations are linear in the derivatives. The method appears to be amenable to numerical computations. (Received November 14, 1949.)

140t. R. C. James: *A non-reflexive separable Banach space isomorphic with its second conjugate space.*

For any sequence $x=(x_1, x_2, \cdots)$ of real numbers, let $\|x\|=[\text{l.u.b.}\ \{\sum_{i=1}^{n}(x_{p_{2i-1}}-x_{p_{2i}})^2+(x_{p_{2n+1}})^2\}^{1/2}$ over all finite increasing sequences of positive integers $p_1, p_2, \cdots, p_{2n+1}]$. Let B be the Banach space of all such x for which $\lim_{n\to\infty} x_n=0$ and $\|x\|$ is finite. Then B can be shown to have a basis $\{z^n\}$, where z^n has all components zero except the nth, which is 1. The sequence of linear functionals $\{f^n\}$, where $f^n(z^m)=\delta^n_m$, is a basis for B^*. B^{**} is a sequence space with a norm equivalent to that of B, but without the restriction $\lim_{n\to\infty} x_n=0$. However, $\lim_{n\to\infty} x_n$ exists, and the natural image of B in B^{**} is a closed maximal linear subspace of B^{**}. B^{**} is isomorphic with B, and all conjugate spaces of B are separable. B does not have an unconditionally convergent basis, since it is known that any Banach space B is reflexive if B has an unconditionally convergent basis and B^{**} is separable. (Received November 14, 1949.)

141t. Herbert Jehle: *On the Majorana representation of Dirac matrices.*

Abandoning the requirement of covariance with respect to reflections, the relativistic Schroedinger equation can be put into a pair of first order differential equations with two complex components (Jehle, Physical Review vol. 75 (1949) p. 1609; Zeitschrift für Physik vol. 100 (1936) p. 702; Bull. Amer. Math. Soc. Abstract 50-1-57) $\gamma^\kappa(\partial/\partial x^\kappa-i\phi_\kappa)\psi=\mu\psi^*$ where $(\gamma^{\kappa*}\gamma^\lambda+\gamma^{\lambda*}\gamma^\kappa)/2=-g^{\kappa\lambda}1$, $\gamma^{\kappa*}=\gamma^\kappa_R-i\gamma^\kappa_I$, $\psi^*=\psi_R-i\psi_I$. The two-by-two matrices γ^κ have been extensively studied by O. Veblen (unpublished manuscript). It has been shown by J. Serpe (Physical Review vol. 76 (1949)) that the above equations are equivalent to an equation with four real components $\Gamma^\kappa(\partial/\partial x^\kappa-\Gamma\phi_\kappa)\Psi=\mu\Psi$, where the first two rows four columns of Γ^κ are γ^κ_R, $-\gamma^\kappa_I$, the second ones $-\gamma^\kappa_I$, $-\gamma^\kappa_R$, and the matrix $\Gamma=\mp\Gamma^1\Gamma^2\Gamma^3\Gamma^4$; Ψ is a column matrix with the four components ψ_R and ψ_I. The Γ^κ satisfy $(\Gamma^\kappa\Gamma^\lambda+\Gamma^\lambda\Gamma^\kappa)/2=-g^{\kappa\lambda}1$, everything being real. The hermitean matrices Γ^1, Γ^2, Γ^3, $i\Gamma^4$ give a Majorana representation (Nuovo Cimento vol. 14 (1937) p. 171) of Dirac matrices (H. A. Kramers, Proc. Amsterdam vol. 40 (1937) p. 814; F. J. Belinfante, Physica vol. 6 (1939) p. 859; W. H. Furry, Physical Review vol. 54 (1938) p. 56). (Received November 25, 1949.)

142. D. E. Kibbey and P. C. Rosenbloom: *A theorem of Doob on the ranges of analytic functions.*

In 1935 Doob introduced the class $K(\rho)$, $0<\rho\leqq 1$, of functions f analytic in $|z|<1$ such that $f(0)=0$ and there is an arc A of length $2\pi\rho$ on $|z|=1$ with the property that if $\{z_n\}$ is an arbitrary sequence approaching a point of A, then $\liminf |f(z_n)|\geqq 1$. He proved that if $0<r_2<r_3<1$, then there is an $r_1=\eta(r_2, r_3, \rho)>0$ such that the range of f covers either $|w|\leqq r_1$ or $r_2\leqq |w|\leqq r_3$. Also there is a $k(\rho)>0$ such that if $f\in K(\rho)$, then the range of f covers a circle of radius $k(\rho)$. Doob's proofs give no method of estimating the constants $\eta(r_2, r_3, \rho)$ and $k(\rho)$. In the present paper, by the use of the elliptic modular function, it is found that $\eta(r_2, r_3,\rho)>287r_2 \exp(-16(3.2-\log(1-r_3))^2 65^{1/\rho}/\rho^2)$ and $k(\rho)>95\exp(-285(65^{1/\rho}/\rho^2))$. (Received September 28, 1949.)

143. B. O. Koopman: *Exponential limiting products in Banach algebras.*

Let $\{a_n\}$ be a sequence of elements in a Banach algebra containing the unit element, and let $\{\omega_n\}$ be a monotonic subsequence of positive numbers with $\omega_n\rightarrow\infty$ and $\omega_{n+1}/\omega_n\rightarrow 1$ as $n\rightarrow\infty$. Then it is shown that if $(a_1+\cdots+a_n)/\omega_n\rightarrow m$ and $[N(a_1)+\cdots+N(a_n)]/\omega_n$ remains finite as $n\rightarrow\infty$, the product $(1+a_1/\omega_n)\cdots(1+a_n/\omega_n)$ approaches e^m. A similarly explicit exponential result is established for the limit of the product $(b_1+a_1/\omega_n)\cdots(b_n+a_n/\omega_n)$, where $\{b_n\}$ are also in the Banach-algebra. These theorems have applications to asymptotic distributions involving dependent probabilities. (Received October 31, 1949.)

144. B. W. Lindgren: *An integral on a space of continuous functions.*

Let C' denote the space of real, continuous functions $x(t)$ defined for $0\leqq t\leqq 1$. Let a "quasi-interval" Q be the set of functions of C' corresponding to real constants $a_1, \cdots, a_n, b_1, \cdots, b_n$, and $0\leqq t_1<\cdots<t_n\leqq 1$, satisfying the n inequalities $a_i<x(t_i)\leqq b_i$ $i=1, \cdots, n$. Let its measure be defined to be the integral in n-dimensional space $m(Q)=[\pi^{n-1}(t_2-t_1)\cdots(t_n-t_{n-1})]^{-1/2}\int_a^b \exp\left[-\sum_2^n(u_k-u_{k-1})^2(t_k-t_{k-1})^{-1}\right] du$. This definition leads to a Lebesgue type measure and integral over C' analogous to those defined by Wiener over the subspace C of $x(t)$ such that $x(0)=0$. Although $m(C')=\infty$, there is a countable set of quasi-intervals of finite measure whose union is C'. The new integral is related to the Wiener integral by the formula $\int_{C'}F[x]dx=\int_{-\infty}^{\infty}d\xi\int_C^w F[y+\xi]d_wy$, where $F[x]$ is summable. Using this relation and certain known formulas for the Wiener integral, the author obtains corresponding integration formulas for the integral over C'. (Received November 10, 1949.)

145*t*. B. W. Lindgren: *Transformation formulas for an integral on a space of continuous functions.*

Let C' denote the space of real, continuous functions $x(t)$ on $0\leqq t\leqq 1$. The author has defined (see previous abstract) an integral on C' related to the Wiener integral on $C=E_x[x\in C', x(0)=0]$ by the formula $\int_{C'}F[x]dx=\int_{-\infty}^{\infty}d\xi\int_C^w F[y+\xi]d_wy$. Using this relation and transformation formulas for the Wiener integral given by R. H. Cameron and W. T. Martin (Trans. Amer. Math. Soc. vol. 58 (1945) pp. 184–219), formulas are obtained for the integral on C' identical in appearance with those for the Wiener integral on C. (Received November 10, 1949.)

146*t*. H. F. MacNeish: *The application of the logarithmic method of integration to the binomial differential.*

From the Chebyshev theorem, $-U=\int x^p(1-x)^q dx$, p and q rational, is integrable in a finite number of terms containing only elementary functions, if at least one of p, q, and $p+q$ be an integer. For the binomial differential this theorem becomes: $I=\int x^m(a+bx^n)^{r/s}dx$, for m, n, r, s rational, $s\neq 0$, is integrable in a finite number of terms containing only elementary functions, if at least one of r/s, $i=(m+1)/n$, $j=(m+1)/n+r/s$ be an integer. The following results are proved by elementary methods: Case 1, i an integer. (a). If i is an integer not less than 1, I is algebraic and can be found by the logarithmic method of integration. (b). If i is an integer less than 1, I is transcendental. Case 2, j an integer. (a). If j is an integer less than 0, I is algebraic and can be found by the logarithmic method of integration. (b). If j is an integer not less than 0, I is transcendental. (Received November 14, 1949.)

147. Szolem Mandelbrojt and H. D. Brunk: *A composition theorem for asymptotic series.*

In a recent theorem (S. Mandelbrojt, *Quelques théorèmes de composition*, C. R. Acad. Sci. Paris vol. 226 (1948) pp. 1155–1157), Mandelbrojt gives conditions sufficient for one of two infinitely differentiable functions of a real variable to vanish identically, provided that the product of their $2k$th derivatives vanishes for each non-negative integer k. The following analogue, for holomorphic functions having asymptotic representations in a half-plane, is proved: if (i) $F(z)$ and $\Phi(z)$ $(z=x+iy)$ are holomorphic in the half-plane $x>0$ and continuous on $x=0$, if (ii) there exists a positive number δ such that $|F(z)|=O(|z|^\delta)$ and $|\Phi(z)|=O(|z|^\delta)$ as $z\rightarrow 0$, if (iii) there exist sequences $\{a_n\}$ and $\{b_n\}$ of complex numbers such that $a_{2k-1}=b_{2k-1}=0$, $a_{2k}b_{2k}=0$ $(k=1, 2, \cdots)$, and sequences $\{M_n\}$ and $\{M'_n\}$ of positive numbers such that $\log M_n$ and $\log M'_n$ are convex functions of n, and such that $|F(z)-\sum_1^n a_k z^{-k}|<M_n|z|^{-n}$ and $|\Phi(z)-\sum_1^n b_k z^{-k}|<M'_n|z|^{-n}$ for $x>0$, and if (iv) $\sum M_n M'_n/M_{n+1}M'_{n+1}=\infty$, then either $F(z)\equiv 0$, in which case $a_k=0$ $(k=1, 2, \cdots)$, or $\Phi(z)\equiv 0$, in which case $b_k=0$ $(k=1, 2, \cdots)$. (Received October 25, 1949.)

148*t*. C. N. Moore: *A convergence factor theorem in general analysis.*

The purpose of the author is to prove a theorem in general analysis which includes as special cases a theorem concerning series due to Hardy and Bohr and an analogous theorem concerning integrals obtained by Hardy. Using the notations introduced in a previous paper (Trans. Amer. Math. Soc. vol. 24 (1922) pp. 79–88), this theorem deals with two classes of functions $\eta(\sigma)$, $\phi(\sigma)$, an operation $(J\eta)(\sigma)$, and the conditions under which the existence of the generalized limit $\lim_\sigma (C_n\eta)(\sigma)$ implies the existence of the limit $\lim_\sigma (C_n[\eta\phi])(\sigma)$. (Received November 4, 1949.)

149. C. N. Moore: *A generalized Tauberian theorem for Dirichlet's series, with application to the prime pair problem.*

In the present paper a Tauberian theorem for Dirichlet's series obtained by Ikehara and later somewhat generalized by Wintner is extended to series of the type $\sum a_n(x)e^{-\lambda_n s}$, which for a particular value of x are Dirichlet series but for a variable x constitute a generalization of such series. It is shown that if the function $f(s, x)$ defined by the above series for $R(s)>1$, in which region the series converges absolutely, tends to a continuous limit as $R(s)\rightarrow 1$, for $|I(s)|>0$, and if further $f(s, x)$

$\sim\phi(x)/(s-1)$ as $s\rightarrow 1$, then $(\sum a_n(x))_{\lambda_n \leqq x} \sim e^s\phi(x)$. This theorem may be applied to the problem of the distribution of prime pairs. (Received December 19, 1949.)

150t. Zeev Nehari: *Conformal mapping of open Riemann surfaces.*

It is shown that open Riemann surfaces of given genus and bounded by a finite number of closed curves can be mapped conformally onto certain types of multiply-covered slit-domains which are analogous to the classical schlicht slit-domains. It is likewise shown that many of the relations between various canonical mapping functions and domain functions have their counterpart in the theory of open Riemann surfaces. (Received December 7, 1949.)

151t. Zeev Nehari: *Extremum problems in the theory of bounded analytic functions.*

Let B be the family of functions $f(z)$ which are regular and single-valued and satisfy $|f(z)| \leqq 1$ in a multiply connected domain D. Generalizing the classical case in which D is the unit circle, it is possible to pose a large number of extremum problems for continuous functionals defined on B. Some problems of this kind were solved by L. V. Ahlfors (Duke Math. J. vol. 14 (1947) pp. 1–11), P. R. Garabedian (Harvard Thesis, 1948) and the author (Duke Math. J. vol. 15 (1948) pp. 1033–1042). In the present paper, a method is developed which permits one to solve a large variety of such problems. The method works equally well when the class B is restricted by various additional conditions. In all cases, the extremal functions yield $(1, m)$ conformal maps of D onto the unit circle, where m depends on the particular problem and can be explicitly stated in each case. The method consists in associating with each particular extremum problem an allied extremum problem within a family of specified functions involving a finite number of parameters. The boundary values of the solution of the associated problem are used to set up a Dirichlet problem, which is then shown to yield the solution of the original problem. (Received November 14, 1949.)

152t. Athanasios Papoulis: *On the strong differentiation of the indefinite integral.*

Given a function $f(x, y)$ integrable on a plane set E of positive measure, form $W(I)=\int\int_{I\cdot E} f(x, y)dxdy$ and $F(I)=\int\int_{I\cdot E}|f(x, y)|dxdy$. It can be seen that from the existence of the strong derivative $F'_s(x, y)$ pp. on E, the existence pp. of $W'_s(x, y)$ follows. The purpose of the author is to show that the converse is not true. (Received December 9, 1949.)

153. F. W. Perkins: *Properties of Nicolesco's means for certain classes of functions.* Preliminary report.

Nicolesco has defined a denumerable set of related integral means of particular interest in connection with the theory of polyharmonic functions (Bull. Soc. Math. France vol. 60 (1932) pp. 129–151). In the present paper the author gives a number of examples illustrating properties of these means as applied to polyharmonic functions and some metaharmonic functions. (Received November 14, 1949.)

154. Tibor Rado and P. V. Reichelderfer: *On n-dimensional concepts of bounded variation, absolute continuity, and generalized jacobian.*

Let f be a bounded continuous mapping from a bounded connected open set in

n-dimensional space into n-dimensional space. The purpose of the authors is to develop from f concepts of essential bounded variation, essential absolute continuity, and essential generalized jacobian which generalize those presently in the literature for the 2-dimensional case [T. Radó, *Length and area*, Amer. Math. Soc. Colloquium Publications, vol. 30, 1948]. Transformation formulas for n-tuple integrals and closure theorems for essentially absolutely continuous transformations are developed, which represent complete generalizations of those presented in Part IV of the above reference. (Received November 14, 1949.)

155. Walter Rudin: *A theorem on subharmonic functions.*

Let P denote a point in the plane. Put $\Delta_r f(P) = m(f; P, r) - f(P)$, where $m(f; P, r)$ denotes the mean of the function f on the circle of radius r, with center at P. The following theorem is proved. Suppose (i) the function u is upper semi-continuous in the finite plane domain D; (ii) $\limsup_{r\to 0} \Delta_r u(P)/r^2 \geqq 0$ for P in $D-E$, where E is a countable subset of D; (iii) $\limsup_{r\to 0} \Delta_r u(P)/r \geqq 0$ for P in E. Then u is subharmonic in D. The theorem is used to extend some of the author's previous results (Bull. Amer. Math. Soc. Abstracts 55-7-303 and 55-11-541). (Received November 14, 1949.)

156. Robert Schatten: *Some Banach spaces of operators.* Preliminary report.

The conjugate space of the Banach space of all linear bounded operators on a Hilbert space is characterized. In it the "trace-class" of operators on a Hilbert space can be imbedded in a "natural" fashion. (Received November 14, 1949.)

157*t*. I. E. Segal: *Equivalences of measure spaces.*

Any one of the following conditions on a measure space M implies all the others, and defines a "localizable" space: (a) completeness of the lattice of all measurable sets modulo null sets, (b) being strongly equivalent to (that is, having a measure ring isomorphic to that of) a direct sum of finite measure spaces, (c) the algebra $A(M)$ of all multiplications by bounded measurable functions on $L_2(M)$ being maximal abelian in the algebra of all bounded linear operators, (d) validity of the conclusion of the Radon-Nikodym theorem for M. Any uniformly locally compact space is localizable, and so also is any "perfect space," this being a regular locally compact measure space in which for each measurable set of finite measure there is a unique continuous function equal nearly everywhere to the characteristic function. Any measure space is "metrically" equivalent (equivalent as far as integration is concerned) to a unique perfect space (for finite spaces the existence part of this is due to Kakutani). Two measure spaces M_1 and M_2 are strongly equivalent if and only if the corresponding multiplication algebras $A(M_1)$ and $A(M_2)$ are algebraically isomorphic, which in turn is the case if and only if they are unitarily equivalent. (Received November 14, 1949.)

158*t*. Seymour Sherman: *Non-negative observables are squares.*

It is shown that the sum of squares of observables is the square of an observable. This makes the hypotheses of Theorem 4 and Corollary 4.1 of I. E. Segal, *Postulates for general quantum mechanics*, Ann. of Math. vol. 48 (1947) pp. 930–948 redundant, so that (1) in a system of observables a pure state of the subsystem coincides on that subsystem with a pure state of the system and (2) a spectral value of an observable equals the value of the observable in some pure state of the system. Thus part of the gap between systems of observables and the collection of self adjoint elements of a C^* algebra is closed. (Received October 27, 1949.)

159. Seymour Sherman: *On a theorem of Banach.*

S. Banach [Studia Math. vol. 10 (1948) pp. 159–179] has shown that if $\mathfrak{A}$ is a denumerably independent family $\{\mathfrak{B}^\gamma | \gamma \in \Gamma\}$ of Borel fields of subsets of Ω, where with each $\mathfrak{B}^\gamma$ there is a measure m^γ such that $m^\gamma(\Omega)=1$, then there exists a common extension measure m of each of the m^γ to $\mathfrak{B}$ the least Borel field containing the $\mathfrak{B}^\gamma$ with respect to which measure the $\mathfrak{B}^\gamma$ are stochastically independent. The set theoretical theorem that $\mathfrak{B}$ is isomorphic to the Borel field generated by the rectangles of a certain product space and standard results on the existence of an independent product measure yield a simple proof of Banach's theorem and clarify its relation to the theory of product measures. (Received September 28, 1949.)

160*t*. Seymour Sherman: *Order in operator algebras.*

The set $\mathfrak{S}$ of self-adjoint elements of C^* algebra $\mathcal{A}$ forms a real vector space which may be ordered by $U \geqq 0$ meaning the spectrum of U is real and non-negative. If the hermitian elements of a self-adjoint algebra (with or without unit, no topological assumptions) of bounded operators on a complex Hilbert space forms a lattice then the algebra is commutative. By example, the various possibilities for lattice ordered subspaces of $\mathfrak{S}$ are explored. (Received November 14, 1949.)

161. Fritz Steinhardt: *On equivalent gauge functions.*

Let $F(x)$, where (x) stands for $(x_1, \cdots, x_n)$, be of class C^1 and be the gauge function of a (smooth) convex body K in E^n. The gauge function $G(x)$ is called ***equivalent*** to $F(x)$ if the convex body K' represented by $G(x) \leqq 1$ is congruent to K under a translation, say by $(b)=(b_1, \cdots, b_n)$. Write $G(x)=F(x; b)$. For the class $\{F(x; b); -b \in \text{int. } K\}$ of all gauge functions equivalent to $F(x)=F(x; 0)$, a system of first-order partial differential equations is obtained (*not* involving the support function of K). $F(x; b)$ is convex in b also (even logarithmically so). Applications to the theory of polar convex bodies are considered, as well as to gauge functions as such. For example, the property $\int_\Omega (\partial F(\xi)/\partial x_i)/F^n(\xi) d\omega = 0$ $(i=1, \cdots, n)$ of (not necessarily convex) gauge functions (where Ω = surface of n-dim. unit sphere; $|\xi|=1$; $d\omega = d\omega(\xi)$) is an immediate consequence. (Received November 14, 1949.)

162*t*. Gabor Szegö: *On a generalization of Dirichlet's integral.*

Let B be a ring-shaped domain bounded by the curves K_0 and K_1 where K_0 is in the interior of K_1. Let K_0 contain the origin. The author denotes by $p(r)$ (r is the distance from the origin) a monotonically decreasing function for which $rp(r)$ is integrable in a neighborhood of $r=0$. Let $\mu=\mu(B)$ be the minimum of $\iint_B \{|\text{grad } u|^2 + p(r)u^2\} d\sigma - \iint_G p(r) d\sigma$ (G is the interior of K_1) where the admissible functions u satisfy the boundary conditions: $u=0$ on K_0, $u=1$ on K_1. Considering now all domains B whose boundary curves K_0 and K_1 have given areas, the quantity $\mu(B)$ will be a minimum if K_0 and K_1 are concentric circles about the origin. This is a generalization of a theorem of Carleman (Math. Zeit. vol. 1 (1918)) which corresponds to the special case $p(r)=0$. (Received November 14, 1949.)

163*t*. W. J. Thron: *Singular points of functions defined by C-fractions.*

In the C-fraction (1) $1+K(d_n x^{\alpha_n}/1)$ let $d_n \neq 0$ for all $n \geqq 1$, $\lim |d_n|^{1/\alpha_n}=1$, and let $\{\alpha_n\}$ be a monotone nondecreasing sequence of positive integers with $\lim \alpha_n = \infty$. It is known that under these conditions (1) converges to a function $f(x)$ which is meromorphic for $|x| < 1$. It is shown in this paper that $f(x)$ has at least one singular point (not a pole) on the circle $|x| = 1$. From this it follows that $f(x)$ has the circle $|x| = 1$ as a natural boundary provided that there exists a sequence of positive integers $\{m_k\}$ with $\lim m_k = \infty$ such that, for every k, m_k divides a_n for all $n > n_k$. This theorem contains the only previously known general results on this problem, due to Scott and Wall (Ann. of Math. (2) vol. 41 (1940) pp. 328–349) as special cases. (Received November 14, 1949.)

164. W. J. Trjitzinsky: *Multidimensional principal integrals, boundary value problems, and integral equations.*

In this work, a study is made of integrals in the sense of principal values, extended over surfaces with edges. Included are related boundary value problems of Hilbert-Riemann type and singular integral equations with principal kernels. (Received November 14, 1949.)

165. C. A. Truesdell: *A form of Green's transformation.*

Let the polyadic nth powers $b^{(n)}$ of a vector b be defined by $b^{(-1)} = 0$, $b^{(0)} \equiv 1$, $b^{(1)} = b$, $b^{(2)} \equiv bb$, $b^{(3)} = bbb$, and so on. Let Σ be any polyadic, and let $\{\Sigma b^{(n)}\} = b^{(n)}\Sigma + b^{(n-1)}\Sigma b + \cdots + \Sigma b^{(n)}$. It is shown that Green's transformation may be put into the form $\oint_S [dS\cdot(bc+cb)r^{(n)} - dSb\cdot cr^{(n)}] = \int_V [b\{cr^{(n-1)}\} + c\{br^{(n-1)}\} - b\cdot c\{r^{(n-1)}I\} + (\text{curl } c\times b + \text{curl } b\times c + b \text{ div } c + c \text{ div } b)r^{(n)}]dV$, where r is the radius vector. The case $n=0$ was given by Burgatti (Bollettino della Unione Matematica Italiana vol. 10 (1931) pp. 1–5). The cases $n=0$and $n=1$ yield as consequences several formulae useful in mechanics and physics. (Received November 14, 1949.)

166. J. L. Ullman: *A moment problem.*

Let $\alpha(t)$ be a nondecreasing function of total variation one, defined on the interval I_t $(-1 \leqq t \leqq 1)$. The moments of the non-negative density function $d\alpha(t)$ are defined to be the numbers $m_k = \int_{-1}^{1} t^k d\alpha(t)$ $(k=0, 1, \cdots)$. For each positive integer n, the numbers $t_{n1}, \cdots, t_{nn}$, which may be complex, are uniquely determined by the conditions that $(1/n)\sum_1^n t_{ni}^k = m_k$ $(k=1, \cdots, n)$. Let Δ be the derived set of the t_{ni} $(n=1, 2, \cdots; i=1, \cdots, n)$. The answer to the question of when Δ lies entirely in I_t is as follows. Let M_j be the constant term in the expression $z^j(\sum_1^\infty m_k/z^k) \exp(-j\sum_1^\infty m_k/kz^k)$, and let $\tau = \liminf \int_{-1}^{1} \log|z-t|\,d\alpha(t)$, for $|z-1| + |z+1| > 2$. The necessary and sufficient condition that Δ be contained entirely in I_t is that $\limsup (\log M_j/j) \leqq \tau$. This theorem is proved using methods developed in the author's *Studies on Faber polynomials*, being prepared for publication. (Received November 15, 1949.)

167*t*. M. A. Woodbury: *A decomposition theorem for finitely additive set functions*. Preliminary report.

Let m be a finitely additive measure defined for a field F of subsets of a base set W. Let $m_0^*(A) = \inf \sum_{i=1}^{\infty} m(A_i)$, where the infimum is over coverings of A by a denumerable family of subsets A_i in F. Then m_0^* is a Carathéodory outer measure and there eixsts a countably additive measure function m_0 defined for a Borel field of sets B which is shown to contain the field F. Since $m_0(A) \leqq m(A)$ for all A in F, the difference $m_1 = m - m_0$ is a purely finitely additive measure. If F consists of all subsets of a denumerable set, then m_1 vanishes for finite sets and W is the union of countably many sets of m_1 measure zero. That the requirement of denumerability of W cannot be relaxed and conclusion retained is shown by an example of B. Jessen who developed the decomposition theorem separately. (Received November 14, 1949.)

168*t*. M. A. Woodbury: *Invariant functionals and measures.*

Let G be a commutative semi-group of linear transformations of a linear space L into itself leaving a subspace L_0 invariant. Let $p(x)$ be a Hahn functional (Banach, *Operations lineaires*, pp. 27–29) defined on L and invariant under G. Let $f(x) \leqq p(x)$ be a linear functional defined on L_0 and invariant under G. Then there exists $F(x)$ defined on L, coinciding with $f(x)$ on L_0 and such that $-p(-x) \leqq F(x) \leqq p(x)$ on L with the additional property that it is invariant under G. Also shown is a related result on bounded finitely additive measures defined over a quorum of subsets [Abstract 35, Ann. Math. Statist. vol. 20 (1949) pp. 141–142]. Specifically if $p(x)$ is such a measure which is invariant under a commutative semi-group of 1-1 transformations of the base set W into itself, then the measure can be extended to the field of all subsets of W and remain invariant under G. This implies a result of Banach (loc. cit. p. 231). (Received November 14, 1949.)

APPLIED MATHEMATICS

169. Nashman Aronszajn: *Approximation methods for eigenvalues of differential problems.*

In recent years the methods of Rayleigh-Ritz and of A. Weinstein were greatly extended and made more precise. These methods present a great advantage as compared with other methods in that they give lower and upper bounds of eigenvalues. The author develops new methods presenting the same advantages, which can be applied in cases where the generalized Rayleigh-Ritz and Weinstein methods are not applicable. (Received December 21, 1949.)

170. J. R. Foote and C. C. Lin: *Some recent investigations in the theory of hydrodynamic stability.*

Two new elements in the theory of hydrodynamic stability of parallel flows are discussed. (1) When the main velocity distribution extends to infinity in both directions, the "viscous" solutions must be rejected. An example of this case has been calculated by Lessen and investigated by Chiarulli from a more general point of view. (2) When the distribution gives rise to two "critical" points the proper branch of the asymptotic solution must be re-examined. It is shown in the paper that for determining the proper asymptotic solution of the stability equation, the path in the plane of the (complex) independent variable must be taken below the critical point where the velocity is increasing and above it where the velocity is decreasing. Specific discussions are restricted to the inviscid case, for which general criteria of stability are worked out. The problem of the instability of zonal winds in the atmosphere is then considered. It is shown that, by a proper transformation of coordinates, this problem can be treated by the present theory. The gradient of "absolute" rotation is shown to be the crucial quantity. (Received November 14, 1949.)

171. A. E. Heins: *A note on a pair of dual integral equations.*

E. C. Titchmarsh [*Theory of fourier integral*, p. 334] solves a pair of dual integral equations by some rather involved analysis. It is shown here that this problem can be solved by the method of Wiener and Hopf [Paley and Wiener, *Fourier transforms* *in the complex domain*, chap. 4]. The special character of the kernels of the integral equations and the application of the Mellin transform to these equations reduces the problem to one of factoring of the Wiener-Hopf type. (Received November 14, 1949.)

172*t*. S. N. Karp: *Conical solutions of the time free wave equation.*

The conical flows, or homogeneous harmonics of degree zero, have been employed by Busemann in the linearized theory of supersonic flow, and by J. Keller in the theory of reflection of pulses in optics and acoustics. (The equation arising in those cases is $\Delta u=0$, one of the variables being pure imaginary.) These solutions are generalized for use with equations $u_{xx}+u_{yy}+u_{zz}+k^2u=0$ and conical surfaces of arbitrary cross section, excited by a point source at the vertex. The semi-infinite triangular lamina is a special case. A new derivation of the diffraction of a plane wave by a semi-infinite plane is obtained. Various applications are considered. (Received December 27, 1949.)

173*t*. S. N. Karp: *Diffraction at wedge shaped boundaries with cylindrical tips.*

A series solution is obtained for the problem of the acoustic or electromagnetic reflection or diffraction of an arbitrary time periodic incident wave by a perfectly reflecting wedge of arbitrary angle whose tip is surrounded by a cylinder of radius a and center at the vertex of the wedge, by a generalization of the method employed by H. M. Macdonald (*Electric waves*, S. 186, Cambridge, 1902) for the same problem in the absence of the cylindrical tip. There are various applications including the transmission of waves over a ridge, the reflection of a wave by a "bump" on a wall, and the comparison of scattered field in the present case with the edge effect for the pointed wedge. (Received December 27, 1949.)

174*t*. S. N. Karp: *Leading and trailing edge solutions for the oscillating airfoil in subsonic compressible flow.*

Solutions of the two-dimensional flutter problem valid near the leading and trailing edge of an airfoil were studied by treating the airfoil as a semi-infinite plane whose leading or trailing edge is in evidence. After Lorentz transformation the linearized theory of subsonic flow leads to the equation $u_{xx}+u_{yy}+k^2u=0$. The velocity potential is represented as a superposition of the singular solution $(e^{ik\rho}/\rho^{1/2})\cos(\phi/2)$, where ρ and ϕ are polar coodinates referred to a variable, origin $(x_0, 0)$. The origin x_0 is distributed along the airfoil cross section in the leading edge case. In the trailing edge case, a distribution over the trailing vortex sheet is added. The resulting integral equation is solved by inspection, for vertical oscillations. The leading edge solution is equivalent to the scattered field in diffraction of a plane wave by a semi-infinite plane. In the case of more general excitations the integral equation may be transformed to an Abel equation by change of variable, and the solution is then well known. (Received December 27, 1949.)

175*t*. S. N. Karp: *Refraction at a wedge shaped boundary.*

An integral equation is obtained for the refraction and reflection of electromagnetic or acoustic waves at a wedge shaped boundary between two media. The integrals involved initially represent single and double layers on the boundary, and mass distributions. An iterative solution is described, and the first approximation is calculated. (Received December 27, 1949.)

176. S. N. Karp: *Wiener-Hopf techniques and mixed boundary value problems.*

The duality between Green's function and eigenfunction procedures for exact solution in simple boundary value problems is shown to carry over the mixed b.v. problems resulting when the coordinate system is not perfectly suited to the boundaries. The Green's function procedure in the latter case has been introduced into diffraction problems by J. Schwinger and generalized by G. Carrier. The problems thus treated hitherto are shown to correspond to use of cartesian coordinates in separation, where the boundary cross section is one or more semi-infinite lines. These are characterized as two part b.v. problems. The three part boundary problem ($y=0$; $-\infty<x<0$; $0<x<1$; $1<x<\infty$) is reduced to a two part boundary in polars (r, θ), and solved in the case of Laplace's equation. (The eigenfunctions are $r^{\pm\nu}e^{\pm i\nu\theta}$, and this suggests Mellin transforms in the Green's function procedure.) A similar treatment may be employed for a circular disc, with Laplace's equation (using spherical coordinates). The question is raised of carrying out the solution for these two geometries when $\Delta u+k^2u=0$. (Received December 27, 1949.)

177. E. D. McCarthy: *The motion of a particle which changes mass when acted upon by a plane central force.* Preliminary report.

Let a large mass M attract a point mass m with a force $(-kv^2Mm/r^2)$ which changes m, the direction of the velocity, but not $|v|$. If this force is added to a gravitational force let $|v(r)|$ be the same as if the gravitational force acted alone. The equations of motion are $D_t m v_i = m a_i + v_i D_t m = f_i$ $(i=1, 2)$. If the added force is Newtonian, $m=m_0 \exp (kM/r)/\exp (kM/r_0)$. Results agree closely with general relativity for two problems of the solar system and have a removable disagreement for the motion of Mercury. The formula from physics $(h\nu=mc^2)$ is used in discussing the frequency (ν) of a light ray in a gravitational field. M is taken as $(2m_0-m)$ when a small particle of mass m is attracted by a second particle and the mass of each is m_0 at $r=r_0$. Gravitation is considered negligible and $|v|$ as constant. It is found that if km_0 is greater than r_0 and the ratio $v_r/|v|$ is not too large, the first particle can be restricted to a closed region. (Received November 28, 1949.)

178*t*. L. F. Meyers and Arthur Sard: *Best approximate integration formulas.*

Among the approximations $c_0x(0)+c_1x(1)+\cdots+c_mx(m)$ of $\int_0^m x(t)dt$ which are exact for degree n, those which are best in the sense of a previous paper (Amer. J. Math. vol. 71 (1949) pp. 80–91) are given for $n=1$, $1\leqq m\leqq 20$; $n=2$, $2\leqq m\leqq 12$; $n=3$, $2\leqq m\leqq 9$. Recursive relations are established which determine the best formulas for $n=1$, all m. (Received November 7, 1949.)

179. L. F. Meyers and Arthur Sard: *Best interpolation formulas.*

Best interpolation formulas in the sense of a previous paper (Amer. J. Math. vol. 71 (1949) particularly p. 90) are studied. The basic relationships are set forth in matrix form. Among formulas which are exact for degree 0 and use any number of ordinates, the conventional formulas (that is, those using fewest ordinates) are best. Likewise among formulas which are exact for degree 1 and use no more than 5 ordinates. Among formulas which are exact for degree 2, the conventional formulas are not best. The best formulas among those which are exact for degree 2 and use no more than 6 ordinates are given explicitly. (Received November 7, 1949.)

180t. K. S. Miller and R. J. Schwarz: *Analysis of a sampling servomechanism.*

An analysis of a sampling servomechanism with an error-clamping device and linear forward and return paths is given. The method leads to a determination of the continuous output of the system in terms of quadratures and gives explicitly the value of the output at the discrete sampling instants. Frequently this is all that is needed in order to study the output variation with time. The question of stability is discussed and a criterion for testing the stability of the system is given in terms of the system parameters. (Received December 15, 1949.)

181t. H. H. Mostafa: *On flows governed by simplified gas dynamical equations.*

The equation of state conidered is a polygonal approximation to the adiabatic equation of state. Flows governed by this equation have been considered before by others. The author obtains the prametric representation $z = A_\nu B_\nu f(\zeta) - 4^{-1}(A_\nu/B_\nu)\int([G'(\zeta)]^2/f'(\zeta))d\zeta$ of the complex coordinate z of the physical plane in terms of an arbitrary analytic function $f(\zeta)$. Here A_ν, B_ν are constants and $G(\zeta)$ is the complex potential of an incompressible flow around a circle in the ζ-plane. This is a generalization of Gelbart's representation [N.A.C.A., T.N. 1170] for a flow under the linearized equation of state. This polygonal approximation is equivalent to considering a step-function approximation to the function $K(t)$ in the equations $\phi_\theta = \psi_t$, $\phi_t = -K(t)\psi_0$, where $K(t) = (1-M^2)/\rho^2$; θ, t the independent variables in a modified hodograph plane, ρ density, M Mach number. In each region for which the step function is constant in t, the partial differential equations can be reduced to Cauchy-Riemann equations. In an unpublished paper, Bers and Gelbart have obtained a parametric representation in terms of the hodograph variables. The author uses their results to determine the coefficients A_ν, B_ν. A practical method is also given to obtain, flows round preassigned bodies by suitably computing $f(\zeta)$. (Received November 14, 1949.)

182. G. G. O'Brien, M. A. Hyman, and Sidney Kaplan: *On the numerical solution of partial differential equations.* I.

In a numerical solution one customarily attacks a partial difference equation approximating the partial differential equation to be solved. Thus an error of approximation is introduced. Additional errors enter during the process of solving the difference equation numerically. Here precise definitions are given of *truncation error* and *numerical error* as well as of the *convergence* and *stability* (*strong* and *weak*) of a given numerical procedure. It is shown that *a satisfactory numerical procedure for solving a partial differential problem (equation plus accessory conditions) must be both convergent and strongly stable.* Detailed investigation is restricted to hyperbolic and parabolic problems, whose numerical solution involves "stepping ahead." Strong stability is a necessary property of a satisfactory numerical procedure but weak stability is more readily analyzed. The gap is bridged here and weak stability is studied, using extensively the Fourier-analytic method outlined by von Neumann but never published. This method gives quantitative information about error growth and emphasizes the advantage to be obtained from use of "implicit" numerical procedures. A large number of calculations have been carried out for the parabolic equation $\phi_t = \phi_{xx}$. It has been found: (a) for a (weakly) stable numerical scheme, the truncation error is small and the numerical error negligible; (b) for a (weakly) unstable numerical scheme, the numerical error is appreciable but completely overshadowed by the truncation error. (Received October 21, 1949.)

183*t*. Eric Reissner: *On the theory of beams on an elastic foundation.*

The problem in question is formulated by means of the following system of equations: (1) $Bd^4w/dx^4 = q(x) - p(x)$, (2) $w(x) = \int_a^b k(|x-\xi|)p(\xi)d\xi$, with appropriate boundary conditions at a, b. The results obtained include (1) determination of a kernel function for the *three*-dimensional elastic foundation, (2) reduction of the problem to an integral equation of the first kind for the foundation pressure distribution p. In this reduction the boundary conditions are taken in the form $w''(a) = w''(b) = w'''(a) = w'''(b) = 0$. It is shown that for a class of kernels which include those for the two-dimensional and for the three-dimensional elastic half-space as foundation, the nature of the integral equation for p is analogous to certain equations arising in aerodynamics so that established techniques may be applied in the solution. The importance of the present reduction lies in the fact that with it the use of relations between divergent series is avoided. Such divergent series are encountered in a direct solution of equations (1) and (2). (Received November 14, 1949.)

184. E. K. Ritter: *Second differential effects on trajectories.*

An earlier paper (*Second differentials of functions in exterior ballistics*, Bull. Amer. Math. Soc. Abstract 55-11-537) established the existence of the second differential of the mapping defined by the equations of motion of the mass-center of a projectile. This second differential is expressible in terms of the second variations of the coordinates of the mass-center. The present paper contains: (1) formulas for the second variations in terms of double Stieltjes integrals; (2) formulas for the second differential effects of abnormal forces on trajectory elements (range, time-of-flight, and so on) in terms of the second variations of the mass-center coordinates; and (3) a theorem on the second differential analogous to that of McShane (*The differentials of certain functionals in exterior ballistics*, Bull. Amer. Math. Soc. Abstract 55-1-37) on the first differential. (Received November 14, 1949.)

185. J. B. Rosser: *A general iteration scheme for solving simultaneous equations.*

Solving simultaneous linear equations and making least squares fit are both special cases of the following general problem. Given a vector b and vectors c_i, to find a linear combination $q = b - \sum \alpha_i c_i$ of minimum length. The following simple procedure is studied. First subtract from b a multiple of c_1, choosing the factor so as to minimize the length. From the result, subtract a multiple of c_2, again minimizing the length. After going through all the c's, start over with c_1 and repeat the process. It is proved that the process converges to the desired q. It is shown that the process is improved if the c's are taken in reverse order on alternate cycles. The rate of convergence is studied, and it is shown that when slow convergence occurs, the process reduces to the summation of a slowly convergent geometric series; using this fact one can modify the process so as to get rapid convergence. The process is essentially free from round off errors, and is admirably suited for use with modern high speed computing machines. Several known methods of solution of equations by iteration are shown to be special cases of the general procedure outlined. (Received November 9, 1949.)

186*t*. J. B. Rosser: *Transformations to speed the convergence of series.*

Numerical instances are given of the speeding of the convergence of series by the Euler transformation. This is even applied advantageously to certain divergent series, and a rigorous justification is given. An example is given of a series for which use of the Euler transformation is not useful. Instances are given of several less widely known methods. Finally, the method of summation by transformation into a continued fraction is illustrated successfully in the case of certain divergent series. The possibility of applying two different methods in succession to a given series is exploited throughout the paper, in spite of the fact that this often requires summing a divergent series. (Received November 9, 1949.)

187. Charles Saltzer: *On the numerical determination of the conformal mapping function of a nearly circular region.*

Theodorsen and Garrick (T. Theodorsen, *Theory of wing sections of arbitrary shape*, N.A.C.A., T. R. No. 411; T. Theodorsen and I. E. Garrick, *General potential theory of arbitrary wing sections*, N.A.C.A., T. R. No. 452) reduced the problem of finding the function which maps a "nearly circular region" whose boundary is given in polar coordinates by $\rho=\rho(\theta)$ $(0\leqq\theta\leqq 2\pi)$ on the interior of the unit circle to the problem of solving the integral equation, $\theta(\phi)-\phi=L\{\log\rho[\theta(\phi)]\}=-(1/2\pi)\int_0^\pi\{\log\rho[\theta(\phi+t)]-\log\rho[\theta(\phi-t)]\}\cot(t/2)dt$. They proposed solving this equation by iteration, replacing the integrals involved by finite sums. Warschawski (S. E. Warschawski, *On Theodorsen's method of conformal mapping of nearly circular regions*, Quarterly of Applied Mathematics vol. 3 (1945) pp. 12–28) proved under certain hypotheses that the sequence of functions $\{\theta_n(\phi)\}_{n=0}^{\infty}$ defined by $\theta_0(\phi)=\phi$, $\theta_{n+1}(\phi)-\phi=L\{\log\rho[\theta_n(\phi)]\}$ converges to the solution of this integral equation. The author shows that the procedure of Theodorsen and Garrick is equivalent to replacing the integral equation by a system of simultaneous equations $\bar{\theta}-\bar{\phi}=H\{\log[\rho(\bar{\theta})]\}$ where $\bar{\theta}=(\bar{\theta}^1, \bar{\theta}^2, \cdots, \bar{\theta}^m)$, $\bar{\phi}=(2\pi/m, 4\pi/m, \cdots, 2\pi)$ and approximating the solution of this system by the sequence $\bar{\theta}_0=\bar{\phi}$, $\bar{\theta}_{n+1}-\bar{\phi}=H\{\log\rho(\bar{\theta}_n)\}$. It is proved under certain conditions that this sequence converges to the solution $\bar{\theta}$ of the above system. Estimates are given for $|\bar{\theta}-\bar{\theta}_n|$ and for the accuracy of the approximation of the solution of the integral equation by the solution of the system of simultaneous equations. (Received November 14, 1949.)

188*t*. B. R. Seth: *Finite elasto-plastic deformation of a rotating disc.*

In the first paper on finite elasto-plastic deformation it was shown that, for a first approximation which corresponds to ordinary small strains, both the plastic flow and deformation theories give similar results for the torsion of a circular cylinder. A second approximation, obtained by using finite components of displacements and strains, showed that unlike the plastic flow theory, the deformation theory gives discontinuities in the stresses across the elasto-plastic boundary, and hence could not be expected to give satisfactory results. In the second paper the rotation of a circular cylinder was discussed and explicit forms for the components of displacements were obtained. In the present paper the method is extended to a rotating disc. (Received November 4, 1949.)

Geometry

189*t*. L. M. Blumenthal: *Generalized euclidean space in terms of a quasi inner product.*

An abstract quasi inner product space Q is formed by an abstract set S in which a binary operation (x, y) on SS to R (the set of real numbers) is defined (quasi inner product) such that (1) $(x, y) = (y, x)$, (2) $(x, x) \geqq 0$, and (3) if $(x, x) = (x, y) = (y, y)$ then $x = y$. The author characterizes generalized euclidean spaces (real, normed linear spaces with an inner product related to the norm in the customary way) among the class of spaces Q by means of three conditions on the quasi inner product and three (existence) conditions on the space. Addition, scalar multiplication, and norm are *defined in such a space Q entirely in terms of the quasi inner product*, and these operations are proved to possess the desired properties. It is noted that the problem solved in this paper is the reverse of the well known problem of defining an inner product in a normed linear space, solutions of which have been given by Fréchet, Jordan and von Neumann, Aronszajn, Birkhoff, and others. (Received December 5, 1949.)

190*t*. Edward Kasner: *The spherical sections of an arbitrary surface.*

The system of ∞^3 plane sections of any surface S has been extensively studied. In the present paper the system of ∞^4 spherical sections of S is investigated. The differential equation, in arbitrary coordinates, is of the form $y'''' = Ay'''^2 + By''' + C$. This form presents itself in many dynamical situations and in the calculus of variations. The fundamental property is that the ∞^1 curves determined by a given point, direction, and curvature have osculating conics whose centers lie on a conic. The spherical curves of S may be regarded as another analogue of circles, to be compared to the standard definitions of Gauss and Minding. (Received November 21, 1949.)

191*t*. Edward Kasner and John DeCicco: *General theorems of physical systems of curves.*

The five characteristic properties of a physical system S_k of ∞^3 curves connected with an arbitrary field of force in the plane are discussed. Additional properties of a system S_k are obtained. If the lineal element $E(x, y, \theta)$ is fixed and k varies, the corresponding focal circles form a hyperbolic pencil, all of which pass through the fixed point $P(x, y)$ of E and a point Q on the line of E. The foci of the osculating parabolas of the actual curves of a system S_k describe the circular arcs PQ. The directrices of the osculating parabolas all pass through a point D. As k varies, D describes the line DQ, which is perpendicular to PQ at Q. As θ changes, D describes a conic section, and the lines to which the osculating parabolas are tangent envelop an algebraic curve of class four. The envelope of the focal circles, where k is fixed and θ varies, is a bicircular quartic with a singular point at P. This reduces to a circle in the case S_0 of dynamical trajectories. The locus of the centers of the focal circles is a conic. Finally the point Q describes a conic which passes through P in the direction of the force vector at P. (Received September 29, 1949.)

192. Edward Kasner and John DeCicco: *The hyperosculating speed for physical systems of curves.*

For a given physical system S_k, there is a single trajectory through a lineal element E such that it has four point contact with its osculating circle at E. The corresponding hyperosculating speed v of the particle, and the curvature κ at E, depend on

(k, x, y, y'). The cases where v and κ are constants, or depend only on (k, x, y), are studied in detail. For the general case, some of the results follow. If v_1 and v_2 are the hyperosculating speeds of the systems S_{k_1} and S_{k_2} at E, then $v_1^2/v_2^2=(3+k_1)/(3+k_2)$. For a given S_k, there are two possible extremal speeds v_1 and v_2 at an ordinary point P, for which the directions λ_1' and y_2' are symmetrical with respect to the line of force through P. If, through P, λ_1 and λ_2 are the angles between a direction y' and the two etremal directions y_1' and y_2', then $(v^2-v_1^2)/v_1^2 \sin^2 \lambda_1=(v^2-v_2^2)/v_2^2 \sin^2 \lambda_2$. Finally the ratio of the hyperosculating curvature of a system S_k to that of a velocity system S_∞ at any direction through P is $(1+k)/(3+k)$. This is another extension of Kasner's theorem which states that the curvature of the rest trajectory $(k=0)$ to that of the line of force is $1/3$. (September 29, 1949.)

193. T. S. Motzkin: *Initial sets in p-adic geometry*. Preliminary report.

Let $f(t)=\sum a_k t^k$, $k=0,1, \cdots$, with coefficients a_k in an algebraically closed field K of characteristic p, and $f(t+c)=\sum g_k(c)t^k$. The set of integers k for which $g_k(c) = 0$ does not hold identically in c is an initial set, that is, together with k it contains every k' whose p-adic digits are not greater than the corresponding digits of k. For $p=0$ write $p=\infty$; the condition for k' becomes $k'\leqq k$. Let $f(x_0)=\sum a_k x_0^k$, $k=0, 1, \cdots, n$, be a form in $x_0, \cdots, x_d$ with coefficients in K. The initial set of f, considered as type of the point $(1,0, \cdots ,0)$ with respect to the hypersurface $f=0$, provides a first classification of singular points on and outside $f=0$. Let $x_i(t)=\sum a_{ik} t^k$, $k\geqq k_i$, $0=k_0<k_1< \cdots <k_d$, be a branch in projective d-space over K. For the branch of an irreducible algebraic curve C at its general point ξ (after $K(\xi)$-reduction), $k_1=1$ and $(k_0, \cdots, k_d)$ is an initial set. If and only if $k_d<p$ (whence $k_i=i$ and $d<p$) does the dual of the dual of C coincide pointwise with C, which is tantamount to saying that the natural correspondence between C and its dual is birational. Case $d=2$ of this is the condition given in Bull. Amer. Math. Soc. Abstract 56-1-25. (Received November 14, 1949.)

194. T. S. Motzkin: *Main exponents and p-adic curve branches*. Preliminary report.

Let $f(t)=\sum a_k t^k$, $k\geqq -n$, with coefficients a_k in an algebraically closed field of characteristic p. Let k_e, $e=0, 1, \cdots$, be the smallest $k\not\equiv 0$ (p^{e+1}) with $a_k\neq 0$, and $k_\infty=0$ if $n=0$, $a_0\neq 0$; these are the main exponents. For unitary parameter transformations $t=u+\sum c_k u^k$, $k\geqq 2$, the general $f(t)$ with given main exponents has a finite number of independent invariants which equals the number of its terms from the first to the last main exponent, both included. Every $f(t)$ can be transformed into a polynomial with this number of terms, which thus furnishes a canonical expansion generalizing the Puiseux expansion with $f(t)=a_0+a_m t^m$ for $p=0$. The main exponents of the projective coordinates vanishing at the origin of a, not necessarily plane or algebraic, branch B define a finite set of projectively covariant linear manifolds belonging to B, including the tangent, quasitangent, and quasiorigin of Bull. Amer. Math. Soc. Abstract 56-1-25. (Received November 14, 1949.)

Logic and Foundations

195. A. H. Copeland: *Logic and Boolean rings*.

A more precise formulation and a deeper understanding of logic can be achieved by means of Stone's representation theorem and the theory of ideals in Boolean rings.

In the present paper the following topics in logic are studied from the point of view of ideal theory: proposition, meaning, truth, falsity, truth tables, strict implication, material implication, non-transitive implication, the nature of proof, propositional function, logical quantifiers, and scientific intuition. (Received November 14, 1949.)

196t. R. L. Goodstein: *The formal structure of a denumerable system.*

A strictly formal mathematical system comprising number and function signs, logical constants and operators, and employing the sentential calculus, is called *denumerable* if the application of the operators is restricted to variables whose range of values is a denumerable class. The system is not fully determined by the restriction on the operators but depends also upon the method of formalisation and the types of function definitions allowed. The denumerable system under consideration is a finitist superstructure built on an extension to a rational field of the Hilbert-Bernays formalisation Z_μ, function theoretic attributes being expressed by the method of proof schemata. All the foundation theorems of classical analysis are shown to have demonstrable analogues in the system. (Received September 23, 1949.)

197. Leon Henkin: *An algebraic characterization of quantifiers.*

A characterization of universal and existential quantifiers is given which is valid for functional calculi of the classical (two valued), intuitionistic, and modal systems of logic. This is accomplished by treating a subsystem H_f common to these three logics, in which the only propositional connective is an implication symbol. An algebraic structure called an *implicative model* is defined as a set of elements, including a special one, 0, on which is defined a binary operation $\dot{-}$ satisfying certain simple axioms. Writing $x \leqq y$ for $x \dot{-} y = 0$ it can be shown that a model is partially ordered with 0 as minimal element. Certain canonical mappings of the formulas of H_f into an arbitrary implicative model are described, in which quantifiers (x) and $(\exists x)$ correspond to operations sup and inf respectively. The principal theorem is to the effect that a formula is formally provable in H_f if and only if it is mapped into 0 by every canonical mapping. This provides an answer to a question posed by Mostowski (Journal of Symbolic Logic vol. 13 (1948) p. 207). (Received December 5, 1949.)

198. Ira Rosenbaum: *Ordered q-ads in an $\aleph_0$-membered universe.*

In a universe with $\aleph_0$ elements, denoted by the positive integers, one q-ad is to precede another if the sum of the q elements of the one is less than the corresponding sum for the other; q-ads in a set each of which has the same q-fold sum are to be ordered by Hausdorff's principle of last differences. In the resulting ordering, the order number, $N(x_1, \cdots, x_q)$, of the ordered q-ad, $(x_1, \cdots, x_q)$, is the sum of the *rank*, $R(x_1, \cdots, x_q)$, and the *height*, $h(x_1, \cdots, x_q)$, of the q-ad, where $R(x_1, \cdots, x_q)$ is the number of q-ads with sum less than $x_1+x_2+\cdots+x_q=s$, and $R(x_1, \cdots, x_q)$ equals the binomial coefficient ${}_{s-1}C_q$, while $(hx_1, \cdots, x_q)$ is the order number of the given q-ad among those with sum s, and is recursively determined by the relations, $h(x_1, \cdots, x_q) = \sum_{j=1}^{s-q+1} d(x_q, j)P(q-1, s-j)+(x_1, \cdots, x_{q-1})$, with $d(x_q, j)=0$ if $x_q \leqq j$, and $=1$ otherwise, $h(x_1, x_2)=x_2$, and $P(q-1, s-j)={}_{s-j-1}C_{q-2}=$ the number of (q-1)-ads with sum $s-j$. The formula $N(x_1, \cdots, x_q)=R(x_1, \cdots, x_q)+h(x_1, \cdots, x_q)$ is easily used and generalizes the familiar formula $N(x_1, x_2)=(x_1+x_2-1)(x_1+x_2-2)/2+x_2$ (Carathéodory, *Vorlesungen uber reele Funktionen*, pp. 28–29) in which x_2 is the height, and the product the rank, of the dyad (x_1, x_2). These processes are reversible; given N, the Nth q-ad is computable. (Received November 14, 1949.)

STATISTICS AND PROBABILITY

199t. H. D. Brunk: *Note on a theorem of Kakutani.*

A recent thorem of Kakutani (S. Kakutani, *On equivalence of infinite product measures*, Ann. of Math. (2) vol. 49 (1948) pp. 214–224), giving conditions necessary and sufficient for the equivalence of direct infinite product measures, is used to derive conditions of an apparently less restrictive nature sufficient in order that one of two sequences of independent random variables shall satisfy one of a large class of limit theorems, provided the other sequence satisfies the limit theorem. An example is given, illustrating the manner in which this result may be regarded as extending known sufficient conditions for limit theorems. (Received October 25, 1949.)

200. D. A. Darling: *On the limiting distribution of sums of random variables with infinite expectations.*

Let $X_1, X_2, \cdots$ be independent, identically distributed random variables which are non-negative and have the (common) absolutely continuous distribution function $F(x)=1-f(x)=\Pr\{X_k<x\}$, and let $S_n=X_1+X_2+\cdots+X_n$. If $E(x_k^\alpha)=\infty$ for all $\alpha>0$ it is well known that $T_n=(S_n-a_n)/b_n$ has no nondegenerate limiting distribution for any two sequences a_n and b_n. The author studies the limiting distribution for S_n under a different normalization, and proves that if $\lim_{x\to\infty} f(ax)/f(x)=1$ for all $a>0$ then $\lim_{n\to\infty} \Pr\{nf(S_n)<\alpha\}=1-\exp(-\alpha)$. He obtains, for example, for the example discussed by P. Levy, where $f(x)=(\log x)^{-1}$, $x>e$, the expression $\lim_{n\to\infty} \Pr\{S_n^{1/n}<\alpha\}=\exp(-(\log \alpha)^{-1})$. It is further shown that if $f(x)$ does not satisfy the above condition, no normalization yielding a nondegenerate limiting distribution for S_n exists. This result is an extension of the necessary and sufficient conditions given by Doeblin for S_n to have a semi-stable limiting distribution. If moments of all orders are infinite then S_n is essentially dominated by its largest term in the sense that if $X_n^+=\max(X_1, X_2, \cdots, X_n)$ then $S_n/X_n^*\to 1$ in the mean. Furthermore, if some moment is finite, but $E(X_k^{1-\delta})=\infty$, then S_n and X_n^* are of comparable magnitude in the sense that given any $\epsilon>0$ there exists an $M=M(\epsilon)$ such that $\Pr\{S_n/X_n^*>n\}\leqq\epsilon$ for all n sufficiently large. If $E(X_k)<\infty$, then $S_n/X^*\to\infty$ stochastically. (Received November 15, 1949.)

201. Einar Hille: *On continuous transition probabilities.*

The transition frequency $f(t; \xi, x)$ of a temporally homogeneous stochastic process normally satisfies the adjoint equations of Kolmogoroff: $b(\xi)S_{\xi\xi}+a(\xi)S_\xi=S_t$, $\{[b(x)T]_x-a(x)T\}_x=T_t$, $b(x)>0$, $-\infty<x<\infty$. The converse problem of deciding when such differential equations determine a unique frequency, leads to problems for semi-groups: C. When is $C[f]=b(\xi)f''+a(\xi)f'$ the generator of a semi-group $\{S(t)\}$, $t\geqq 0$, of positive contraction operators in $C[-\infty, \infty]$, leaving $g(x)=1$ invariant, and strongly continuous in t with $S(0)=I$? L. When is $L[f]=\{[b(x)f]'-a(x)f\}'$ the generator of a semi-group $\{T(t)\}$, $t\geqq 0$, of positive contraction operators in $L(-\infty, \infty)$, isometric on positive elements, and strongly continuous in t with $T(0)=I$? Necessary and sufficient conditions for the solvability of these problems are found in terms of integrability properties of $W(x)=\exp\{-\int_0^x[a(s)/b(s)]ds\}$ in the intervals $(-\infty, 0)$ and $(0, \infty)$. If L is solvable, so is C, but not vice versa. If both problems are solvable, then there is a unique transition frequency satisfying Kolmogoroff's equations. If C has a solution, but L does not, there is a transition frequency satisfying both equations, unique as far as the first equation is concerned. Cauchy's problem for the second equation, however, does not have a unique solution in $L(-\infty, \infty)$. (Received December 19, 1949.)

202*t*. R. B. Leipnik: *On the number of zeros of a random polynomial.*

By means of a result of Schur, upper bounds are obtained in terms of integrals for the probability that $a(0)z^n + \cdots + a(n)$ has p real zeros, where $a(0), \cdots a(n)$ are independent random variables with the same distribution F. These integrals are evaluated for several choices of F. (Received November 14, 1949.)

203*t*. Herman Rubin: *On the existence of nearly locally best unbiased estimates.*

For any family $\mathfrak{F}$ of distributions, and any distribution F_0 of $\mathfrak{F}$, there exists a bilinear function κ whose arguments are all parameters defined for all distributions of $\mathfrak{F}$ and for which there exist unbiased estimates which have finite variance if F_0 is the true distribution, and which has the following properties: (1) If θ is any parameter in the domain of κ, and t is any unbiased estimate of θ, then var $(t | F_0) \geqq \kappa(\theta, \theta)$. (2) This result is best possible, that is, for any θ there is an unbiased estimate t of θ whose variance differs from $\kappa(\theta, \theta)$ by less than any preassigned amount. (Received November 14, 1949.)

204*t*. Jacob Wolfowitz: *Minimax estimates of the mean of a normal distribution with known variance.*

The classical estimation procedures (point and interval) are proved to be minimax solutions of properly formulated estimation problems, sequential and non-sequential. For example, for the problem considered by Stein and Wald (Ann. Math. Statist. vol. 18 (1947) pp. 427–433), it is proved that, for any choice of parameters in the classical (fixed-sample) procedure C, there exists a positive constant c with the following property: Let G be the generic designation of the sequential estimation procedure, $\alpha(\xi, G)$ the probability under G that the estimating interval will contain the mean ξ, and $n(\xi, G)$ the expected number of observations. Then $1-\alpha(\xi, C)+cn(\xi, C) = \inf_G \sup_\xi [1-\alpha(\xi, G)+cn(\xi, G)]$. The result of Stein and Wald is an immediate consequence. Other such optimal properties are obtained. (Received September 12, 1949.)

205*t*. L. A. Zadeh and J. R. Ragazzini: *An extension of Wiener's theory of prediction.*

The authors give an extension of Wiener's theory of prediction which differs from the latter in the following respects: I. The signal (message) component of the given time series is assumed to consist of two parts: (a) a non-random function of time which is representable as a polynomial of degree not greater than a specified number n, and about which no information other than n is available; and (b) a stationary random function of time which is described statistically by a given correlation function $\psi_M(\tau)$. (In Wiener's theory, the signal may not contain a non-random part except when such a part is a known function of time.) II. The impulsive response of the predictor or, in other words, the weighting function $W(t)$ used in prediction, is assumed to vanish outside of a specified time interval $0 \leqq t \leqq T$. (In Wiener's theory T is assumed to be infinite.) The determination of $W(t)$ reduces to the solution of a modified Wiener-Hopf equation $\int_0^T w(\tau)\psi_{M+N}(t-\tau)d\tau = \lambda_0 + \lambda_1 t + \cdots + \lambda_n t^n$

$+\int_{-\infty}^{\infty} k(\tau)\psi_M(t-\tau)d\tau$, where the λ's are Lagrangian multipliers; $k(t)$ is the prescribed prediction operator (in the time domain), and $\psi_{M+N}(\tau)$ is the correlation function of the random part of the time series. An explicit solution of this equation is given. (Received November 11, 1949.)

TOPOLOGY

206. R. D. Anderson: *Monotone interior dimension-raising mappings.*

The author proves two theorems: (I) There exist a one-dimensional continuous curve M in the plane and a monotone interior mapping of M onto the plane; and (II) For any positive integer n, there exist a one-dimensional continuum M_n in Euclidean three-space and a monotone interior mapping of M_n onto Euclidean n-space. (Received November 14, 1949.)

207. R. H. Bing: *A characterization of 3-space by partitionings.*

It is known that if S is a compact continuous curve, there exists a decreasing sequence of regular partitionings $H_1, H_2, \cdots$ such that (a) H_i is a finite collection of mutually exclusive connected open sets whose sum is dense in S, (b) each element of H_i equals the interior of its closure, (c) H_{i+1} is a refinement of H_i, and (d) as i increases without limit, the diameters of the elements of H_i approach 0. A necessary and sufficient condition that a compact continuous curve be a simple solid (set topologically equivalent to a 3-sphere in Euclidean 4-space) is that one of its decreasing sequences of regular partitionings have the following 4 properties: (1) The boundary of each element of G_i is a simple surface. (2) If the boundaries of two elements of $\sum G_i$ intersect, this intersection is a 2-cell. (3) The intersection of the closures of three elements of G_i is 1-dimensional at each of its points. (4) The elements of G_i may be ordered $[g_{i1}, g_{i2}, \cdots, g_{in_i}]$ such that if g is an element of G_{i-1}($g=S$ if $i=1$) and j is an integer less than n_i, then $\bar{g}_{i,j+1}$ intersects $S-g+\bar{g}_{i1}+\bar{g}_{i2}+\cdots+\bar{g}_{ij}$ in a connected set. Unicoherence cannot be substituted for condition (4) because projective 3-space is unicoherent and has a decreasing sequence of regular partitionings satisfying conditions (1), (2), and (3). (Received November 14, 1949.)

208*t*. A. L. Blakers and W. S. Massey: *The homotopy groups of a covering.* Preliminary report.

Let $(X; A, B)$ be a triad (see A. L. Blakers and W. S. Massey, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) p. 323) with $x \in A \cap B$, $A \cup B = Y$. Let S^n be an n-sphere, $n>0$, E^n_+, W^n_- the "upper" and "lower" hemispheres, and p a point of the equator $E^n_+ \cap E^n_-$. Denote the function space of maps f: $(S^n; E^n_+, E^n_-, p) \rightarrow (Y; A, B, x)$ by $G^n(A/B, x)$ and the homotopy classes of $G^n(A/B, x)$ by $\pi_n(A/B, x)$. If $n>1$, $\pi_n(A/B, x)$ can be made into a group, called the ***nth homotopy group of the covering** A/B*. These groups are abelian if $n>2$ and fit into exact sequences: $\cdots \rightarrow \pi_n(A, x) \rightarrow \pi_n(A/B, x) \rightarrow \pi_n(B, A\cap B, x) \rightarrow \pi_{n-1}(A, x) \rightarrow \cdots$, and $\cdots \rightarrow \pi_{n+1}(X; A, B) \rightarrow \pi_n(A/B) \rightarrow \pi_n(X) \rightarrow \pi_n(X; A, B)$. Denote the function space of maps $(S^n, p) \rightarrow (X, x)$ by $F^n(X, x)$; let $G^n_0(A/B, x)$ and $F^n_0(X, x)$ be the components of $G^n(A/B, x)$ and $F^n(X, x)$ respectively which contain the constant map. Then the homotopy sequence of the pair $[F^n_0(X, x), G^n_0(A/B, x)]$ is naturally isomorphic to that of the triad $(X; A, B)$; for example, $\pi_q[F^n_0(X, x), G^n_0(A/B, x)] \approx \pi_{n+q}(X; A, B)$. This isomorphism enables one to define a natural homomorphism of the homotopy sequence of the pair (R_{n-1}, R_{n-2}) (where

R_m = rotation group of S^m) into that of the triad (S^n; E^n_+, E^n_-). This is useful in studying the higher homotopy groups of spheres. (Received November 29, 1949.)

209. L. W. Cohen and Casper Goffman: *On completeness and category in uniform space.*

A uniform space S, metrizable by an ordered abelian group, is called complete in the sense of Archimedes (a-complete) if certain associated spaces, analogous to factor groups, are topologically complete (t-complete). It is shown that an a-complete space is of the second category. The authors have shown elsewhere (*A theory of transfinite convergence*, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 65–74) that a t-complete space may be of the first category. (Received October 10, 1949.)

210. E. E. Floyd: *The decomposition maps associated with* 0-*dimensional transformation groups.*

Let X be a locally compact metric space; let G be a compact 0-dimensional transformation group operating on X such that no element of G except the identity leaves any points of X fixed. Let Y be the orbit decomposition space, and f: $X \rightarrow Y$ the decomposition map. Analogues of theorems concerning covering maps are proved for f. For example, the conclusion of the covering homotopy theorem holds here. As an application, it is proved that if X is an n-manifold and G is infinite, then Y contains no $(n-1)$-cell. (Received November 14, 1949.)

211. David Gale: *Compact sets of functions and function rings.*

Let Y^X denote the space of continuous functions from the space X to the space Y in the compact-open topology. The author investigates conditions under which a subspace $F \subset Y^X$ is compact. The classical conditions when Y is a metric space involve equicontinuity and uniform boundedness of F. If Y is not metric equicontinuity may be replaced by a sort of mutual continuity requiring that if U is open in Y, then $\bigcap_{f \in F} f^{-1}(U)$ be open in X. Boundedness is replaced by the condition that $\bigcup_{f \in F} f(x)$ be compact. Using these ideas one obtains simple, necessary and sufficient conditions for the compactness of F. The result is then applied to obtain a duality theorem for the ring $R(X)$ of real valued continuous functions on a space X. Namely, if $R(X)$ is given the compact-open topology and H is the space of all continuous homomorphisms of $R(X)$ onto the real numbers with the same topology, then, with mild restrictions on x, it is shown that H is homeomorphic with X. (Received November 9, 1949.)

212. F. B. Jones: *An elementary two color problem.*

Consider a unit square in the number plane with the ends of a diagonal at (0, 0) and (1,1). Let M_x denote the set of all straight line arcs spanning the square parallel to the y-axis and let M_y denote a similar set parallel the x-axis. Color the intersection of a line in M_x with a line in M_y either red or green, except along the line $y=x$ which is left uncolored; furthermore the coloring is symmetrical with respect to $y=x$. THEOREM: *If H_x is the subset of M_x with rational abscissas and H_y is the corresponding subset of M_y, then H_x contains an infinite subset H'_x such that H'_x intersects H'_y (the image in H_x with respect to $y=x$) in only one color.* QUESTION: *Does M_x contain a subset M'_x of the same number of elements as M_x such that M'_x intersects M'_y in only one color?* (Received December 23, 1949.)

213. A. N. Milgram: *A solution of the frame problem for the three sphere.*

H. Rademacher raised the question as to whether every bounded subset of E_n (Euclidean n-space) can be circumscribed by an n-dimensional cube. A solution for subsets of E_3 was given by S. Kakutani (Ann. of Math. (2) vol. 43, pp. 739–741). This note provides an affirmative answer to the four-dimensional case. Specifically we prove that if f is a continuous function defined on the sphere S_3 with center O in E_4, there exist points $P_i \in S_3$, $i=1, 2, 3, 4$, such that the vectors OP_i are mutually orthogonal and $f(P_i)=f(P_j)$, $i, j=1, \cdots, 4$. The proof makes essential use of the lemma: If $f(x, y)$ is a doubly periodic function of the real variables x, y, with period 1, there exist x_0, y_0, z_0 such that $f(x_0, y_0)=f(x_0+1/4, y_0)=f(z_0, y_0+1/2)=f(z_0+1/4, y_0+1/2)$. (Received November 15, 1949.)

214. L. T. Ratner: *An extended theory of semi-continuity.*

X and Y are Hausdorff spaces satisfying the first axiom of countability; further, Y is simply ordered by a proper transitive relation $\prec$ such that $\{y' \mid y' \preceq y\}$ and $\{y' \mid y \preceq y'\}$, for any $y \in Y$, are closed subsets of Y. Let f be a single-valued transformation of X into Y. For U any open set containing x, denote by $l(x; f; U)$ the set of *lower bounds* for $f(U)$ and by $L(x; f; U)$ the set of *upper bounds* for $f(U)$; let $l(x; f)$ be the closure of the union of all $l(x; f; U)$, $L(x; f)$ the closure of the union of all $L(x; f; U)$. f is said to be *lower semi-continuous*, *upper semi-continuous*, or *continuous at* x provided that $f(x) \in l(x; f)$, $f(x) \in L(x; f)$, or $f(x) \in l(x; f) \cap L(x; f)$, respectively. These definitions provide a generalization of the notions of semi-continuity employed in real variable theory and elsewhere; continuity reduces to the normal concept. A sequence of standard theorems is carried over for this theory of semi-continuity. (Received November 15, 1949.)

215t. Russell Remage: *Invariance and periodic properties of non-alternating transformations in the large.* Preliminary report.

Extensions of some of the results of cyclic element theory to compact connected Hausdorff spaces are obtained by means of nodal set theory. If X is a non-vacuous subset of a continuum S, a D-chain, $D(X)$, is the set $C(X)$ defined by Wallace as the intersection of all nodal sets containing X. Let X in nondegenerate $Z(X)$ be the set of points separating a pair of points of X in S. A nondegenerate prime chain P is contained in $D(X)$ if and only if either $P \cap (X \cup \overline{Z(X)})$ is nondegenerate or $P \cap \overline{Z(X)}$ is a single point which fails to separate P from X. End-elements which are not cut points are prime chains not contained in $D(z_1 z_2)$ for any pair of cut points z_1, z_2. Any power of a n.a.l. transformation T of a continuum S onto itself is n.a.l. If PT is not a cut point unless P is, and either $P \subset PT$ or $PT \subset P$, P is called nonvariant, and contains an invariant set. If a nodal set meets its image, it contains a nonvariant P. If R is the union of all nonvariant prime chains, then $z \in Z(R)$ is either fixed or has no recurrence properties. Applications yield extensions and analogues of several theorems dealing with element-wise recurrence. (Received November 14, 1949.)

216t. Jane C. Rothe: *The topological degree of certain mappings in Euclidean n-space.*

Mappings in complex and real Euclidean n-space E^n which are defined by n polynomials in n variables, the polynomials homogeneous of degree k in the n variables, are studied. By using resultant theory, it is proved that if E^n is complex and the topological degree at zero of such a mapping is defined, the topological degree is k^n. By elementary methods, it is proved that if E^n is real and $n=2$, the topological degree is one of the following values: if k is odd, $\pm 1, \pm 3, \cdots, \pm k$; if k is even, $0, \pm 2, \pm 4, \cdots, \pm k$. The results are applied to obtain existence theorems for certain types of non-linear integral and differential equations. (Received November 4, 1949.)

217*t*. G. T. Whyburn: *Dimension raising on surface sets.*

A locally connected generalized continuum every true cyclic element of which is a 2-manifold (open or closed) is called a surfacoid. It is shown that if A is a locally connected generalized continuum on a surfacoid and $f(A)=B$ is a light open mapping, for any branch point y of B, every point of $f^{-1}(y)$ is an isolated point of $f^{-1}(y)$ and $\dim B=\dim A$. Further, if A is a locally connected continuum on a cactoid and $f(A)=B$ is a quasi-monotone mapping, then $\dim B \leqq 2$. Also it is shown that if A and B are compact metric spaces of dimension greater than 0 and $f(A)=B$ is a light open mapping, there exist locally connected continua A' and B' topologically containing A and B respectively, with $\dim A'=\dim A$, $\dim B'=\dim B$, and a light open extension $f(A')=B'$ of f. (Received October 28, 1949.)

T. R. Hollcroft,
Associate Secretary

APPENDIX

EXCERPTS FROM REPORT OF TREASURER

December 22, 1949

TO THE BOARD OF TRUSTEES OF THE
AMERICAN MATHEMATICAL SOCIETY

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1949, with certain pertinent comments.

Investment Portfolio and Income

On November 30, 1949, the market value of securities held for Invested Funds exceeded book value by $20,403, but the market value of securities held for Current Funds was less than book value by $4,333. On the whole portfolio, the market value therefore exceeded book value by $16,070. Reserves held in accounts "Reserve for Investment Losses" ($4,386) and "Profit on Sales of Securities" ($15,851) may still be considered adequate protection against contingent depreciation in market value.

The following is a summary of the changes in security holdings made during the year.

Acquired	
10 shares	American Can Co. common
100 shares	Borg-Warner Corp. common
100 shares	Sterling Drug, Inc. common
$2,000	U. S. Treasury 2's 1954/52
Sold	
100 shares	General Electric Co. common
Redeemed	
$8,000	U. S. Savings Bonds 1949

The investment portfolio, valued at market November 30, 1949, now includes Government bonds 32.6 per cent, other bonds 7.6 per cent, preferred stock 15.3 per cent, common stock 39.6 per cent, cash in savings banks 4.9 per cent.

Income received during the year from investment of Current Funds amounted to $2,991. This represents a return of 3.1 per cent computed on average book value of investments. Income on Invested Funds amounted to $8,832, representing a return of 4.6 per cent. Total investment income from all sources was $11,823, representing a return of over 4.1 per cent.

Income from the Henderson Estate was $5,200; in 1948 it was $4,990.

The Year's Operations

The budget for 1949 adopted by the Trustees authorized an expenditure of $47,000 in excess of anticipated income. Actually, for a number of reasons it was not necessary to expend the full amount authorized. The principal reasons were the inability of the Society to appoint an Executive Director until November 1 and delay in the publication of certain material, particularly the Birkhoff papers.

Another factor which should be noted is that the Society has incurred obligations of approximately $25,000 for work in process for which bills have not been rendered or were rendered so late in the fiscal year that it was possible to defer payment until 1950. The Society could have paid these bills had they been rendered only by substantially liquidating invested current funds. In other words, while the assets of the Society decreased only approximately $3,000, had all of the obligations incurred during the year been paid during the year, assets would have declined by nearly $30,000.

The Society has incurred a new fixed obligation through the employment of an Executive Director and will in all probability have to incur additional fixed obligations in the near future to provide adequate office, working, and storage space. Unless additional sources of income are developed, it would appear to be financially impossible to expand the program of publication or even to maintain it at its present level unless the Society wishes to borrow from its invested funds or from banks. However, the sale of mathematical publications, while dependable, is so slow that it is unlikely that any commercial bank would regard such a loan as attractive.

Respectfully submitted,
ALBERT E. MEDER, JR.
Treasurer

BALANCE SHEET

	November 30, 1949	November 30, 1948
Assets		
CURRENT FUNDS:		
Cash	$ 12,057.26	$ 16,292.01
Due from Invested Funds	31.58	
Account Receivable from United States Government	3,440.49	480.21
Investments	76,292.42	77,500.19
	$ 91,821.75	$ 94,272.41
INVESTED FUNDS:		
Cash		$ 531.01
Investments	$192,153.46	192,102.21
	$192,153.46	$192,633.22
TOTAL ASSETS	$283,975.21	$286,905.63
Liabilities		
CURRENT FUNDS:		
Publications	$ 23,928.48	$ 41,167.18
International Congress	18,089.17	5,484.84
Policy Committee	(160.70)	97.64
Prize Funds and Other Special Funds Accumulated Income	9,553.82	8,238.39
Sinking Fund	1,244.97	1,156.05
Profit on Sales of Securities	2,032.98	2,032.98
Miscellaneous	986.90	746.46
Surplus	36,146.13	35,348.87
	$ 91,821.75	$ 94,272.41
INVESTED FUNDS:		
Endowment Fund Principal	$ 71,000.00	$ 71,000.00
Prize Funds and Other Special Funds	33,033.22	33,033.22
Life Membership and Subscription Reserve	2,851.70	3,073.09
Mathematical Reviews	65,000.00	65,000.00
Reserve for Investment Losses	4,385.89	4,385.89
Profit on Sales of Securities	15,851.07	16,141.02
Due to Current Funds	31.58	
	$192,153.46	$192,633.22
TOTAL LIABILITIES	$283,975.21	$286,905.63

Note: Inventories of books and periodicals are not included in the balance sheet. Books on hand may be expected to yield eventually a net income from sales of approximately $40,000. Back issues of periodicals are difficult to value; average annual sales over the last five year period were approximately $7,500.

SUMMARY STATEMENT OF INCOME AND EXPENDITURES

1948–1949

	1949	1948
GENERAL RECEIPTS		
Dues—Ordinary Memberships	$ 30,017	$ 29,120
Dues—Contributing Memberships	746	712
Dues—Institutional Memberships	9,408	9,582
Initiation Fees	1,091	1,880
Investment Income	12,526	11,953
Miscellaneous	122	191
	$ 53,910	$ 53,438
PUBLICATIONS RECEIPTS	73,775[1]	77,680[2]
OTHER RECEIPTS	1,884	4,470
TOTAL RECEIPTS	$129,569	$135,588
GENERAL EXPENSE	$ 25,507	$ 19,684
COST OF PUBLICATIONS AND SALES	118,837	110,144
OTHER EXPENSE	533	1,212
TOTAL EXPENSES	$144,877	$131,040
EXCESS OF RECEIPTS OVER EXPENSES		$ 4,548
EXCESS OF EXPENSES OVER RECEIPTS	$ 15,308	

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[1] This includes $7,140 received under a contract with the Office of Naval Research for support of Mathematical Reviews.

[2] This includes $22,560 received under a contract with the Office of Naval Research for support of Mathematical Reviews.

BOOK REVIEWS

Quantum mechanics. By L. I. Schiff. New York, McGraw-Hill, 1949. 404 pp. $5.50.

The subject matter of quantum mechanics is emphasized in different ways by different professional groups. To the philosopher the field has a personal interest because of its bearing on the relationship between man and the natural world about him. To the mathematician the subject represents a rich field of problems dealing with function space and operator theory. To the physicist it is important as an extension of classical physics into the domain of the microscopic world and as an indispensable tool for solving the many problems of interest.

Introductory books written for each group will exhibit corresponding differences in emphasis. That for the philosopher will stress the statistical interpretation of quantum mechanics and will deal only with the most elementary physical situations. That for a mathematician will stress the character of the boundary value problems and operators which appear, placing emphasis upon mathematical rigor. That for the physicist should expand the connection between classical and quantum mechanics, give ample illustrations of the applications of quantum mechanics to physical problems for which it was invented, and which it has been eminently successful in solving, and should at least lead to that frontier region, dealing with high energy particles, in which quantum theory is being extended at present.

This excellent book is written primarily and emphatically for the physics student so that it falls definitely in the third category. Yet the writer has managed throughout to preserve interest in the philosophical applications of the subject and in mathematical rigor; any reader will recognize that the subject is rich in philosophical connotation and that mathematical neatness and rigor have an important place in its evolution.

Perhaps the importance of the book in contemporary graduate education is best illustrated by the fact that about a half dozen theoretical physicists with whom the reviewer has spoken have commented on the fact that the text is closely similar to the lecture notes they have employed for a number of years in teaching quantum mechanics to graduate students. All regard it as one of the most important recent additions to graduate training in physics. The writer has no hesitation in saying that it probably will become the standard reference text in this country for a substantial period of time in the future.

This relatively unique position rests not only on the fact that it is a clear and accurate version of quantum mechanics, for other books such as those of Rojansky and Kemble have the same virtues. This book has the additional virtue that it is intended to prepare the reader for an understanding of the frontier problems of quantum mechanics as they are being attacked at the present time by theoretical physicists. That is, the book has what might be termed vector content in the sense that the student not only obtains an appreciation of the problems in which non-relativistic quantum mechanics has triumphed in treating atomic and nuclear problems, but is also guided systematically to the frontier problems of relativistic field theories which play so important a role in current developments of the physics of high energy particles. Perhaps the last introductory account of the subject which attempted to do this is Pauli's *Die allgemeine Prinzipien der Wellenmechanik* which appeared in the Handbuch der Physik, vol. 24, in 1933.

The book contains 14 chapters and may be divided into three parts. In Schiff's own words, "The first three chapters constitute an introduction to quantum mechanics, in which the physical concepts are discussed and the Schrödinger wave formalism is established. The next eight chapters comprise the central part of the book. This part presents exact solutions of the wave equation for both energy-level and collision problems, the Heisenberg matrix formalism and transformation theory, approximation methods, radiation theory, and some applications to atomic systems. Since the first eleven chapters correspond to a typical one-year graduate course, it seemed desirable to include a semi-classical treatment of electromagnetic radiation in the central part of the book (Chapter X) even though some of the results are obtained again in Chapter XIV. The last part of the book corresponds to a short course in what is often called advanced quantum mechanics. It consists of relativistic particle theory and an introduction to quantized field theory and quantum electrodynamics."

FREDERICK SEITZ

Plastic deformation. By L. N. Kachanov, N. M. Beliaev, A. A. Ilyushin, W. Mostow, and A. N. Gleyzal. Ed. by H. H. Hausner. New York, Mapleton, 1948. 192 pp. $8.00.

This book consists of a set of seven independent papers on plasticity. Five of these appeared originally in Russian periodicals and the two remaining ones are based on technical reports prepared under the auspices of the U. S. Navy. Advance notices of this book gave the impression that it forms a coherent account of the current state of art in the field of plasticity. This is far from being the case. The rationale followed in the selection of papers is not clear and the choice appears to be almost random.

The book is divided into seven chapters corresponding to seven distinct papers reproduced in it. The first five of these are photo-offset reproductions of mimeographed free translations of original articles prepared by the Applied Mathematics Group of Brown University for the use of the personnel of the David Taylor Model Basin. They appear as:

Chapter 1, *On the mechanics of plastic solids*, a 5-page largely synoptic article originally published in Prikladnaia Matematika i Mekhanika vol. 4 (1940) pp. 37–42, by L. N. Kachanov.

Chapter 2, *Theories of plastic deformation*, Izvestia, Acad. Sci. USSR, Technical Series (1937) pp. 49–70, by N. M. Beliaev.

Chapters 3, 4, and 5 consist of three significant papers by A. A. Ilyushin, (a) *Some problems in the theory of plastic deformations*, Prikladnaia Matematika i Mekhanika vol. 7 (1943) pp. 245–272. (b) *Relation between the theory of Saint Venant-Levy-Mises and the theory of small elastic-plastic deformations*, ibid. vol. 9 (1945) pp. 207–218. (c) *The theory of small elastic-plastic deformations*, ibid. vol. 10 (1946) pp. 347–356.

The two remaining chapters are drawn from a report entitled *Plastic deformations of thin plates under hydrostatic pressure*, prepared by W. Mostow for the U. S. Navy Bureau of Ships, and a report by A. A. Gleyzal on *Plastic deformation of a thin circular plate under pressure*, which originally appeared as David Taylor Model Basin Report No. 532 (1946).

The reproduction is below average in quality. Some mimeographed material did not photograph well and the publishers had it retyped and introduced so many errors in the process as to make several pages unintelligible. An indication of the careless way in which the book was thrown together may be gleaned from the short page of introduction which contains no less than a dozen errors including the misspellings of the names of three authors of this book. For some reason the backbone and cover carry only the name of L. N. Kachanov whose contribution to the volume consists of a paper occupying less than five typewritten pages.

The usefulness of this book lies only in the fact that it makes translations of several Russian papers readily available.

The reviewer feels impelled to call attention to the questionable ethical conduct of the publishers. Despite the fact that all papers reproduced in this volume are in the public domain, the book bears the notice of copyright by the Mapleton House. It is certain that no authors' or translators' permission was obtained by the publisher to reproduce this material. Considering the inexpensive photo-offset mode of reproduction of existing material, the price of the book is exorbitant.

I. S. Sokolnikoff

Mathematical biophysics. By Nicolas Rashevsky. Rev. ed. University of Chicago Press, 1948. 23+669 pp. $7.50.

The wide attention and considerable acclaim given to the first publication of Rashevsky's *Mathematical biophysics* in 1938 was due in large part to the rich promise contained in that book. Those mathematicians who loved to see mathematics in the role of fathoming nature's secrets viewed mathematical biophysics as a new area of conquest. Those biologists who were "monists" or "reductionists" at heart hailed it as the much needed link between their own methods and those of mathematical physics.

The revised edition gives ample evidence that the promise is being fulfilled. Almost twice as large as the original publication, the revised edition includes in its pages ten years' progress in mathematical biology.

The progress is largely due to the work of Rashevsky himself and that of his principal collaborators, who are or have been members of the Committee on Mathematical Biology at The University of Chicago. The work covers a wide range of topics, but essentially it can be viewed as pursuing two main directions: (1) The extension of the theory and methodology; (2) The applications of the theory.

With regard to the first direction, an important extension of the previous approach is the approximation method used in computing the forces of metabolic origin acting on a cell of "arbitrary" shape. The method was sketched in the appendix to the first edition and is more fully developed in the new book. The previous line of attack was through the diffusion equation, a partial differential equation which could be solved explicitly only in very special cases involving highly symmetrical boundary conditions. By means of a bold approximation method, Rashevsky was able to reduce the partial differential equations to ordinary ones and to express the forces in terms of parameters amenable to "rough and ready" measurement: the "long" diameter and the "short" diameter of an "oblong" cell.

The inclusion of a kinetic theory of diffusion also appears as a worthwhile extension of the old avenues of attack. One would guess that the "true" picture of the diffusion processes of a metabolizing cell lies somewhere between the two extremes—the hydrodynamic, continuous model and the kinetic, discontinuous one. A two-sided attack on the problem seems the logical way to proceed.

A third, extremely promising, methodological innovation is the introduction of Boolean algebra (or logical calculus, or symbolic logic) methods in the construction of models for neural nets with specified properties. The method was initiated in a paper by McCulloch and Pitts published in The Bulletin of Mathematical Biophysics in 1943. In the hands of Rashevsky and his collaborators, notably Householder and Landahl, the method was greatly extended and enriched. In particular the previous "continuous" theory of neural nets was reinterpreted as a limiting case of the "discontinuous" theory involving a large number of neural elements. One is inclined to regret that a more extended development of the method was not included in the revised edition.

The new applications of mathematical biology form the other direction in the extended work of Rashevsky. It is particularly gratifying to see the inclusion of a considerable amount of experimental data, which were only scantily represented in the first edition. Significantly, the reason for the paucity of experimental evidence in the first edition is that a great deal of it was obtained *since* 1938. It thus forms to some degree a corroboration of the *predictions* of the theory.

In part I, the most interesting data are those dealing with the rates of cell division, exhibiting an interesting relation between the rate of elongation and the rate of constriction, predicted by Rashevsky's theory of cell division based on the approximation method.

The most abundant extensions of applied mathematical biophysics are found in Part III, which deals with the central nervous system, especially in the chapters on discrimination, delayed reflexes, error elimination and learning, and visual perception. These chapters now include an impressive amount of experimental evidence in sufficient agreement with the Rashevsky two-factor theory of nervous excitation to put beyond doubt the usefulness of the theory.

A. RAPOPORT

Equazione differenziali. By Francesco Tricomi. Turin, Einaudi, 1948. 312 pp.

This text contains a somewhat unusual but interesting and well integrated sequence of topics. A Picard type existence theorem is given and a singular point for a first order differential equation is investigated. Most of the discussion is concerned with second order equations or the equivalent systems. For these the linear dependence of solutions, the Wronskian theory, the variation of parameters method, the circuit of a singularity in the complex plane, Fuchs theorem, solution by power series, the Sturm-Liouville theory and the asymptotic behaviour of characteristic functions and characteristic values are given. The last chapter also contains a Cauchy type existence theorem, based on majorants.

There is a valuable emphasis on individual functions, whose propeties are derived from the fact that they are solutions of a differential equation. For instance the circular and elliptic functions are treated in this way. The asymptotic behaviour of the Laguerre and Legendre polynomials and the Bessel functions are used to illustrate the characteristic function theory. The hypergeometric series is developed in the last chapter. On the other hand as a matter of policy the usual methods for the integration of first order equations are omitted.

The style is clear and the book should prove a valuable reference. The author claims, quite justly, that this corresponds to a "modern course" in differential equations and there is quite a contrast with the American courses on "methods of solution" and "theory." The pressure from applications and the results of theoretical developments are clearly present. However the existence theory is not the most general possible and the elementary methods and the constant coefficient linear equations are worth considering. The need of two courses seems clear but they should be carefully organized for maximum usefulness.

F. J. MURRAY

Tables of generalized sine- and cosine-integral functions. Parts I and II. (Annals of the Computation Laboratory of Harvard University, vols. 18, 19.) Harvard University Press, 1949. Part I, 38+462 pp. $10.00. Part II, 8+560 pp. $10.00.

DEFINITIONS:

$$S(a, x) = \int_0^x \frac{\sin u}{u}\,dt; \qquad C(a, x) = \int_0^x \frac{1 - \cos u}{u}\,dt;$$

$$Ss(a, x) = \int_0^x \frac{[\sin u]\sin t}{u}\,dt; \quad Sc(a, x) = \int_0^x \frac{[\sin u]\cos t}{u}\,dt;$$

$$Cs(a, x) = \int_0^x \frac{[\cos u] \sin t}{u} dt; \quad Cc(a, x) = \int_0^x \frac{[\cos u](1 - \cos t)}{u} dt$$

where

$$u = (a^2 + t^2)^{1/2}.$$

The above six functions are tabulated to six decimals over the range of both a and x from 0 to 25. The intervals in the arguments vary from 0.01 in both a and x when they are less than unity, to 0.2 in both arguments for $a \geqq 10$. In addition to the main table, there are a few pages of coefficients which occur in the various formulas that were employed for computing the functions.

The introduction by J. Orten Gadd, Jr., and Theodore Singer gives the more familiar properties of the tabulated functions, the method used to compute the entries on the automatic sequence controlled calculator, and methods of interpolation in the tables. Some of the derivations in this introduction are correct for positive values of a and x only, although this limitation is not stated. For example, it is implied that

$$S(a, \infty) = \int_0^\infty \{[\sin a(1 + t^2)^{1/2}]/(1 + t^2)^{1/2}\} dt;$$

this is of course true only for positive values of a. Moreover, the authors derived their integration formula from a system of linear algebraic equations. It should be pointed out that the formula is well known, and can be derived more simply from the integration of the interpolation polynomial. In fact, the coefficients of this formula have been extensively tabulated. These may be minor flaws, but the definitions of the functions deserve improvement in any future edition of the tables. For instance, the authors write

$$Cc(a, x) = \int_0^x \frac{\cos u(1 - \cos x)dx}{u},$$

with $u = (x^2 + a^2)^{1/2}$. All the other functions are defined in the same spirit, and at best these definitions are ambiguous.

The method of computation described in the introduction emphasizes rigorous accuracy, and in the light of past performances of the Harvard Computation Laboratory, there is every reason to believe that the entries are trustworthy. The format of the tables is pleasing; reproduction is by a photo offset process.

GERTRUDE BLANCH

Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik. By W. Magnus and F. Oberhettinger. (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, vol. 52.) 2d. ed Berlin-Göttingen-Heidelberg, Springer, 1948. 8+230 pp. 24.60 D.M.

Much of the toil of mathematical physics consists in the transformation of mathematical expressions which have been arrived at by a conceptual process (of theoretical physics) to other more applicable forms. The book under review is intended to be an aid to this process, and should be of great value to the toilers of mathematical physics.

There are no numerical tables or graphs in the book. Except for the necessary preliminary definitions, the collected material is limited to formulas which are likely to be usable in the solution of special problems. Thus all proofs are omitted. Most physical scientists will probably find the omission of questions relating to expansions in orthogonal systems of functions to be more serious. It does not seem that such an addition would entail much greater length of the text. In that fine synoptic treatment by E. Madelung, *Die Mathematischen Hilfsmittel des Physiker*, one finds the relevant material condensed into twlve pages. A treatment on the same order, but extended to cover the additional functions considered, would make the present book more nearly self-contained, and probably unrivaled in comprehensiveness. A mere repetition of the Madelung material would help. Even old worn descriptions, like Mark Twain's diamonds, are better than none at all.

The following suggestions for future editions may also be made:

(A) Every root of an algebraic equation can be expressed as a hypergeometric function of the coefficients of the equation (a fact considered at least as early as 1914 by Mellin); it would be of value to give the corresponding formulae in the chapter on the hypergeometric function.

(B) There are a few slips in the otherwise excellent typography. The appropriate limits of integration and superscripts are missing in several places, as, for example, on pages 2, 98, and 208.

Of the functions common in mathematical physics, only the Lamé functions are omitted altogether, and the treatment of the Mathieu functions is perhaps unduly brief in view of their growing importance in applications.

Chapter I deals with the gamma function, its logarithmic derivative or Ψ function, and the beta function. Chapter II, on the hypergeometric function, treats of the series and its generalizations, independent solutions of the hypergeometric differential equation, and of its generalized form known as the Riemann differential equation. The specialization of the generalized series on the one hand, and the equation and its solutions on the other, to the many simpler and more familiar forms which they take is concisely and clearly described throughout the book. Chapter III on the cylinder functions, the longest in the book, gives results on the functions of Bessel, Struve, Anger, Weber, Lommel and Mathieu, Kapteyn, Schlömilch, and mixed cylinder-elementary-function types of series are also given.

It is interesting to note that the cylinder functions, for which so much theoretical material has been developed, have also been by far the most worked on numerically. This may be seen by a quantitative survey of the amount of tabular material in a book like Jahnke and Emde's *Funktionentafeln*. In fact, the only outstanding competitor among the non-elementary functions in this matter of popularity with the computers seems to be the family of elliptic and theta functions. If one enlarges the category of cylinder functions to that of the confluent hypergeometric functions, of which it is a special case, one finds that the vast majority of all the numerical-graphical material is shared between the confluent hypergeometric family and the elliptic-theta family. There remains mainly the small number of tables associated with the gamma and Legendre functions.

In Chapter IV on sphere functions (spherical harmonics) the various special forms are discussed, as well as the functions of Gegenbauer in their connection with many-dimensional sphere functions. We remark that an interesting physical interpretation may be given to Hecke's theorem on integrals of functions over hyperspheres (end of Chapter IV). Imagine a general system of dipoles (a *multipole*) at the origin of a unit hypersphere in an n-dimensional space. Suppose further that on the unit hypersphere there is a continuous distribution of charge, arbitrary in every way except that it is rotationally symmetric with respect to a given axis of the sphere. Then the contribution to the total energy, of the continuous charge distribution in the field of the multipole, is entirely equivalent to that for a certain single point charge located at the place where the symmetry axis intersects the hypersphere. Now, the really interesting feature of the theorem is that, no matter how we vary the structure (but not the multiplicity) of the multipole at the center, it is the *same* point charge which may replace a given continuous distribution, as far as its energy contribution is concerned.

Orthogonal polynomials, the confluent hypergeometric function, elliptic integrals, theta functions, and elliptic functions are taken up in the next three chapters. The relations between the various functions are clearly given and transformations and degenerations are carefully listed.

The two last chapters depart from the treatment of special functions to more general considerations. Chapter VIII treats of integral transforms and their inverses. Included are concise accounts and tables of Fourier, Laplace, Hankel, Mellin, and Gauss transforms. All of these transform relations may be regarded as integral equations of the first kind, with essentially singular kernels and infinite domains of integration. Further analogous integral equations are given, among which are several with finite integration interval, for example, Hilbert's cotangent-kernel equation and Abel's integral equation. Chapter IX is a summary of, for the most part, conventional material on coordinate transformations. A novel feature here is the material on the system of many-dimensional polar coordinates.

Several appendices, one on linear second order differential equations, one on Fourier series, partial-fraction and product representations of some elementary functions, and one on certain summations, complete this comprehensive, scholarly, and useful compilation.

M. AVRAMY MELVIN

Sequential analysis. By Abraham Wald. New York, Wiley, 1947. 12+212 pp. $4.00.

This book could be reviewed from the point of view of the general mathematician, the probability theorist, the mathematical statistician, the theoretical statistician, or from the point of view of any of many kinds of users. We shall try to provide a somewhat composite view.

The central feature of this book is the sequential probability ratio test, abstractly a random walk between adsorbing barriers. We have only to think of a "particle" moving in "steps," the amount of each step being determined by chance in the same way, the stepping process ending whenever the particle, which started from the origin, passes certain "barriers" and leaves an assigned interval. Some 40 pages of the mathematical appendix is devoted to exact and approximate results for such walks. The theory of sequential analysis is growing, and more complex processes are entering; yet this simple random walk is still the core.

To read the body of the book, the reader needs, explicitly, a knowledge of calculus, and, implicitly, a feeling of comfort with sentences where the mathematical characters outweigh the punctuation by three to one.

Destructive inspection of military equipment, where every item tested was a scarce item lost to use, called the SPR test into being. Milton Friedman and Allen Wallis's keen appreciation of the need for, and probable value of, sequential methods "gave the incentive for the author's [Wald's] investigations which then led to the development of the sequential probability ratio test" (page 2). This test solved the problem of deciding whether a lot had distribution A or distribution B with given risks of error (there are *two* ways to be wrong) with the smallest average number of items tested. (The proof that the number of items tested is a minimum was given by Wald and Wolfowitz [1], a year after the appearance of *Sequential analysis.*)

The solution to this one problem, in the hands of Wald and the other workers at Statistical Research Group, Columbia University, gave usable solutions for testing one mean, or comparing two means, from binomial, normal, and chi-square distributions, where the alternative distributions are really infinite in number, and the true problem is still unmanageable. The theoretical background of these working tests is discussed in considerable and careful detail, as is their connection to the general theory. The practical problem of distinguishing between distributions has learned much from the SPR test, but reducing the average number of trials is not always the consideration; it may be worth more to schedule testing neatly than to reduce it to a minimum. The SRG-C manual on sampling inspection [2], and Wald's later work [3, 4], show some directions in which the subject of sequential testing has broadened out.

In Chapters 4, 10, and 11, Wald opens a strong attack on an adequate treatment of problems with more than two alternatives. Whenever the alternatives, as in testing composite hypotheses, or the decisions themselves, as in multiple decision problems, are more than two in number, there is a great increase in the difficulty of the problems at the philosophical, theoretical, mathematical, and computational level. This is an important open frontier. Wald's work in these chapters, and his frankly heuristic discussions at appropriate points is of great interest to both the theoretical and the mathematical statistician and should be a continuing stimulus to new work.

In its present form, the book is unsatisfactory for a reader wishing to study the advanced theory of sequential analysis, whether from the point of view of probability or mathematical statistics. Three facts contribute to this: (1) The relegation of the more complex details to the appendices, which makes the body of the book accessible to a wider class of readers, has inevitably made a smooth and coherent presentation almost out of the question. (2) The discussion of the various applications are quite repetitive, and while this may be desirable for the general reader, it does not help the mathematician. (3) The book is now two years old, and was written when many parts of the field were undergoing very active development. It is to be hoped that future editions, or a book at a higher level, will provide a more coherent account, covering such new work as that of Wolfowitz [**5**] and Seth [**6**] on sequential estimation, Wald [**3**, **4**] and Arrow, Blackwell, and Girshick [**7**], on multiple decision functions.

The book is relatively free from typographical errors. We may mention that a "Δ" should appear in the denominator of the fraction appearing at the very end of page 9, that in expression (3.3) on page 38 the inequality should be reversed, that in expression (4.22) on page 85 the fraction on the right should be inverted, and that the summand of the sum in the third line of page 133 should be squared.

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MELVIN P. PEISAKOFF AND JOHN W. TUKEY

Variétés abéliennes et courbes algébriques. By A. Weil. Paris, Hermann, 1948. 165 pp.

This is the second of a series of papers with which the author promised to follow his book, *Foundations of algebraic geometry*, American Mathematical Society, 1946. The first paper, entitled: *Sur les courbes algébriques et les variétés qui s'en déduisent*, is concerned in particular with the theory of correspondences of an algebraic curve and contains the author's proof of the Riemann hypothesis over a finite field. In a sense this study is generalized to the case of general abelian varieties in the present work.

In the classical sense an abelian variety is a variety in a complex projective space whose nonhomogeneous coordinates are $2n$-ply periodic meromorphic functions in n complex variables. An essential feature is the possibility of defining a group operation on the variety, which the author takes as point of departure in his theory of abelian varieties over a field of characteristic p. Varieties are abstract varieties in the sense of the author, and a function in a variety U, with values in another V, is defined by a sub-variety Z of $U\times V$ such that $[Z:U]=1$. The values of the function are defined only at points where the projection of Z into U is regular. With the notion of function so defined, a group variety G is one such that a function is defined everywhere in $G\times G$, with values in G, with the usual group axioms satisfied. If the variety is complete, then the group operation has to be commutative, a result analogous to compact complex Lie groups. An Abelian variety is a complete group variety.

The classical theory of abelian varieties, in the hands of Scorza, Rosati, Lefschetz, and Albert, is essentially a theory of Riemann matrices, the period matrices of the periodic functions, and one of the main problems is to study the complex multiplication of Riemann matrices. To arrive at these concepts in the abstract approach we consider functions f defined in an abelian variety B, with values in an abelian variety A. Up to an additive constant f preserves group addition. Such a function is called a homomorphism of B into A, and an endomorphism if $B=A$. With addition defined by adding the function values and multiplication defined as successive application of the endomorphisms all endomorphisms of A form a ring $\mathcal{A}(A)$, the ring of endomorphisms. This ring $\mathcal{A}(A)$ can be imbedded into a ring $\mathcal{A}_0(A)$ over the rational field Q. $\mathcal{A}_0(A)$ is a semi-simple algebra over the rational field. If A is a simple variety, that is, one which contains no proper abelian sub-variety, $\mathcal{A}_0(A)$ is a field of finite rank over Q. The study of the rings $\mathcal{A}(A)$ and $\mathcal{A}_0(A)$ takes the place of complex multiplication.

The construction of abelian varieties of a given dimension is achieved by the consideration of the Jacobian variety of a curve. Let Γ be a complete curve without multiple point, of genus $g>0$. It is possible to construct an abelian variety J^g and a function ϕ, defined on Γ, with values in J, such that if $M_1, \cdots, M_g$ are g independent generic points of Γ (with respect to a field of definition K for Γ, J, and ϕ), the point $z=\sum_{i=1}^{g}\phi(M_i)$ is generic on J with respect to K. The Jacobian varieties furnish concrete examples of abelian varieties of a given dimension and give a link between abelian varieties and curves.

But the more important link is furnished by the application of abelian varieties to the theory of correspondence of curves. In fact, there exists an isomorphism between the module of classes of correspondences between Γ and Γ' and the module of homomorphisms of J into J'. In the case $\Gamma=\Gamma'$ this isomorphism is one between the ring of classes of correspondences on Γ and the ring of endomorphisms of J. Thus the study of endomorphisms of an abelian variety is a generalization of the theory of correspondences of a curve.

The above gives perhaps a broad outline of some results of this book in relation to the classical theory. The author has not only generalized the classical theory into a more profound new theory, with new results, but has presented the results in such a way that the development seems most natural. Among other things the book gives ample justification of the struggle one has to go through in reading the *Foundations* of the author.

SHIING-SHEN CHERN

Lattice theory. By G. Birkhoff. (American Mathematical Society Colloquium Publications, vol. 25.) Rev. ed. New York, American Mathematical Society, 1949. 14+280 pp. $6.00.

In the preface to the first edition of *Lattice theory*, Professor Birkhoff remarked that one of the attractive features in writing such a book was "fitting into a single pattern ideas developed independently by mathematicians with diverse interests." Thus the first edition contained a quite exhaustive account of those topics in mathematics which make extensive use of lattice operations. The same philosophy prevails in the new edition though it is a complete revision of the original. Due to the large number of contributions to the subject in the intervening years, the new volume is nearly twice the size of the old, and yet many important topics are barely mentioned.

The general plan of the book is unchanged. Beginning with partially ordered sets (Birkhoff now uses the term "partly ordered set," though he was not completely successful in changing every "partially" into "partly"), the author treats successively more special systems concluding with chapters on lattice-ordered groups and vector lattices. This method of presentation has the advantage that results for particular lattices often follow as natural specializations of results already proved for more general lattices. On the other hand, this order is sometimes awkward in that the structure of the more general lattices is often characterized in terms of the structure of special lattices which are treated later in detail. For example, the most useful tool for the study of semi-modular lattices is the fact that independent covering elements generate a Boolean algebra.

There are several additions that should be particularly noted. A new chapter on chains and chain conditions contains a detailed account of the various forms of the Axiom of Choice with their interrelations and the applications to transfinite induction. A separate chapter has been devoted to the rapidly developing subject of semi-modular lattices. The inclusion of a proof of Ore's theorem on direct decompositions rectifies a serious omission in the first edition. Finally, there are new chapters on lattice ordered groups and semigroups, the study of which had just been initiated when the original edition was published.

Another feature which considerably improves the new edition is the addition of an extensive collection of exercises. These serve both to give the reader a chance to try the various lattice techniques for himself and to acquaint himself with further results which, for reasons of space, could not be given a more formal treatment.

The new *Lattice theory* will be invaluable as a reference book for workers in the field. The fact that it contains an account of nearly all of the work to date in the subject makes it ideal for this purpose. The theorems have been carefully formulated, the proofs are concise and, for the most part, well presented. Since the author has emphasized the applications of lattice methods to other fields of mathematics, the book should prove particularly helpful to researchers in other fields who are interested in the connection between lattice theory and their specialties. On the other hand the book will not provide a satisfactory introduction to the subject for the beginner. The comprehensive character of the treatment which makes it useful to the expert is likely to confuse the beginner. It will also be difficult for him to recognize the methods and techniques which are best suited to each of the various branches of lattice theory. For example, it may not be clear that formal algebraic methods using the basic lattice operations are appropriate for the study of distributive and modular lattices while more local and structural methods are appropriate for semi-modular and general relatively complemented lattices.

The emphasis upon the many points in which lattice theory touches on other mathematical fields tends to obscure the fact that there is a considerable body of pure lattice theory which is motivated in a natural manner from the postulates, and which has produced many results and problems which are fascinating and quite difficult. This is in contrast with many of the applications where the lattice theory is often quite trivial and the difficulties are those associated with the field in which the application is being made. Thus the lattice theory involved in the study of vector lattices is of a very elementary nature, while many of the results are not at all easy.

As a minor criticism, it appears that several of the "unsolved" problems should have been considered more carefully. For example, problem 82 asks for a characterization of Boolean algebras which are isomorphic with the lattice of all regular open sets of a suitable T_1-space. But such a Boolean algebra must be complete and any complete Boolean algebra is isomorphic with the lattice of regular open sets of its associated Boolean space. Also the answer to problem 93 is trivially in the negative. It should be pointed out in general that the problems vary widely both in difficulty and importance.

Clearly, the new edition is a great improvement over the original, and it should be the definitive work in the subject for some time to come, particularly, since the rapid growth of the field makes it unlikely that another such comprehensive account will be written.

R. P. DILWORTH

Calcolo tensoriale e applicazioni. By B. Finzi and M. Pastori. Bologna, Zanichelli, 1949. 8+427 pp, 2000 Lire.

Despite the wealth of the literature which concerns itself with vector and tensor analysis and allied subjects, the present volume is a unique and worthwhile addition to the field. It is a broad survey of tensor algebra and tensor calculus, skillfully interwoven with geometric considerations, followed by applications to the mechanics of deformable continua, electromagnetic fields, and the theory of relativity.

The exposition is lucid throughout, proceeding from particular intuitive ideas to general abstract concepts. The account is replete with illustrative examples. The many applications make it of equal interest to the mathematician and theoretical physicist.

As is natural in such a broad, undetailed treatment, little attention is given to questions of rigor (as, for example, Duschek-Mayer, *Lehrbuch der Differentialgeometrie*, vol. 2, 1930) and the discussion of many topics appears to be compressed excessively. Even the survey of differential geometry, which comprises four chapters and is an excellent introduction to the subject must, of necessity, restrict itself to the high points of the field. In the opinion of this reviewer, the inclusion in later editions of a chapter on the classical mechanics of particles and rigid bodies would be especially desirable.

Vector fields in euclidean 3-space are treated in the first chapter. The concepts of circulation, gradient, curl, flux, divergence are developed. Harmonic fields, the integral theorems, Green's lemmas are discussed and the chapter concludes with various formal properties of differential operators.

The second chapter develops tensor algebra in a euclidean space, first by the use of cartesian coordinates and then more generally. The fundamental tensors are introduced. In the third chapter, linear vector functions are considered. The connection with second order tensors is exhibited. Special homographies, decomposition, and invariants are discussed.

The next two chapters concern themselves with tensor fields in euclidean and Riemannian n-dimensional spaces, respectively. The latter chapter also contains some material about non-Riemannian spaces such as the space defined by the Weyl connection. Differentiation of vectors and tensors, the Riemann tensor, Gaussian curvature, divergence, gradient, curl are among the topics considered.

The preceding results are used to develop the differential geometry of a surface and of a Riemann space in the sixth and seventh chapters, respectively. Aspects of the subject which are treated are the geometric properties of the first and second fundamental forms, parallelism, geodesics, curvature, asymptotic lines and lines of curvature, congruences.

The last three chapters deal with various aspects of theoretical physics. The statics and dynamics of deformable continua including elastic bodies, fluids, plastic continua are developed in Chapter VIII. The following chapter contains a discussion of electromagnetic fields and includes Maxwell's equations, wave propagation, the Lorentz transformation, the energy tensor. An outline of the theory of relativity, both restricted and general, is given in the last chapter. Among the topics discussed are inertia, gravitation, energy, statics, dynamics, cosmology.

A bibliography and an adequate index are given at the end of the book.

A. D. FIALKOW

Cardinal algebras. By A. Tarski. New York, Oxford University Press, 1949. 12+326 pp. $10.00.

Professor Tarski's latest book is built around an axiomatic study of cardinal numbers (including zero) under finite and countable addition. Besides the commutative and associative laws in their general (countably infinite) form, these operations satisfy a Refinement Postulate ($a+b=\sum c_i$ implies the existence of a_i, b_i such that $\sum a_i=a$, $\sum b_i=b$, and $a_i+b_i=c_i$), and a Remainder Postulate (if $a_1=b_1+b_2+b_3+\cdots+b_n+a_{n+1}$ for all finite n, then c exists such that $a_1=\sum_{i<\infty} b_i+c$).

A "cardinal algebra" is a system satisfying these postulates. Among cardinal algebras, we may include: Boolean σ-algebras; the non-negative elements of any complete l-group, if $+\infty$ is adjoined; and relation numbers under cardinal addition. On the other hand, since $3|2\cdot 2\cdot 2\cdots$ (countable multiplicands) without dividing any factor, cardinal numbers do *not* form a cardinal algebra under multiplication. The first part of the book is devoted to the formal properties of such cardinal algebras. After defining $a\leqq b$ to mean that $a+x=b$ has a solution, and $n\cdot a$ as $a+\cdots+a$ (n summands), the following typical results are proved: $a\leqq b$ and $b\leqq a$ imply $a=b$ (Schroeder-Bernstein Theorem); if $a+n\cdot c\leqq b+n\cdot c$, then $a+c\leqq b+c$—whence $n\cdot a=n\cdot b$ implies $a=b$; if $a_i\leqq b_j$ for all i, j of a finite or countable set, then c exists such that $a_i\leqq c\leqq b_j$ for all i, j. Again, if $a\cap b$ exists in the sense of lattice theory, then $a\cup b$ exists and $a+b=(a\cap b)+(a\cup b)$; under other existential hypotheses, $a\cap\sum_{i<\infty} b_i=\sum_{i<\infty}(a\cap b_i)$ and $a+\bigcap_{i<\infty} b_i=\bigcap_{i<\infty}(a+b_i)$. In fact, since $\sum_{i<\infty} a_i=\text{Sup}\ \{\sum_{i<n} a_i\}$, one can define countable sums in terms of binary sums; but the postulates in terms of binary sums alone would be more awkward.

Since the proof of the postulates for addition of cardinal numbers does not involve the Axiom of Choice for uncountable sets, deductions from the postulates are also independent of this Axiom, provided no further appeal to it is made. Hence the book is an important axiomatic contribution to the foundations of set theory.

On the other hand, some readers may find it difficult to accept the author's use of "the class of all sets" (p. 215). When it comes to "generalized cardinal algebras," in which closure under the operations is not assumed, there are further logical difficulties, noted in the addenda.

However, many noteworthy theorems about generalized cardinal algebras are proved. Thus every generalized cardinal algebra can be extended to a cardinal algebra; again the ideals of any cardinal algebra form, under set-union, a cardinal algebra in which $a+a=a$.

Another interesting feature of the book is the construction, given a cardinal algebra $\mathfrak{A}$ and a group G of "partial automorphisms" of $\mathfrak{A}$, of a "refinement algebra," very similar to a cardinal algebra, in which elements of $\mathfrak{A}$ equivalent under G (directly or by decomposition) are identified. In this direction, various abstract analogs of theorems relating to the existence of measure, and to the Banach-Tarski paradox, are proved. (Let $\mathfrak{A}$ consist of Borel sets, and let G be the group of isometries of space.) The algebra of cardinal numbers under addition is deduced from the algebra of sets by the same construction.

Another section deals with the relation between "cardinal algebras" and other types of algebraic systems—especially semigroups and distributive lattices. Finally, an appendix discusses "cardinal products of isomorphism types"—that is, with the direct factorization of general algebraic systems with a binary operation and a zero. The results here are related to the monograph *Direct decompositions of finite algebraic systems*, by the author and Bjarni Jónsson.

It seems certain, to the reviewer, that the postulates and ingenious deductions of Professor Tarski will permanently enrich modern algebra—at the same time that they show once more the value of considering infinitary operations. On the other hand, it seems less clear that the particular combinations of conditions labelled by the author "cardinal algebra," "generalized cardinal algebra," and "refinement algebra" will survive without modification. At all events, the author is to be congratulated for penetrating deep into new and heretofore uncharted mathematical territory; the book is a "must" for everyone seriously interested in modern algebra or set theory.

GARRETT BIRKHOFF

Theory of functions. By J. F. Ritt. Rev. ed. New York, Kings Crown Press, 1947. 10+181 pp. $3.00.

A student's introduction to the theory of functions has certain aspects in common with a youth's emergence into the adult world. Here he meets directness and subtlety, power and simplicity, beauty and rigor in quantity and proportion not previously experienced. A mathematician who undertakes to mediate this introduction by means of a book must be in control of these properties. Beyond that he needs the tact which prevents him from overwhelming the student with needless detail or deluding him with insufficient mathematical content. In the book under review these conditions have been met with ease. The teacher in search of a textbook will find, agreeably, that each chapter of the book has one theme and may be presented in one lecture.

The book seems to fall into five parts:

1. Real numbers and real functions occupy nine chapters covering 51 pages. The real numbers are identified with the non-terminating decimals; the arithmetic for terminating decimals being extended to them by means of the least upper bound theorem. There follow theorems on limits and on linear point sets. For real functions there are considerations of continuity, uniformity, monotoneity, and the basic theorems of the differential and integral calculus. This portion concludes with a study of the convergence, integrability, and differentiability of infinite series and the exhibition of an integrable function which is discontinuous on a dense set.

2. Functions of two variables, regions and curves in the plane, and curvilinear integrals are discussed in the next 40 pages. An interesting feature here is the distinction between the equivalence and inverse equivalence of two curves according as one is obtained from the other by an increasing or decreasing transformation of the parameter interval. This is neatly exploited in orientation questions involved in curvilinear integrals. In these pages are introduced the complex numbers, complex functions and their differentiation, the Cauchy-Riemann equations, and the conformal mapping property of analytic functions.

3. The next 26 pages deal with the Cauchy integral theorem and formula. This material presents a most serious problem of exposition to anyone writing on analytic functions. A treatment of this problem on the clean mathematical level set for himself by the author in this work would have led to a digression disrupting the pattern of the book. It is his choice to give an elegant discussion of the topology of the triangle and refer to the literature for the extension to Jordan curves. This manner of dealing with the difficulty offers to the lecturer using the text the inviting opportunity to expound his favorite proof of the Jordan curve theorem, a matter on which there is a wide range of taste.

4. With the machinery developed up to this point the analytic product is now obtained in 43 pages covering the series of Taylor and Laurent, the fundamental theorem of algebra as a corollary to Liouville's theorem, the zeros and poles of analytic functions. There is a treatment of the classical elementary functions, periodicity, and infinite products.

5. The concluding 25 pages expound such results as the Weierstrass factorization theorem, the Mittag-Leffler theorem on mero-

morphic functions, the neighborhood preserving property of analytic functions as a corollary to Rouché's theorem, the theory of residues. The concept of multiple valued function is prepared for by a neat discussion of the variation in amplitude of a function along a curve. The Riemann surface is mentioned and there is a concluding chapter on analytic continuation.

If we consider a presentation of function theory which is elegant without being formidable and simple without being trivial, then we have in Professor Ritt's *Theory of functions* a simple and elegant book.

L. W. Cohen

NOTES

The establishment of a new journal, Zeitschrift fuer Angewandte Mathematik und Physik, has been announced. It is published at Basel, Switzerland by Verlag Birkhäuser.

New York University announces a symposium on the *Theory of electromagnetic wave propagation*, to be held on June 6–8, 1950.

Cambridge University has announced that the 1949 award of the Adams Prize has been shared equally by J. C. Burkill of the University of Cambridge; Subrahmanyan Chandrasekhar of the University of Chicago; W. K. Hayman of University College of the Southwest of England, who is this year at Brown University; and J. M. Whittaker of Liverpool University. The subject assigned for the 1949 prize was Mathematical Analysis.

The French Academy of Sciences announces the award of its Carrière Prize to Professor M. Gevrey of the University of Dijon, the Thébault prize to Professor Raymond Estève of the Lycée Jacques-Decour and Professor H. Mitault of the Lycée Saint-Louis, the Dickson prize to Professor René de Possel of the University of Algiers, the Bordin Prize to Professor Charles Ehresmann of the University of Strasbourg, the Petit D'Ormoy prize to J. Leray, the Saintour prize to Professor S. Carrus of the University of Algiers.

Professor H. S. M. Coxeter of the University of Toronto has been awarded the Tory Medal by the Royal Society of Canada.

Dr. Ilse L. Novak of Wellesley College has received a Frank B. Jewett post-doctoral fellowship.

Associate Professor Henry Wallman of Massachusetts Institute of Technology who is now on leave at Chalmers Institute of Technology in Gothenburg, Sweden, has been made a foreign member of the Royal Swedish Academy.

Professor G. T. Whyburn of the University of Virginia has been awarded an honorary doctorate of science by Washington and Lee University.

Dr. M. M. Andrew of Massachusetts Institute of Technology has accepted a position with the Applied Mathematics Laboratories of the National Bureau of Standards.

Assistant Professor J. L. Brenner of the University of California has been appointed to an assistant professorship at the State College of Washington.

Professor Claude Chevalley of Princeton University has been appointed to a professorship at Columbia University.

Assistant Professor W. J. R. Crosby of the University of Saskatchewan has been appointed to an assistant professorship at the University of Toronto.

Dr. Werner Fenchel of the Technological University of Denmark has been appointed to a visiting professorship at the University of Southern California.

Professor M. G. Gaba of the University of Nebraska has retired with the title emeritus.

Dr. C. C. Hurd of Carbide and Carbon Chemicals Corporation, Oak Ridge, Tennessee, has accepted a position as director of the Applied Science Department, International Business Machines Corporation, New York, New York.

Mr. J. S. Isenberg of Brown University has accepted a position with Bell Aircraft Corporation of Buffalo, New York.

Dr. Shizuo Kakutani of the Institute for Advanced Study has been appointed to an assistant professorship at Yale University.

Professor W. R. Longley of Yale University has retired with the title emeritus.

Professor H. W. March of the University of Wisconsin has retired.

Mr. H. L. Merritt of Johns Hopkins University has accepted a position with the Ballistics Research Laboratory, Aberdeen, Maryland.

Dr. J. H. Mulligan Jr., of Allen B. DuMont Laboratories, Inc., has been appointed to an assistant professorship of electrical engineering at New York University.

Dr. J. P. Nash of the Kimberly-Clark Corporation has accepted the position of research assistant professorship at the University of Illinois.

Miss Ruth E. Porter of Berea College has been appointed to an assistant professorship at John Brown University, Siloam Springs, Arkansas.

Assistant Professor R. W. Shephard of New York University has accepted a position with the Rand Corporation, Santa Monica, California.

Mr. David Singer of City College, New York, New York, has accepted a position as a structural engineer with M. H. Treadwell Company, New York, New York.

Dr. Vivian E. Spencer of the Bureau of Census has accepted the position of Chief of the Minerals Section.

Professor W. D. A. Westfall of the University of Missouri has retired.

Associate Professor A. J. Zanolar has been appointed to the presidency of St. Joseph's College, Collegeville, Indiana.

The following promotions are announced:

H. H. Mostafa, Fouad I University, Guiza, Egypt, to an assistant professorship.

E. G. Begle, Yale University, to an associate professorship.

F. L. Brown, University of Notre Dame, to an assistant professorship.

W. L. Chow, Johns Hopkins University, to a professorship.

H. F. DeBaggis, University of Notre Dame, to an assistant professorship.

Watson Fulks, University of Minnesota, to an assistant professorship.

R. E. Gilman, Brown University, to a professorship.

E. L. Godfrey, Defiance College, to an associate professorship.

Harriett Griffin, Brooklyn College, to an associate professorship.

L. M. Klauber, San Diego Gas and Electric Company, to Chairman of the Board.

J. P. LaSalle, University of Notre Dame, to an associate professorship.

J. C. R. Li, Oregon State College, to an associate professorship.

A. T. Lonseth, Oregon State College, to a professorship.

C. W. Mathews, Washington University, to an assistant professorship.

W. L. Moore, University of Louisville, to a professorship.

J. P. Murray, Fairfield University, to an assistant professorship.

Helen K. Nickerson (Mrs. W. J.), Wheaton College, Norton, Massachusetts, to an assistant professorship.

A. R. Poole, Oregon State College, to an associate professorship.

E. L. Post, City College, New York, New York, to a professorship.

O. W. Rechard, Ohio State University, to an assistant professorship.

E. H. Rothe, University of Michigan, to an associate professorship.

Hans Samelson, University of Michigan, to an associate professorship.

T. H. Southard, Wayne University, to an associate professorship.

R. L. Swain, Ohio State University, to an assistant professorship.

D. F. Votaw, Yale University, to an assistant professorship.

Margaret F. Willerding, Harris Teachers College, Saint Louis, Missouri, to an assistant professorship.

Frances M. Wright, Triple Cities College of Syracuse University, to an assistant professorship.

The following appointments to instructorships are announced:

Brown University: Leila R. Raines (Mrs. Herbert Federer); University of Buffalo: Mr. Benjamin Lapidus; Santa Barbara College, University of California: Dr. D. D. Wall; Cooper Union: Mr. P. L. Chessin; Harvard University: Mr. S. F. Neustadter; University of Michigan: Dr. C. J. Titus; University of Notre Dame: Mr. J. C. Smith; Ohio State University: Dr. E. D. Cashwell; University of Tennessee: Miss Annette Sinclair; State College of Washington: Mr. P. A. Clement; Yale University: Mr. R. R. Bernard, Mr. W. H. Mills.

Professor Alberto Dias Coimbra of the University of Coimbra, Coimbra, Portugal, died November 24, 1949.

Professor S. E. Brasfield of Rutgers University died October 14, 1949, at the age of seventy-six years. He had been a member of the Society for eighteen years.

Professor F. F. Decker of Syracuse University died November 28, 1949, at the age of sixty-nine years. He had been a member of the Society for forty-two years.

Mr. B. F. Groat died June 15, 1949, at the age of eighty-three years. He had been a member of the Society for fifty-one years.

Professor Emeritus A. C. Lunn of the University of Chicago died November 19, 1949, at the age of seventy-two years. He had been a member of the Society for forty-six years.

Professor Emeritus D. A. Rothrock of Indiana University died September 2, 1949, at the age of eighty-five years. He had been a member of the Society for fifty years.

Professor Emeritus Lao G. Simons of Hunter College died November 25, 1949, at the age of seventy-nine years. She had been a member of the Society for twenty-seven years.

Professor Emeritus Virgil Snyder of Cornell University died January 4, 1950, at the age of eighty-one years. He had been a member of the Society for fifty-four years.

NEW PUBLICATIONS

ARLEY, N., and BUCH, K. R. *Introduction to the theory of probability and statistics*, New York, Wiley, 1950. 12+236 pp. $4.00.

BERKELEY, E. C. *Giant brains or machines that think*, New York, Wiley, 1949. 16+270 pp. $4.00.

BOREL, E. *Eléments de la théorie des ensembles*, Paris, Michel, 1949. 320 pp. 720 f.

BUCH, K. R. See ARLEY, N.

LE CORBEILLER, P. *Matrix analysis of electric networks* (Harvard Monographs in Applied Science, no. 1.) Cambridge, Harvard University Press, 1950, 12+112 pp. $3.00.

DUSCHEK, A. *Vorlesungen über höhere Mathematik*. Vol. 1. Vienna, Springer, 1949. 10+395 pp. $8.70.

FRANKLIN, P. *Fourier methods*. New York, McGraw-Hill, 1949, 10+289 pp. $4.00.

LANCZOS, C. *The variational principles of mechanics*. (Mathematical Expositions, no.4.) University of Toronto Press, 1949. 26+307 pp. $5.75.

MANDELKER, J. *Principles of a new energy mechanics*. New York, Philosophical Library, 1949. 8+73 pp. $3.75.

VON NEUMANN, J. *Functional operators*. Vol. 1. *Measures and integrals*. (Annals of Mathematics Studies, no. 21.) Princeton University Press, 1950. 261 pp. $3.50.

SARTORI, A. *Problemi di secondo grado risolti e discussi*. Turin, Paravia, 1949. 5+163 pp. 600 Lire.

SCHILD, A. See SYNGE, J. L.

SCHILPP, P. A., ed. by. *Albert Einstein: Philosopher-scientist*. (Library of Living Philosophers, vol. 7.) Evanston, Library of Living Philosophers, 1949. 16+781 pp. $8.50.

SIEGEL, C. L. *Transcendental numbers*. Princeton University Press, 1949. 8+102 pp. $2.00.

SYNGE, J. L., and SCHILD, A. *Tensor calculus*. (Mathematical Expositions, no. 5.) University of Toronto Press, 1949. 12+324 pp. $6.00.

Tables of the Bessel functions of the first kind of orders sixty-four through seventy-eight. (The Annals of the Computation Laboratory of Harvard University, vol. 13.) Cambridge, Harvard University Press, 1949. 2+566 pp. $8.00.

Tables of the confluent hypergeometric function and related functions. (Applied Mathematics Series, no. 3.) Washington, National Bureau of Standards, 1949. 22+73 pp. $.35.

Tables of the function sin ϕ/ϕ *and of its first eleven derivatives*. (The Annals of the Computation Laboratory of Harvard University, vol. 22.) Cambridge, Harvard University Press, 1949. 19+241 pp. $8.00.

Tables of the generalized exponential-integral functions. (The Annals of the Computation Laboratory of Harvard University, vol. 21.) Cambridge, Harvard University Press, 1949. 25+416 pp. $8.00.

WALKER, R. J. *Algebraic curves*. (Princeton Mathematical Series, vol. 13.) Princeton University Press, 1950. 11+201 pp. $4.00.

INTERNATIONAL CONGRESS OF MATHEMATICIANS

Several donations to the International Congress of Mathematicians have been received since the most recent announcement was published in this BULLETIN and mailed to members of the Society. The Society wishes to express its thanks at this time to the following organizations and individuals who have made contributions.

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AREA AND REPRESENTATION OF SURFACES

LAMBERTO CESARI

What is surface and what is area? This question is not a trivial one, because study of the literature reveals that there are several relevant concepts of surface and area, just as there are several relevant concepts of curve and length (see [1, I.I, II.5, V.4]; numbers in square brackets refer to the bibliography at the end of this paper). These concepts are not only related to one another, but they are also related to the solution of many problems in which they play an essential part. My purpose is to discuss the connection of such concepts with calculus of variations and to show how the slow development of some of them is closely related to advances in calculus of variations.

During the war years I succeeded in developing a complete theory for Fréchet surfaces and Lebesgue area, which was related to previous work of McShane and Radó. It was with great pleasure that I learned after the war that, at the same time and independently, Radó and other mathematicians in this country had developed a theory founded upon the same fundamental ideas. The results obtained are partly overlapping and partly complementary. This allows us to combine our results into a single theory. Now this parallel development is not due to chance, but rather to an underlying common aim that has been a constant guide in our respective efforts: the aim to obtain a more general basis for the theory of double integrals in parametric form in calculus of variations. Indeed, the theory for such integrals has not yet been finally settled, as has been noticed by both Radó and Tonelli.

The concepts of curve and surface, length and area, were presented at other meetings of this society by J. W. T. Youngs [59] from a general point of view. Therefore I will limit myself here to showing only how much our concepts owe to calculus of variations, and then I will discuss the most recent and important results concerning the problem of area and representation of surfaces.

A complete list of all papers which are related to this topic would be too extensive for the purposes of the present paper. The bibliography at the end is not meant to be complete in any sense. The recent book of Radó, *Length and area* [1], will be used as a general refer-

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An address delivered before the Annual Meeting of the Society in Columbus, Ohio, on December 29, 1948, by invitation of the Committee to Select Hour Speakers for Summer and Annual Meetings; received by the editors February 21, 1949.

ence. Several papers that have been published since that book appeared are included in the bibliography, provided that they are directly related to our main topic.

The first question which led to the modern theory of calculus of variations was the problem of Dirichlet, that is, the problem of determining a function $f(x, y)$ whose values are given on the boundary of a Jordan region A and for which the integral

$$\iint_A \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dxdy$$

assumes its minimum value. From the fact that the set of the values assumed by such an integral has a finite lower bound, mathematicians inferred the existence of the minimum. This became a general principle, called the principle of Dirichlet, and it gave rise immediately to numerous applications. The most general of these was certainly the one made by Riemann, who based his theory of abelian functions entirely on this principle. After criticism by Weierstrass the principle of Dirichlet was abandoned and the question of determining a harmonic function that assumed assigned values on the boundary of a Jordan region was settled in another way by Neumann, Schwarz, and Poincaré.

The possibility of giving a direct demonstration of the principle of Dirichlet was recognized by Arzelà, who was the first to try to prove the principle of Dirichlet in connection with concepts of the functional analysis of Volterra. Arzelà did not succeed in this, but his research, continued by Hilbert, led the latter in 1900 to the first proof of the principle. This achievement gave rise to a good deal of research between 1900 and 1904. We mention the work of Lebesgue, Bolza, Carathéodory, who gave existence theorems for problems of calculus of variations under hypotheses which were still rather restrictive. However, the method created by Hilbert did not yield further results, so that several authors later turned to other procedures for the solution of maximum and minimum problems in calculus of variations. Of these authors I would like to mention Hadamard and Levi.

The reason for this fact was realized later when Tonelli observed first that the concept of semicontinuity introduced in 1906 by Baire in the theory of real functions should be introduced as the fundamental concept. Tonelli noted that the most common functionals of a curve in calculus of variations are lower or upper semicontinuous. In particular the "length" of a curve is the simplest example of a functional of a curve that is lower semicontinuous. Indeed all the regular positive functionals of a curve are lower semicontinuous.

Arising from the functional analysis of Vito Volterra and the notion of semicontinuity introduced by Tonelli into calculus of variations, the direct method of calculus of variations for functionals of a curve was developed for the last thirty years and, with Menger, we may say that the results obtained are now classical. This work was carried on by mathematicians all over the world. I mention only Graves, McShane, Reid in the United States and Tonelli, Baiada, Cinquini, Manià in Italy.

A new step is due to Bouligand and Menger. The former observed that, upon introducing a suitable metric in the space of curves, each integral along a curve, which is positive, regular, and therefore lower semicontinuous, can be considered as a "generalized length" for the curve. The latter observed that by means of this approach, combined with the Weierstrass integral, it is possible to extend the direct method to functionals of a curve in any abstract space. With this modern extension the ordinary length of a curve becomes only the most important "functional of comparison," but these functionals are always lower semicontinuous.

I would like to mention, however, that the concept of lower semicontinuity, which also had a fundamental role in Marston Morse's topological theory of calculus of variations, has been replaced in his more recent work by the concept of upper reducibility. Also in this connection operational concepts and metrization seem to have increasingly wide application.

Let us now turn our attention from curves to surfaces, from line integrals to surface integrals, from length to area.

Of all definitions of surface area which were known at the beginning of this century there was only one which was completely general and which satisfied the principle of lower semicontinuity: the one proposed by Lebesgue in his memoire in 1900 [**33**]. Geöcze worked a long time on this definition during the period from 1905 to 1916. Tonelli [**53**] used it in 1915 for a demonstration of the minimum property of the sphere known as the isoperimetric inequality. This inequality was proved recently by Radó [**44**] for general closed continuous surfaces. As proved by Radó, the fundamental isoperimetric inequality states that

$$V^2(S) \leq L^3(S)/(36\pi),$$

where $L(S)$ is the Lebesgue area of the closed surface S and $V(S)$ is the volume enclosed by S according to the following definition. Let $i(x, y, z)$ be the "topological index" of the point (x, y, z) with respect to S, where $i(x, y, z)=0$ for points on S. Then $V(S)$ is defined as the integral of $|i(x, y, z)|$. Just as the Lebesgue area $L(S)$ is a lower semicontinuous area, so also is the volume $V(S)$ considered by Radó a lower semicontinuous volume.

Of course, if the surface S is an ordinary surface or, at least, admits of a lipschitzian representation, then all definitions of area and of volume agree and the above inequality is to be expected. But with the concept of Lebesgue area and the preceding definition of volume, this inequality is completely general and holds for general surfaces even in cases in which not all definitions of area and volume agree.

Let us recall the main concepts of the theory of curves

$$C: \quad x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad a \leqq t \leqq b.$$

A curve C has finite length $l(C)$ if and only if the functions that represent it are of bounded variations; the length of a curve is always greater than or equal to the value of the classical integral:

$$l(C) \geqq \int_a^b (x'^2 + y'^2 + z'^2)^{1/2} dt;$$

the length of a curve is equal to the classical integral if and only if the functions that represent the curve are absolutely continuous.

These three fundamental facts have been extended by Tonelli [54] to the Lebesgue area of nonparametric surfaces

$$z = f(x, y), \qquad (x, y) \in A.$$

Tonelli introduced appropriate definitions for bounded variation and absolute continuity of the function $f(x, y)$. The main results are the following: the surface S has finite Lebesgue area if and only if the function $f(x, y)$ is of bounded variation; the Lebesgue area $L(S)$ of S is always greater than or equal to the value of the classical integral:

$$L(S) \geqq \iint_A (1 + p^2 + q^2)^{1/2}, \qquad p = \frac{\partial f}{\partial x}, \qquad q = \frac{\partial f}{\partial y};$$

the Lebesgue area of the surface S is equal to the classical integral if and only if the function $f(x, y)$ is absolutely continuous [54].

It is also remarkable that these concepts of bounded variation and absolute continuity introduced by Tonelli for the problem of Lebesgue area for nonparametric surfaces S have had, from 1926 to the present, extensive and varied applications in the theory of differential equations, in potential theory, and in the theory of double and multiple Fourier series. But one of the main applications is certainly that in calculus of variations to double integrals over a nonparametric surface S [56]:

$$I_S = \iint_A f(x, y, z, p, q)dxdy.$$

A complete theory based on the concept of semicontinuity has been created since 1929 by Tonelli, Cimmino, Cinquini, Scorza in Italy and by Graves, Bliss, McShane, Morse, Reid in the United States. In this theory the problem of Dirichlet and also the more difficult problem of Plateau were settled by Courant, Douglas, McShane, Morse, Radó, and Tonelli [55]. At the same time some more general variational problems for surfaces in parametric form

$$S: \quad x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v), \quad (u, v) \in A,$$

were studied by McShane [34, 35, 36, 37] and Radó [45].

From the above discussion of applications in calculus of variations it appears advisable to choose for general surfaces a concept of area which satisfies the fundamental principle of semicontinuity, namely the Lebesgue concept of area, which was so useful for nonparametric surfaces.

Naturally for applications in which the principle of semicontinuity is not so important, and even for calculus of variations if we do not base it on the concept of semicontinuity, other definitions of area can also be considered. In this connection I mention the definitions of Minkowski and Carathéodory. Carathéodory introduced a general and convenient concept of linear measure or *length*, and of superficial measure or *area*, of any point set in space. A complete theory for rectifiable sets is due to A. S. Besicovitch. This theory was extended by A. P. Morse and J. F. Randolph by use of a more general concept of Carathéodory measure; it was further extended to point sets of finite area by Besicovitch, Busemann, Federer, and other authors. This theory has been very carefully worked out and is related to recent trends in the theory of integration. Another definition of area is due to Banach [4] and Vitali [57]; the theory for this area was carried out by Banach and Schauder.

Of course all definitions of area coincide in ordinary cases, and they also coincide when the surface has a lipschitzian representation; in the general case they no longer coincide.

Today we have various definitions of area which satisfy the principle of semicontinuity, but in all such cases it follows from the research of Radó and myself that they coincide with the Lebesgue area. First of all let us mention the so-called Geöcze area $G(S)$ in the sense used by McShane [34] and Morrey [41], which is introduced by means of the topological index of a point with respect to a closed plane curve. I have shown [16] that the lower semicontinuous Geöcze area $G(S)$ coincides with the Lebesgue area $L(S)$. Radó [1, V.1] has introduced a "lower area" that is also lower semicontinuous. I have completed Radó's proof that it coincides with the Lebesgue area [1, V, 4.8]. Quite recently Radó modified previous definitions of Cauchy and Favard in order to obtain two new definitions of area satisfying the principle of semicontinuity. Jointly with Helsel [46] and Mickle [47], he has shown that these two concepts of area also coincide with the Lebesgue area.

In 1928, in his communication at the International Mathematics Meeting of Bologna, Radó [48] had already stated the conjecture that each lower semicontinuous functional over a surface which coincides with the elementary area for polyhedral surfaces must coincide with the Lebesgue area for any surface, at least under very weak and natural conditions. From this point of view, which we shall call the tendency for an axiomatic definition of area, contributions have been made by Kempisty, Fréchet, Scorza, Zwirner, Stampacchia.

On the same occasion Radó [48] also observed that it would be desirable to define axiomatically the area $\alpha(S)$ of the surface S by means of a Dedekind cut in the field of real numbers. Indeed, if we require $\alpha(S)$ to be a lower semicontinuous functional coinciding with the elementary area upon polyhedral surfaces, then it is easy to prove that $\alpha(S) \leqq L(S)$. Therefore $L(S)$ is an upper bound for the area $\alpha(S)$. If we require $\alpha(S)$ to be greater than or equal to the value of the Geöcze area $G(S)$, which is intuitively the smallest possible interior area, then $\alpha(S)$ also has a lower bound

$$G(S) \leqq \alpha(S) \leqq L(S).$$

Radó's conjecture that $G(S) = L(S)$ has now been proved by myself [16], and therefore the Lebesgue area is, in this sense, determined by an intuitively motivated Dedekind cut.

The notion of surface S that we use is not the set of points covered by it, but comes from the consideration of all possible "transformations" or "representations"

$$(1) \qquad T:\quad x = x(u, v),\quad y = y(u, v),\quad z = z(u, v),\qquad (u, v) \in A,$$

that are equivalent in the Fréchet sense. Each of these equivalent transformations gives a representation of the surface S. It is well known that the Lebesgue area $L(S)$ does not depend upon the representation, but is a property of the surface itself. If we require the notion of Fréchet equivalence to conserve the orientation of the boundary A^* of the Jordan domain A, as McShane did [34], then we speak of "oriented Fréchet surfaces"; otherwise, simply of "Fréchet surfaces" or "surfaces."

Important research in set topology, related to this concept of a surface, was carried out by Kerékjárto, R. L. Moore, Whyburn, Wiener, and Wilder. Of this research I point out the theory of upper semicontinuous collections and the theory of cyclic elements; these led to the topological notions of the cactoid of R. L. Moore and of the hemicactoid of Morrey. A complete revision of the applications of this theory to surface area theory has been made by Radó and his school, so that we now have a very simple and clear theory of the topology of Fréchet surfaces [1, II]. Each transformation T is the product of a monotone transformation M and of a light transformation L: $T=LM$. This is a monotone-light factorization of the transformation T. M maps A onto a Peano space $\mathfrak{M}$ that is called the "middle space"; L maps $\mathfrak{M}$ onto S. For two different representations T_1 and T_2 of the same surface S, we have

$$T_1 = LM_1, \qquad T_2 = LM_2;$$

that is, we have different monotone factors but the same light factor L. The middle space is the cactoid or hemicactoid of the previous theory. Youngs has succeeded in giving a complete intrinsic topological characterization of the notion of Fréchet equivalence. These results of Youngs are now in the process of publication [62].

If we divide A into a finite number of Jordan domains $A_1, A_2, \cdots, A_n$ by curves $\gamma_1, \gamma_2, \cdots, \gamma_m$ and let $S_1, S_2, \cdots, S_n$ be the continuous surfaces defined by (1) upon $A_1, A_2, \cdots, A_n$ respectively, then we have

$$L(S) \geqq \sum_{i=1}^{n} L(S_i).$$

It is of the greatest importance to have conditions sufficient for equality to hold in this relation. Theorems of this type, or *additivity theorems*, have been proved and applied by Radó [1] and Youngs [59].

In 1924, in order to extend to surfaces the theory of Jordan and Tonelli for length of curves, Banach [4] and Vitali [57] gave, independently of each other, a theory for the area of surfaces in parametric form:

$$(2) \qquad S:\ x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \quad (u, v) \in A.$$

They were the first to observe that properties of a surface, for instance that of having finite area, depend upon properties, not of the single functions x, y, z of the representation, but of pairs of these functions:

$$T_1:\ \begin{cases} y = y(u, v), \\ z = z(u, v), \end{cases} \quad T_2:\ \begin{cases} z = z(u, v), \\ x = x(u, v), \end{cases} \quad T_3:\ \begin{cases} x = x(u, v), \\ y = y(u, v); \end{cases}$$

that is, they depend upon properties of the three "flat surfaces" or the plane transformations which are the three projections of the surface upon the three coordinate planes. From this point of view the results obtained by Banach and Vitali constitute a formal extension of the theory of arc length, in the sense of Jordan and Tonelli, to general continuous surfaces given in parametric form. However, they made use of a concept of surface area which differs from Lebesgue area and lacks the property of lower semicontinuity.

In quite different ways, and independently, Radó [1, IV.4, IV.5] and I [8, 12, 14] introduced concepts of plane transformations of bounded variation. These concepts, however, are completely equivalent, as Radó [50] has shown.

Furthermore, during the war I [8] proved the following

THEOREM 1. *The Lebesgue area of a general surface S is finite if and only if the three plane transformations T_1, T_2, T_3, which are obtained by projecting the surface upon the three coordinate planes, are all of bounded variation.*

This result can also be expressed by saying that the Lebesgue area of the surface S is finite if and only if these three projections T_1, T_2, T_3, considered as flat surfaces, all have finite Lebesgue area.

One result in Radó's theory, as well as in mine, is that each continuous surface of finite Lebesgue area has generalized Jacobians $J_1(u, v)$, $J_2(u, v)$, $J_3(u, v)$ almost everywhere in A. For such surfaces these generalized Jacobians are L-integrable functions in A, generalize the ordinary Jacobians

$$\frac{\partial(y, z)}{\partial(u, v)}, \qquad \frac{\partial(z, x)}{\partial(u, v)}, \qquad \frac{\partial(x, y)}{\partial(u, v)}$$

formed with ordinary first partial derivatives, and coincide with them provided the ordinary Jacobians exist almost everywhere.

Radó, Reichelderfer, and I, at the same time and independently, have each proved the following

THEOREM 2. *If the surface S has finite Lebesgue area, then the plane* *transformations* T_1, T_2, T_3 *are of bounded variation and have generalized Jacobians* $J_1(u, v)$, $J_2(u, v)$, $J_3(u, v)$ *a.e. in* A; *furthermore the Lebesgue area is greater than or equal to the classical integral*

$$L(S) \geqq \iint_A (J_1^2(u, v) + J_2^2(u, v) + J_3^2(u, v))^{1/2} dudv.$$

This result is also in agreement with the classical theory of Banach concerning functions of a set.

In completely different ways Radó and I have each introduced concepts of absolutely continuous plane transformations. These concepts are also equivalent, as Radó [50] has shown. Radó and I have each proved the following

THEOREM 3. *The Lebesgue area is equal to the classical integral if and only if the three plane transformations* T_1, T_2, T_3, *projections of the surface upon the coordinate planes, are absolutely continuous.*

I [11, 25] have also proved several tangential properties for the most general surfaces of finite Lebesgue area.

The classical results of Jordan and Tonelli for curves and the results of Tonelli for surfaces in nonparametric form are now completely extended by these theorems concerning Lebesgue area to surfaces in parametric form. These theorems also establish the fact that the concepts of total variation and absolute continuity for plane transformations which were introduced by Radó, Reichelderfer, and myself are completely adequate.

It is of interest to note that the concept of plane transformation of bounded variation and the concept of absolutely continuous plane transformation have been applied to the important problem of the transformation of areas or of double integrals by Radó, Reichelderfer, Helsel, and Mickle here in the United States, by myself [24] in Italy, obtaining equivalent and very general formulas of transformation.

The Kolmogoroff principle has been proved and generalized by Mickle and Helsel [32] and has been applied to the stretching process for surfaces and in the proofs of additivity theorems. The main results that I have mentioned solve the fundamental problems for the Lebesgue area of general surfaces. The following is another very important problem.

We know that each continuous rectifiable curve has at least one representation for which the length is equal to the classical integral and such a representation is obtained in a very simple way by choosing as parameter the length of the arc from a fixed point to a variable point of the curve. We also know that for non-rectifiable curves the problem of a special representation has been solved by Marston Morse.

For surfaces we have the following two problems, which may be termed representation problems:

(i) Given a surface S of finite Lebesgue area, does there exist a representation for which the Lebesgue area is given by the classical integral calculated with generalized Jacobians?

(ii) Given a surface S of finite Lebesgue area, does there exist a representation for which the functions x, y, z have first partial derivatives a.e. in A and for which the Lebesgue area is given by the classical integral calculated with ordinary Jacobians?

McShane studied these problems because of important applications to the calculus of variations and, in particular, to the Plateau problem. McShane [36] showed that such a special representation exists for the so-called "saddle surfaces." More generally, Morrey [43] found such a representation for the surfaces that we call "open nondegenerate," that is, according to the nomenclature of Radó, those for which the middle space is a 2-cell. For such surfaces McShane and Morrey found a new type of representation, the so-called "almost conformal representation." A representation (1) of a surface S is called "almost conformal" if

(a) the functions x, y, z are absolutely continuous in the sense of Tonelli in A;

(b) the first partial derivatives x_u, x_v, $\cdots$, z_v are L^2-integrable functions in A;

(c) we have $E=G$, $F=0$, a.e. in A, where $E=x_u^2+y_u^2+z_u^2$, $G=x_v^2+y_v^2+z_v^2$, $F=x_ux_v+y_uy_v+z_uz_v$.

For such representations the Lebesgue area is necessarily equal to the classical integral. The theorem of Morrey can now be rephrased as follows: Each open nondegenerate surface of finite Lebesgue area admits of at least one almost conformal representation upon A for which the area is given by the classical integral.

We know that a Jordan domain can be represented upon a circle in a conformal way. We also have the Schwarz theorem: each polyhedral surface which is open nondegenerate can be represented upon a circle in a conformal way; that is, it admits of a representation (1) for which properties (a) and (b) hold and also for which the relations $E=G$, $F=0$ hold everywhere in A with the exception of a finite set of points, the points which correspond to the vertices of the polyhedral surface. By an almost conformal representation of any open nondegenerate surface we mean one for which the exceptional points form a set of measure zero. The theorem of Morrey thus appears as a generalization of Schwarz's theorem.

The almost conformal representations of an open nondegenerate surface of finite Lebesgue area satisfy a minimum principle [15, **17**]: Among all possible representations of the surface, which are equivalent in the sense of Fréchet and which satisfy conditions (a) and (b), there exist some for which the double integral of the sum of the squares of all the first partial derivatives of the functions x, y, z is a minimum:

$$\iint_A (x_u^2 + y_u^2 + z_u^2 + x_v^2 + y_v^2 + z_v^2)dudv.$$

This minimum is equal to the Lebesgue area of the surface itself. The minimal representations are all almost conformal and the almost conformal representations are all minimal [15, **17**].

This problem is considered here as a problem of calculus of variations and is very similar to the problems of Dirichlet and Plateau. Indeed, the proofs of McShane and Morrey contain ideas similar to that used by Lebesgue, Radó, Courant, Tonelli for such variational problems. Using the direct method of calculus of variations, I have proved a theorem of the type of Schwarz's for the representation of polyhedral surfaces, and I have also proved the theorem of Morrey directly [15].

Starting from this result I have, moreover, succeeded in proving the following

THEOREM 4. [23] *Each general surface S of finite Lebesgue area admits of a representation upon a circle A such that the Lebesgue area is equal to the classical integral calculated with generalized Jacobians*

$$L(S) = \iint_A (J_1^2 + J_2^2 + J_3^2)^{1/2} dudv.$$

In such a representation the nondegenerate parts of the surface, that is, in the nomenclature of Radó, the parts that come from the cyclic elements of the middle space, are represented almost conformally. This solves the first representation problem.

Now by means of an observation that was suggested by a previous simpler remark of Youngs, I have proved the following

THEOREM 5. [29] *Each general surface S of finite Lebesgue area admits of a representation upon a circle A such that*

(a*) *The functions x, y, z have ordinary first partial derivatives a.e.*

in A;

(b) *the first partial derivatives* $x_u, x_v, \cdots, z_v$ *are* L^2*-integrable functions in* A;

(c) *we have* $E=G$, $F=0$ *a.e. in* A, *where* $E=x_u^2+y_u^2+z_u^2$, $G=x_v^2+y_v^2+z_v^2$, $F=x_ux_v+y_uy_v+z_uz_v$;

(d) *the Lebesgue area is given by the classical integral calculated with ordinary Jacobians*:

$$L(S) = \iint_A \left(\left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(z, x)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2\right)^{1/2} dudv.$$

This theorem solves the second representation problem. Here we have replaced the condition (a) of the theorem of Morrey by the weaker condition (a*). This fact is related to a remark of Mickle [40]: there exists a surface S of finite Lebesgue area which admits of no representation satisfying the strong condition (a). In this new weaker sense the above theorem can be expressed by saying that each continuous general surface of finite Lebesgue area admits of an almost conformal representation.

These concepts and results are already being applied to the notion of integral upon surfaces in parametric form. I recall here that the notion of integral upon a surface

$$I_S = \iint_A F(x, y, z, J_1, J_2, J_3)dudv$$

as a Lebesgue integral was studied in 1932 by McShane [34, 35, 37] for surfaces S that are given by representations satisfying conditions (a) and (b). More generally Radó [45] studied the same concept in 1944 under the weaker hypotheses that the functions x, y, z have first partial derivatives a.e. in A and that the Lebesgue area is finite and given by the classical integral with ordinary Jacobians.

Recently, in 1946, I [21] introduced the notion of integrals I_S as a Weierstrass integral for any continuous surface of finite Lebesgue area given by any continuous representation. I proved that the Weierstrass integral I_S is equal to the Lebesgue integral for each representation for which the Lebesgue area is given by the classical integral and, finally, that the necessary conditions [26] and the sufficient [27] conditions determined by McShane and Radó for semicontinuity hold for the class of the most general continuous surfaces. From the above solution of the second representation problem we know now that there always exists at least one representation of the surface S for which the Weierstrass integral I_S can be calculated as a Lebesgue integral with ordinary Jacobians, that is, as one of the integrals studied by Radó [45].

With all this research in which the efforts of so many mathematicians have been concentrated, perhaps the way is now open for further applications in analysis.

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NORMED LINEAR SPACES OF CONTINUOUS FUNCTIONS

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1. **Introduction.** In addition to its well known role in analysis, based on measure theory and integration, the study of the Banach space $B(X)$ of real bounded continuous functions on a topological space X seems to be motivated by two major objectives.

The first of these is the general question as to relations between the topological properties of X and the properties (algebraic, topological, metric) of $B(X)$ and its linear subspaces. The impetus to the study of this question has been given by various results which show that, under certain natural restrictions on X, the topological structure of X is completely determined by the structure of $B(X)$ [**3**; **16**; **7**],[1] and even by the structure of a certain type of subspace of $B(X)$ [**14**]. Beyond these foundational theorems, the results are as yet meager and exploratory. It would be exciting (but surprising) if some natural metric property of $B(X)$ were to lead to the unearthing of a new topological concept or theorem about X.

The second goal is to obtain information about the structure and classification of real Banach spaces. The hope in this direction is based on the fact that every Banach space is (equivalent to) a linear subspace of $B(X)$ [**1**] for some compact (that is, bicompact Hausdorff) X. Properties have been found which characterize the spaces $B(X)$ among all Banach spaces [**6**; **2**; **14**], and more generally, properties which characterize those Banach spaces which determine the topological structure of some compact or completely regular X [**14**; **15**]. These properties are defined in terms of concepts which are meaningful in all Banach spaces; in particular, no lattice [**10**] or ring [**8**; **9**; **11**] structure is presupposed.

I propose here to outline and supplement some recent results along the above two lines, using methods developed in [**14**]. In one or two instances details of proofs are given; these are brief and have not previously appeared in print.

2. **The mapping** $C(X)$. Let X be a completely regular topological space. The set of all real-valued, bounded continuous functions on X, with the usual laws of addition and multiplication by real num-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

bers, is made into a Banach space $B(X)$ over the reals by defining $\|b\| = \sup_{x\in X} |b(x)|$. Let B be a closed linear subspace of $B(X)$. Consider the following possibilities:

(I) B is separating over X; that is, given $x_1, x_2 \in X$, $x_1 \neq x_2$, there exists a $b \in B$ such that $b(x_1) \neq b(x_2)$.

(II) Given $x \in X$, and a closed set $X_1 \subset X$ not containing x, there exists a finite set $b_1, \cdots, b_n \in B$ and a $\delta > 0$, such that, for each $x_1 \in X_1$, $|b_i(x) - b_i(x_1)| \geqq \delta$ for at least one value of i.

(III) B is *completely regular over* X; that is, given $x \in X$ and a closed set $Y \subset X$ not containing x, there exists a $b \in B$ such that $b(x) = \|b\|$, $\sup_{y\in Y} |b(y)| < \|b\|$.

It is easy to see that (III)→(II)→(I). If X is compact, (I) = (II).

Now let B_W^* be the conjugate space to B, provided with the weak-* topology, let E_W^* be its compact [1; 4] solid unit sphere. For each $x \in X$, let x^* be the point in E_W^* defined by $x^*(b) = b(x)$ for all $b \in B$. The mapping C of X into E_W^* defined by $C(x) = x^*$ for all $x \in X$ is always continuous, $C(X)$ is a total subset of B_W^*, and for each $b \in B$, $\|b\| = \sup_{f\in C(X)} |f(b)|$. C is one-to-one if and only if B satisfies (I). It is a homeomorphism if and only if B satisfies (II).

Given any Banach space B, there always exists a compact X such that B is equivalent to a closed linear subspace of $B(X)$ satisfying (II); for example, X can be taken as E_W^* in B_W^*. However, the class of Banach spaces satisfying (III) with some completely regular X is, as we shall see, a proper subclass of all Banach spaces, and itself contains as a proper subclass the class of Banach spaces satisfying (III) with some *compact* X. We shall characterize both of these classes. Also, we shall see that a given Banach space B satisfies (III) with at most one compact X.

3. **Spaces completely regular over** X [14; 15]. Let B be an arbitrary Banach space. Each $b \in B$ is contained in a T-set—that is, a subset of B maximal with respect to the property that for each of its finite subsets $(b_1, \cdots, b_n)$, $\|\sum b_i\| = \sum \|b_i\|$. The intersections of T-sets with S, the surface of the unit sphere in B, are the maximal convex subsets of S used by Eilenberg [7]. For each T-set T we define a functional F_T by the formula $F_T(b) = \inf_{t\in T}(\|b+t\| - \|t\|)$. Each F_T is continuous over B and is convex, and $|F_T(b)| \leqq \|b\|$, with $F_T(t) = \|t\|$ for $t \in T$. F_T is linear over B if and only if $F_T(b) = -F_T(-b)$ for all $b \in B$. Also, F_T is linear if and only if there exists a unique $f \in S_W^*$ such that $f(t) = \|t\|$ for all $t \in T$, and in this case $f = F_T$ over B; this follows from a theorem on unique norm-preserving extensions of a linear functional, which is stated and proved in §8. Thus

if F_T is linear, given any point $f \in E_W^*$ different from F_T there is a $b \in B$ such that the continuous function over B_W^* defined by b according to $b(F) = F(b)$ for all $F \in B_W^*$ has the properties $b(F_T) = \|b\|$, $b(f) < \|b\|$; further, $b(f) = -\|b\|$ only if $f = -F_T$. If all the functionals F_T are linear, they form a total subset M of B_W^*, contained in S_W^*, such that for each $b \in B$, $\|b\| = \sup_{f \in M} |f(b)|$; B is equivalent to a closed linear subspace of $B(M)$. If furthermore M is the union of two disjoint closed antipodal subsets Ω and $-\Omega$, then B is equivalent to a closed linear subspace of $B(\Omega)$ completely regular over Ω. If M is closed in E_W^*, Ω is compact.

Conversely, let X be a compact space, and let B be a closed linear subspace of $B(X)$. For each T-set T in B there is an $x \in X$ such that either $T = \{b \in B \mid b(x) = \|b\|\}$ or $T = \{b \in B \mid b(x) = -\|b\|\}$. Now assume B is completely regular over X; then the x of the previous statement is unique, and also every set of the form $\{b \in B \mid b(x) = \|b\|\}$ or $\{b \in B \mid b(x) = -\|b\|\}$ is a T-set. We denote the T-sets corresponding to x by T_x and $-T_x$. *A fundamental fact is that for every $b \in B$ and $x \in X$,*

$$b(x) = \inf_{t \in T_x} (\|b + t\| - \|t\|) = F_{T_x}(b).$$

It follows that each F_T is linear. The set of linear functionals F_T, considered as points in S_W^*, is the union of two disjoint closed antipodal subsets, one of which is $C(X)$ and the other $-C(X)$.

Thus we have the following set of necessary and sufficient conditions that there exists a compact X such that a given Banach space B is equivalent to a linear subspace of $B(X)$ completely regular over X:

(A₁)
(1) All the functionals F_T are linear.
(2) They form a closed subset M of E_W^*.
(3) M is the union of 2 disjoint antipodal closed subsets Ω and $-\Omega$.

Next, to study the case of a non-compact X, it is natural to attempt to compactify X so as to reduce the problem to the compact case. Let B be a closed linear subspace of $B(X)$ satisfying (II). We define a *compactification of X with respect to B* to be a compact space $\overline{X}$ in which X is dense and such that all the functions in B are extendable so as to be continuous over $\overline{X}$ and form a set which is separating over $\overline{X}$. Then there is a unique compactification of X with respect to B; it is simply the closure of $C(X)$ in E_W^*. Unfortunately, if B is completely regular over X, it will not in general be completely regular over $\overline{X}$ (see §5 for an example), so that we cannot completely reduce the problem to the compact case. Of course, if

B is the whole of $B(X)$, $\bar{X}$ is the Tychonoff-Čech-Stone [17; 5; 16] compactification of X and B is c.r. over $\bar{X}$.

Let B be a closed linear subspace of $B(X)$ completely regular over X. The sets $T_x = \{b \in B \mid b(x) = \|b\|\}$ and the sets $-T_x$ are still T-sets, but there may be "free" T-sets not of such form. However, considering B as a subspace of $B(\bar{X})$, each T-set is of the form $T_{\bar{x}}$ or $-T_{\bar{x}}$ for some $\bar{x} \in \bar{X}$. For each $x \in X$, $b(x)$ can be evaluated by $\inf_{t \in T_x} (\|b+t\| - \|t\|)$, so that the functionals F_{T_x} and $-F_{T_x}$ are linear. The set $M \subset S^*_W$ of all those functionals F_T which are linear is not in general closed in E^*_W; but it is the union of two disjoint closed antipodal subsets Ω and $-\Omega$, which respectively contain as dense subsets $C(X)$ and $-C(X)$. Ω is homeomorphic to a dense subset of $\bar{X}$, which is the whole of $\bar{X}$ if B is completely regular over $\bar{X}$.

Conversely, let B be any Banach space. If the set $M \subset S^*_W$ of those functionals F_T which are linear is large enough to have the property that, for each $b \in B$, $\|b\| = \sup_{f \in M} |f(b)|$, and if M is the union of two disjoint closed antipodal subsets Ω and $-\Omega$, then B is equivalent to a closed linear subspace of $B(\Omega)$ completely regular over Ω.

Thus a given Banach space B is equivalent for some completely regular X to a closed linear subspace of $B(X)$ completely regular over X if and only if the set $M \subset S^*_W$ of those functionals F_T which are linear satisfies the following conditions:

(1) for each $b \in B$, $\|b\| = \sup_{f \in M} |f(b)|$.
(A_2) (2) M is the union of two disjoint antipodal closed (in M) subsets.

4. **The whole space** $B(X)$. Necessary and sufficient conditions that B be equivalent to the *whole* of $B(X)$ for some compact X can also be phrased in terms of the functionals F_T [14]. They are as follows.

(1) All F_T are linear.
(A_3) (2) There exists a unit element $e \in B$; that is, an element e of norm 1 such that $|F_T(e)| = 1$ for all F_T.
(3) For each $b_1 \in B$, there exists a $b_2 \in B$ such that $F_T(b_2) = |F_T(b_1)|$ for all F_T for which $F_T(e) = 1$.

The sufficiency of these conditions is based on Kakutani's [10] analogue for Banach lattices of the Stone-Weierstrass approximation theorem [16]; namely, if a closed linear sublattice of the lattice $M(X)$ of all continuous functions on a compact X contains the constant functions and is separating over X, then it is the whole of $M(X)$. Here the set of F_T such that $F_T(e) = 1$, with the weak-* topology, serves as X.

Condition (1) can be replaced by

(1)′ The linear extension of each T-set in B is the whole of B. Condition (1)′ is stronger than (1) for the general Banach space B. In fact, (1)′ does not usually hold even if B is equivalent to a subspace of $B(X)$ completely regular over X, although, as we have seen, (1) does hold.

Condition (2) can be replaced by

(2)′ There exists an $e \in B$ such that for each $b \in B$ either $\|b+e\| = \|b\|+1$ or $\|b-e\| = \|b\|+1$.

Condition (3) can be replaced by

(3)′ For each b_1, $b_2 \in B$ there exists a $b_3 \in B$ such that $F_T(b_3) = F_T(b_1) \cdot F_T(b_2)$ for all F_T such that $F_T(e) = 1$.

Obviously (3)′ enables us to use the Stone-Weierstrass approximation theorem.

Clarkson's solution [6] of the problem of characterizing $B(X)$ consists essentially of condition (2)′, together with the demand that the half cone in B with vertex e and directrix E has the property that the intersection of any two of its translates is itself a translate. This latter condition is sufficient to define a lattice structure on B and to show that it is an (M)-space in the sense of Kakutani.

The solutions given by Arens and Kelley [2] to the same problem do not appeal directly to any lattice theorem to guarantee that B is the *whole* of $B(X)$. One of their sets of conditions consists essentially of (1)′ and (2)′ plus a condition which insures that there exists a $b \in B$ which "separates" any given pair of disjoint closed sets in X; this turns out to guarantee that B is the whole of $B(X)$. Their other set of conditions is stated in terms of geometrical properties of E_W^*, and involves a proof that the set $C(X)$ in $E_W^*(X)$ is simply the set of extreme points in $E_W^*(X)$ (see also [13]).

5. **Examples.** Here are a few examples to show the various possibilities when B is completely regular over a non-compact X, and the more elegant results when B is completely regular over a compact X.

Let X be the open interval $\pi/4 < x < 3\pi/4$ of the x-axis, and let B be the two-dimensional Banach space consisting of all the functions $b(x) = c \sin x + d \sin 2x$ as c, d range over the reals, with norm defined as $\sup_{x \in X} |b(x)|$. The compactification of X with respect to B is the closed interval $\overline{X} = [\pi/4 \leqq x \leqq 3\pi/4]$. B is completely regular over X, but not over $\overline{X}$; for every $b \in B$ such that $b(\pi/4) = \|b\|$ has the property $b(3\pi/4) = -\|b\|$. The sets $\{b \in B \mid b(\pi/4) = \|b\|\}$ and $\{b \in B \mid b(\pi/4) = -\|b\|\}$ are T-sets, but the corresponding functionals F_T are not linear. The set M of those functionals F_T which are linear forms a subset of S_W^* consisting of two disjoint homeomorphic images of X whose weak-* closures are disjoint sets in S_W^* homeomorphic to $\bar{X}$.

Now let X be the interval $2^{1/2}-1<x\leqq 1$, and let B be the set of functions $b(x)=cx^2+dx$ with $\|b\|=\sup_{x\in X}|b(x)|$. Here $\bar{X}=[2^{1/2}-1\leqq x\leqq 1]$. $b(2^{1/2}-1)=\|b\|$ if and only if $d=-2c(2^{1/2}-1)$, $c<0$, in which case $b(1)=-\|b\|$. B is completely regular over X but not over $\bar{X}$. However, whenever $d<-2c(2^{1/2}-1)$ and $c<0$, we have $b(1)=-\|b\|$ and $-\|b\|<b(2^{1/2}-1)<\|b\|$. Hence $\{b\in B \mid b(2^{1/2}-1)=\|b\|\}$ is not a T-set, and the functionals F_T are all linear; as in the previous example they form a subset M of S_W^* consisting of two disjoint homeomorphs of X, whose weak closures are disjoint sets in S_W^* homeomorphic to $\bar{X}$.

If $X=[0<x<1]$, let B be the set of all continuous functions on X such that $\lim_{x\to 0} b(x)=\lim_{x\to 1} b(X)=0$. Here $\bar{X}$ is homeomorphic to a circle. B is completely regular over X, but not over $\bar{X}$. The functionals F_T are all linear and form a subset of S_W^* whose closure in E_W^* is obtained by adjoining the origin in B_W^*.

The set of functions ax^2+bx+c is completely regular over $X=[0<x<1]$, and also over $\bar{X}=[0\leqq x\leqq 1]$. The functionals F_T are all linear, and form a subset of S_W^* closed in E_W^* consisting of two disjoint subsets each homeomorphic to $\bar{X}$.

6. Some classes of Banach spaces. Let

β = class of all Banach spaces,
β_1 = class of Banach spaces satisfying (I) with some compact X,
β_2 = class of Banach spaces satisfying (A_2),
β_3 = class of Banach spaces satisfying (A_1),
β_4 = class of Banach spaces satisfying (A_3).

Then we have seen that $\beta=\beta_1\supset\beta_2\supset\beta_3\supset\beta_4$. These are all proper inclusions.

Another class is that of all Banach spaces which have the property that all F_T are linear. This class properly contains β_3. In addition, it contains all euclidean spaces and all Hilbert (inner product) spaces. The latter spaces are strictly convex, and for strictly convex spaces it can be shown that the linearity of all F_T is equivalent to the existence of a Gateaux differential for the norm at every point of the space.

7. The uniqueness theorem and relations between X and $B(X)$. If B is completely regular over a compact X, we have seen that the set $M\subset S_W^*$ of functionals F_T is the union of two disjoint closed antipodal subsets each homeomorphic to X. If there exist two different decompositions of M into a pair of disjoint closed antipodal subsets, say Ω, $-\Omega$ and Ω', $-\Omega'$, it is easy to show that Ω and Ω' are homeomorphic. Hence a given Banach space can be completely regular over at most one compact space X [14]. We say that B *determines the topology* of X. As a special case, if $B=B(X_1)=B(X_2)$ for compact X_1, X_2, then X_2 is homeomorphic to X_1 (Banach-Stone theorem) [3; 16]; "compact" can be replaced by "completely regular and 1st countable" [7].

Thus the problem naturally arises of translating properties of X into properties of $B(X)$ or of subspaces of $B(X)$ completely regular over X, and conversely. In this direction, the following theorems can be proved.

If X is completely regular, $B(X)$ is separable if and only if X is compact and metrizable [12].

If X is completely regular, $B(X)$ contains a separable linear subspace completely regular over X if and only if X is separable metric [15].

If X is completely regular, $B(X)$ is reflexive if and only if X consists of a finite number of points [15].

If X is completely regular, $B(X)$ is n-dimensional if and only if X consists of n points (n finite).

X is connected if and only if there does not exist a decomposition of $B(X)$ as a direct sum [7].

A separable metric space X is finite-dimensional if and only if there exists a finite-dimensional closed linear subspace of $B(X)$ which is completely regular over X [15]. If n is the smallest dimension of a euclidean space in which X is homeomorphically imbeddable, there is a closed linear $(n+2)$-dimensional subspace of $B(X)$ containing the constant functions and completely regular over X.

This last result indicates that topological properties of a compact finite-dimensional metric space should be translatable into purely metric properties of a finite-dimensional Banach space. For example, let X be a closed subset (not on a circle or line) of the unit disc in the (x, y)-plane. Let B_X be the set of functions $b=a_1(x^2+y^2)+a_2x+a_3y+a_4$ on X, with $\|b\|=\sup_{x\in X}|b|$. Then B_X is a closed linear 4-dimensional subspace of $B(X)$ completely regular over X. As X ranges over the subsets of the disc, B_X remains unchanged both algebraically and topologically, only its norm varies.

Another result is the following,

A compact space X is a Peano space (compact, locally connected,

connected metric space) if and only if $B(X)$ is equivalent to a linear subspace B of $B(I)$ (I = closed interval) with the property that every T-set in $B(I)$ intersects B in a T-set in B.

This is a consequence of the following theorem.

If X_1 and X_2 are compact spaces, there exists a continuous mapping of X_1 *onto* X_2 if and only if $B(X_2)$ is equivalent to a linear subspace B of $B(X_1)$ with the property that every T-set in $B(X_1)$ intersects B in a T-set in B.

We prove this as follows. Let $f(X_1)=X_2$, where f is continuous. The mapping F of $B(X_2)$ into $B(X_1)$ given by $b_2 \to b_2 f$ for all $b_2 \in B(X_2)$ is an equivalence between $B(X_2)$ and a closed linear subspace B of $B(X_1)$. A T-set in $B(X_1)$ is either of the form $\{b_1 \in B(X_1) \mid b_1(x_1) = \|b_1\|\}$ or $\{b_1 \in B(X_1) \mid b_1(x_1) = -\|b_1\|\}$. The intersection of such a T-set with B is either $\{b_2 f \in B(X_1) \mid b_2 f(x_1) = \|b_2\|\}$ or $\{b_2 f \in B(X_1) \mid b_2 f(x_1) = -\|b_2\|\}$. These are T-sets in B because they are the images under F of the sets $\{b_2 \in B(X_2) \mid b_2(f(x_1)) = \|b_2\|\}$ or $\{b_2 \in B(X_2) \mid b_2(f(x_1)) = -\|b_2\|\}$ which are T-sets in $B(X_2)$. Conversely, let F be an equivalence between $B(X_2)$ and a linear subspace B of $B(X_1)$ with the property that every T-set in $B(X_1)$ intersects B in a T-set in B. The mapping H which takes each T-set in $B(X_1)$ into its intersection with B is a mapping *onto* the set of T-sets in B since each T-set in B (by Zorn's Lemma) is extendable to one in $B(X_1)$. Let $e=F(e_2)$, where $e_2 \in B(X_2)$ is the function identically 1 over X_2. Let Ω be the set of functionals $F_T \in B^*$ corresponding to the set K of T-sets in B which contain e, and let Ω_1 be the set of functionals F_T in $B^*(X_1)$ corresponding to the set of T-sets $H^{-1}(K)$. Then H induces a mapping $h(\Omega_1)=\Omega$. If T_1 is a T-set in $B(X_1)$ intersecting B in T, $F_{T_1}(b)=F_T(b)=\|b\|$ for all $b \in T$; since F_{T_1} and F_T are both linear functionals over B of norm 1, they are identical over B. From the definition of the weak-* topology, it follows that h is continuous. But Ω is homeomorphic to X_2, and Ω_1 is homeomorphic to X_1. Thus h induces a continuous mapping of X_1 onto X_2.

8. **A theorem on extension of linear functionals.** We give now a proof of the following result, a special case of which was used in §3.

Let L be a linear substance of a Banach space B, and let α be a linear functional over L of norm A. Then there is a unique norm-preserving linear extension f of α over B if and only if the functional $F(b)=\inf_{l \in L} [A\|b+l\| - \alpha(l)]$ is linear, in which case $f=F$ over B.

PROOF. Over L, $\alpha=F$. For the expression $A\|b+l\|-\alpha(l)$ takes on the value $\alpha(b)$ when $l=-b$, and for a fixed $b \in L$ this is its minimum value since $A\|b+l\|-\alpha(l) \geqq \alpha(b+l)-\alpha(l)=\alpha(b)$.

Now

$$\begin{aligned} -F(-b) &= -\inf_{l\in L}\,[A\|-b+l\| - \alpha(l)] \\ &= -\inf_{l\in L}\,[A\|-b-l\| + \alpha(l)] \\ &= \sup_{l\in L}\,[-A\|-b-l\| - \alpha(l)]. \end{aligned}$$

Referring to Banach's proof [2] of the Hahn-Banach theorem, we see that if b is outside L every linear extension f of α of norm A must have the property $-F(-b)\leqq f(b)\leqq F(b)$, and also for each r satisfying $-F(-b)\leqq r\leqq F(b)$ there is a linear extension f of α with norm A and with $f(b)=r$. It follows that if F is linear, f is unique and equals F, and conversely if f is unique, $f=F$.

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UNIVERSITY OF MICHIGAN

THE FEBRUARY MEETING IN EAST LANSING

The four hundred fifty-fourth meeting of the American Mathematical Society was held at Michigan State College on Friday and Saturday, February 24–25, 1950. There was a joint session with the Industrial Mathematics Society on Saturday morning.

The total attendance was about 135, including the following 75 members of the Society:

J. H. Bell, Gerald Berman, R. A. Beth, H. L. Black, M. I. Blyth, Richard Brauer, R. W. Bryant, R. E. Carr, Joshua Chover, R. V. Churchill, Harvey Cohn, A. H. Copeland, Max Coral, M. L. Curtis, D. A. Darling, Ben Dushnik, Benjamin Epstein, J. S. Frame, Cornelius Groenewoud, V. G. Grove, H. A. Hanson, Frank Harary, Gerald Harrison, H. L. Harter, G. E. Hay, Fritz Herzog, T. H. Hildebrandt, J. D. Hill, W. E. Jenner, Leo Katz, L. M. Kelly, D. E. Kibbey, W. M. Kincaid, K. J. Klein, Leo Lapidus, H. D. Larsen, J. H. Levin, D. J. Lewis, A. J. Lohwater, O. T. McMillan, Saunders MacLane, A. M. Mark, Imanuel Marx, D. C. Morrow, A. L. Nelson, W. J. Nemerever, E. A. Nordhaus, J. I. Northam, J. D. Novak, R. S. Pate, M. H. Payne, Sam Perlis, George Piranian, J. E. Powell, R. P. Reid, R. A. Roberts, E. H. Rothe, J. C. Rothe, Hans Samelson, W. C. Sangren, M. F. Smiley, W. F. Smith, T. H. Southard, G. L. Spencer, M. D. Springer, R. G. Stanton, B. M. Stewart, P. C. Sweetland, R. M. Thrall, Leonard Tornheim, E. F. Trombley, C. P. Wells, G. S. Young, J. W. T. Youngs, J. W. Zimmer.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, three addresses were delivered on Friday afternoon at 2:30 P.M. The speakers were Professor J. S. Frame, Michigan State College; Professor Sam Perlis, Purdue University; and Professor M. F. Smiley, State University of Iowa. The titles to the addresses were, respectively: *Induced monomial representations of a finite group*; *The holomorph of an algebra*; and *Some questions concerning alternative rings*. The presiding officer was Professor Saunders MacLane of the University of Chicago, who led the discussion period at the completion of each of the half hour addresses.

Sections for contributed paper were held at 11 A.M. Friday, with Professor H. D. Larsen presiding, and at 10:30 A.M. Saturday, with Professor T. H. Southard in charge.

On Friday evening there was a dinner for the Society in the Union Building of Michigan State College. Professor T. H. Osgood, of Michigan State College, welcomed the Society to the campus and Professor H. F. Smiley responded on behalf of the Society.

Abstracts of all papers presented at the Meeting are given below. Papers given by title are indicated by the letter "*t*." Mr. Nims was introduced by Professor A. W. Jacobson, Dr. Elliott and Mr. Johnson by Professor Benjamin Epstein.

ALGEBRA AND THEORY OF NUMBERS

218. J. H. Bell: *A note on the solution of unilateral matrix equations.*

M. H. Ingraham has given (Bull. Amer. Math. Soc. vol. 47 (1941) pp. 68–70) an algorithm for the solution of the unilateral matrix equation when the coefficient matrices are square and have elements in a field of characteristic zero. In this note it is shown that the algorithm is applicable when the coefficient matrices are $m\times n$ matrices and $m\neq n$. If $m>n$, the problem reduces to one of solving a unilateral matrix equation with square coefficients of order n. In this case a corollary obtained by W. E. Roth (Trans. Amer. Math. Soc. vol. 32 (1926) p. 67) is immediately obtained. If $m<n$, it is shown that the equation has either no solutions or an infinite number of solutions. In particular, it is shown that if $m<n$ and a certain equation, with square coefficient matrices of order m, has a solution, then the original equation has an infinite number of solutions. (Received January 9, 1950.)

219*t*. R. C. Buck: *Factoring of homomorphisms.*

Let A, B, and C be algebras having an addition $+$, and a zero 0, and with B simple. Let $\phi_1, \phi_2, \cdots, \phi_n$ be homomorphisms of A onto B, let ψ be a homomorphism of A into C, let $K_i=\text{kern}\ (\phi_i)$, and $K=\text{kern}\ (\psi)$. Then, if $\bigcap_1^n K_i\subset K$, there exist homomorphisms θ_i of B into C such that $\psi=\sum_1^n \theta_i\phi_i$. As an application, the Jacobson density theorem for irreducible rings of endomorphisms follows (Trans. Amer. Math. Soc. vol. 57 (1945) pp. 228–245). (Received December 8, 1949.)

220. Harvey Cohn: *Periodic algorithm for cubic forms.* Preliminary report.

The basis vectors x'_1, x'_2, x'_3 are called a reduced basis of a lattice in three-space if these vectors together with $x'_4=-\sum x'_i$ lie each in one octant of space or its negative octant. Then the tetrary cubic form $\prod \sum n_j x'_j$ is likewise called reduced. If this form is the norm in a cubic module, then there are only a finite number of reduced forms under change of basis, and they can be arranged into chains by a suitable definition of neighbors. There are three to six neighbors according to which one of 40 sets of inequalities prevails. The chains have the structure of a cube (8 forms) for the integers of the field with $D=49$, and are more complicated structures of 16 and 28 forms respectively for $D=81$ and $D=148$. The abelian or non-abelian character of these totally real cubic fields is reflected in symmetries in these chains. (Received January 6, 1950.)

221*t*. D. O. Ellis: *Autometrized Boolean algebras.* II. *The group of motions of B.*

Let B be a Boolean algebra with meet, join, and complement denoted by ab, $a+b$, and a', respectively. Define distance in B by $d(x, y)=xy'+x'y$. The paper is concerned with the group of motions of B (self-isometries of B) and it is shown that: 1. The group of motions of B is a subgroup of the group of complementation-preserving biuniform mappings of B onto itself and has only the identity in common with the group of automorphisms of B. 2. The group of motions of B is isomorphic to the additive group of the Boolean ring associated with B (cf. M. H. Stone, *Subsumption of Boolean algebras under the theory of rings*, Proc. Nat. Acad. Sci. U.S.A. vol. 20 (1934) pp. 197–202). The paper is part of a series of studies the first of which (David Ellis, *Automatrized Boolean algebras*) is to appear in the Canadian Journal of Mathematics. (Received January 9, 1950.)

222. B. M. Stewart: *Class equation for row-equivalent matrices over a Galois field.*

This note is concerned with the determination and the development of the properties of functions $Q(r, x)$ and $P(r, n, x)$ in terms of which it is possible to write the class equation for row-equivalent matrices of order n with elements in a Galois field of order $x=p^m$ as follows: $x^{n^2}=\sum Q(r, x)P^2(r, n, x)$, summed from $r=0$ to $r=n$. The function $P(n, x)=\sum P(r, n, x)$, which gives the total number of classes, is studied. Some of these functions are related to enumerative functions previously known for finite projective geometries, but their independent development from the standpoint of matric theory reveals in a natural way some of their recursive properties. For example, $Q(r, x)$ and $P(r, n, x)$ are shown to be interesting generalizations of the familiar factorial and combinatorial symbols, respectively. (Received January 11, 1950.)

ANALYSIS

223t. Evelyn Frank: *Convergence of C-fractions.*

This paper is concerned with parabolic and circular convergence regions of *C*-fractions $1+a_1z^{\alpha_1}/1+a_2z^{\alpha_2}/1+\cdots$ (cf. Bull. Amer. Math. Soc. Abstract 55-3-142). Corresponding value regions are found, as well as the types of functions to which the *C*-fractions converge. Special convergence regions are obtained for special *C*-fractions. (Received January 10, 1950.)

224t. W. S. Gustin: *Asymptotic behavior of mean value functions.*

The mean value $M(t)$ of order t of n positive real numbers x_ν weighted by n positive real numbers a_ν with $\sum a_\nu=1$ is, except for the three singular values $t=-\infty, 0, +\infty$, given by the formula: $M(t)=(\sum a_\nu x_\nu^t)^{t^{-1}}$. The exact behavior of $M(t)$ at its three singularities is obtained and used to give a simple proof of the result that both $M(t)$ and $\log M(t)$ as functions of t are convex near $t=-\infty$, and concave near $t=+\infty$ (see Shniad, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 770–776). (Received January 27, 1950.)

225t. Josephine M. Mitchell: *A theorem in the geometry of matrices.*

Let Z be a complex $m\times n$ matrix, $\bar{Z}'$ its transposed conjugate, I the identity matrix, $\sigma(Z)$ the trace of Z, and dZ its differential. It is known that in the domain defined by $I-Z\bar{Z}'>0$ the quadratic differential form $\sigma[(I-Z\bar{Z}')^{-1}dZ(I-\bar{Z}'Z)^{-1}d\bar{Z})]$ is invariant with respect to the conjunctive group of motions of signature (m, n) [Hua, Amer. J. Math. vol. 56 (1944)]. Using Hua's techniques we prove that $\sigma[(I-Z\bar{Z}')^{-1}dZ(I-\bar{Z}'Z)^{-1}d\bar{Z}']=\sum_{i,k=1}^{m}\sum_{j,l=1}^{n}(\partial^2 \log [\det (I-Z\bar{Z}')]^{-1}/\partial z_{ij}\partial\bar{z}_{kl})\cdot dz_{ij}d\bar{z}_{kl}$, and the latter invariant quadratic form may be compared with a quadratic form, obtained by Bergman in the case of two complex variables, which is invariant with respect to pseudo-conformal transformations [Bergman, Mémorial des Sciences Mathématiques vol. 106 (1947)]. Further it has been shown in the case of 2×2 matrices that the function $\pi^{-2n}n!(n+1)![\det (I-Z\bar{Z}')]^{-2-n}$ possesses the characterizing properties of the Bergman kernel function with respect to analytic functions of $2+n$ complex variables. (Received January 13, 1950.)

226t. A. E. Ross: *Markoff chains associated with transformations of the circle and some related stability problems.*

Let $\psi(\alpha, t)$ be a one-parameter family of transformations of the circle, and let $\psi^k(\alpha_0, t)$ be the smallest power (iterate) of $\psi(\alpha_0, t)$ with fixed points. If τ_1 is a fixed point of $\psi^k(\alpha_0, t)$ such that the difference $d(\alpha_0, t) = \psi^k(\alpha_0, t) - t$ changes sign at τ_1, then $\psi^k(\alpha, t)$ has fixed points for all values of α in a neighborhood of α_0. Write A_0 for the largest such neighborhood of α_0. If $d(\alpha_0, t) > 0$ (<0) for $t < \tau_1$, then we call τ_1 a stable (unstable) equilibrium point. Every closed chain $\tau_1, \cdots, \tau_k [\tau_i = \psi(\alpha\tau_{i-1}), \tau_1 = \psi(\alpha\tau_k)]$ of k stable fixed points is separated by a closed chain $\tau_1^*, \cdots, \tau_k^* [\tau_i^* = \psi(\alpha\tau_{i-1}^*), \tau_1^* = \psi(\alpha\tau_k^*)]$ of unstable fixed points. Assume next that the mechanism which carries out the transformation $\psi(\alpha_0, t)$ is not precise, but is subject to small errors Δ of certain probability density $p(\Delta)$ in the interval $\Delta' < \Delta < \Delta''$. If points τ_i do not lie too close to the points τ_i^*, then under quite general conditions all but a finite number of members of the Markoff chain (1) t_0, $t_1 = \psi(\alpha, t_0) + \Delta_0$, $t_2 = \psi(\alpha, t_1) + \Delta_1, \cdots$ are trapped in the suitably determined k neighborhoods (τ_i) of τ_i. This usually occurs when α is near the middle of the interval A_0. If, however, every τ_i has in its proximity a point τ_i^*, then there occur (intermittent) precessions of the members of the chain (1) either in the clockwise or in the counterclockwise direction. The precessions take place for the values of the parameter α near the ends of the interval A_0. There exist no similar stability properties for the non periodic $\psi(\alpha, t)$. (Received January 13, 1950.)

227. W. C. Sangren: *Expansions over finite intervals when coefficients are discontinuous.*

The solution of boundary value problems involving interfaces or surfaces of separation of two media calls for expansions of an arbitrary function $f(x)$ which generalize the ordinary Sturm-Liouville series. These expansions and necessary conditions imposed upon $f(x)$ are found by using the standard method of the Laplace transform. The expansions are also found formally by the classical method of separation of variables. The expansions in the case of arbitrary coefficients which are discontinuous at the interfaces is shown to be equi-continuous with corresponding expansions where the coefficients are step functions. (Received January 9, 1950.)

228t. W. C. Sangren: *Expansions over semi-infinite and infinite intervals when coefficients are step functions.*

In the case of boundary value problems involving coefficients which due to interfaces are step functions, it is necessary to have expansions of an arbitrary function $f(x)$ which generalize the ordinary Fourier integrals. The expansions and necessary conditions imposed upon $f(x)$ are found by using the method of the Laplace transform. The expansions are also found formally by using a method of separation of variables. (Received January 9, 1950.)

229t. I. E. Segal: *Decompositions of operator algebras.* I.

An algebra of operators on a Hilbert space can be decomposed relative to a Boolean algebra of invariant subspaces as a kind of direct integral, similar to the decomposition as a direct sum of algebras of linear transformations on finite-dimensional spaces. This decomposition results from a decomposition formula for the "states" of operator algebras. If the Boolean algebra is maximal, and with a certain separability restriction, the constituents in the integral are almost everywhere irreducible. It follows that in the case of a separable Hilbert space, a weakly closed self-adjoint algebra is a direct integral of factors. Any continuous unitary representation of a separable locally compact group G is a direct integral of irreducible such representations. If G is unimodular, then its two-sided regular representation is the direct integral of irreducible two-sided representations. Any measure on a compact metric space which is invariant under a group of homeomorphisms of the space is a direct integral of ergodic measures. Our basic results are closely related to theorems of von Neumann and Mautner, but there are important differences in both techniques and results which yield considerable simplification and facilitate the study of algebras derived from groups. (Received January 26, 1950.)

230*t*. J. E. Wilkins: *Neumann series of Bessel functions*. II.

Theorems on representation of a function $f(x)$, defined when $0 \leq x$, by a Neumann series of the form $\sum a_{n\nu} J_{\nu+2n+1}(x)$, in which $a_{n\nu} = \int_0^\infty t^{-1} f(t) J_{\nu+2n+1}(t) dt$, which were proved earlier (Trans. Amer. Math. Soc. vol. 64 (1948) pp. 359–385) for a certain class of functions $f(x)$, are here extended to the (best possible) class of functions $f(x)$ for which all the coefficients $a_{n\nu}$ exist. (Received January 11, 1950.)

APPLIED MATHEMATICS

231*t*. Nathaniel Coburn: *A method for constructing correlation tensors in isotropic turbulence*.

In this paper, a tensor method for constructing correlation tensors in isotropic turbulence is developed. The method depends upon: (1) a decomposition of the turbulence velocity vector, at any point P, along three mutually orthogonal unit vector fields (these fields are defined by parallelism); (2) a requirement that the correlation tensors be invariant under the orthogonal group acting on vector fields perpendicular to the line joining the two points involved in the correlations; (3) a decomposition of the metric tensor with respect to the three mutually orthogonal vector fields introduced in (1). The method is such that other types of symmetry than those encountered in the usual isotropic turbulence can be exploited. As an example, the correlations transverse to a specified direction (which can be interpreted as the direction of the mean flow) are studied. Here, the previous condition (2) is reformulated. It is shown that: (1) a two-dimensional analogue of the continuity equation for isotropic turbulence exists; (2) the third order correlation tensor vanishes; (3) the equations of motion lead to the non-steady heat equation in four dimensions. (Received December 7, 1949.)

232. P. T. Nims: *Design criteria for automatic controls*.

The differential equations of a linear automatic controls system may be studied by means of the Laplace transformation and useful information about the behavior of the control obtained without the necessity for finding the inverse transformation or even finding the poles of the transform. Gardiner in *Transients in linear systems* has shown how to compute the final value of the inverse by a limiting process, and if this method is combined with the integration operation, we may compute the net final area A_{10} enclosed between a step function disturbance and the response to that disturbance. The well known theorem on derivatives of a transform permits calculation of the weighted area A_{11}; that is, the integral of t times the difference between the step input and the response. Several examples are given of the use of these expressions for areas in the design of automatic control systems. (Received January 13, 1950.)

GEOMETRY

233. L. M. Kelly: *Distance sets.*

With each subset S of a distance space is associated a set of non-negative real numbers $d(S)$ called the distance set of S, a number being an element of $d(S)$ if and only if it is a distance in S. This mapping induces an inverse mapping $d^{-1}(N)$ where N is a set of non-negative numbers including zero. This paper is primarily concerned with this inverse mapping. Thus it establishes: (1) every set of non-negative numbers and zero is the distance set of some metric space; (2) every countable set of non-negative real numbers and zero is the distance set for some subset of Hilbert space; (3) every set of $n+1$ non-negative numbers including zero is a distance set of some subset of E_n. On the other hand, it is shown that: (4) there exist sets of non-negative real numbers including zero which are not distance sets for any separable metric space; (5) there exist countable such sets not "realizable" in any E_n; (6) there exist sets of $n+2$ such numbers not realizable as the distance set in E_n for all n. The work is inspired by, and overlaps to some extent, the study of S. Piccard (*Sur les ensembles de distances*, Mémoirs de l'Université de Neuchatel, 1940). (Received January 13, 1950.)

STATISTICS AND PROBABILITY

234. D. R. Elliot: *A location estimator for non-Gaussian distributions.*

An expression has been found for an estimator of the location parameter of a non-Gaussian distribution. The estimator g satisfies the conditions (1) $g(X_1, X_2, \cdots, X_n) = g(X_1+U, X_2+U, \cdots, X_n+U) - U$ and (2) variance of $E=$ a minimum, where $X_1, X_2, \cdots, X_n$ are the n sample variates ordered according to magnitude and not in general just any arbitrary ordering, and U is a parameter of location. Both the median and the mean satisfy equation (1) but not necessarily equation (2). g for a Gaussian distribution gives $g=\bar{X}$; g for a rectangular distribution gives $g=$ (smallest variate $+$largest variate)/2; more complicated expressions are obtained for the Cauchy and single "saw-tooth" distributions. For some distributions the use of the above estimator may be considerably more efficient than the average of the sample. (Received January 13, 1950.)

235. L. G. Johnson: *The median ranks of sample values in their population with an application to certain fatigue studies.*

Let ${}_0x_1, {}_0x_2, {}_0x_3, \cdots, {}_0x_n$ be a set of "n" ordered observations from a population whose cumulative distribution function is $F(x)$. The author determines in a nonparametric fashion the median of each of the values $F({}_0x_1)$, $F({}_0x_2)$, $F({}_0x_3), \cdots,$ $F({}_0x_n)$ for all possible samples of size "n" from the given population. The use of these medians is illustrated in a graphical method of estimating population parameters from ball bearing fatigue data. Because of the skewness of extreme values, it is pointed out that the mean of each of the quantities $F({}_0x_1)$, $F({}_0x_2)$, $F({}_0x_3), \cdots,$ $F({}_0x_n)$ is graphically misleading and hence not as useful as the median. (Received January 13, 1950.)

TOPOLOGY

236. E. W. Rothe: *A relation between the type numbers of a critical point and the index of the corresponding field of gradient vectors.*

Let $I(x)$ be a real-valued function of the point x of the real Euclidean n-space E^n which has continuous second derivatives in some neighborhood of the origin o of E^n. Let o be an isolated critical point of I, and m^r $(r=0, 1, \cdots, n)$ the corresponding Morse type numbers. The vector field $g(x) = \text{grad } I$ has then an isolated singularity at o. Let j be its index. The result of the paper is the relation $j = \sum_{r=0}^{n} (-1)^r m^r$ which is proved under the following assumption: there exists a neighborhood U of o such that for all $x \neq o$ of the intersection $U \cap \{x \mid I(x) = I(0)\}$ the vectors grad I and $x - o$ are not parallel. (Received January 11, 1950.)

237*t*. A. D. Wallace: *A theorem on endpoints.*

Let X be a compact Hausdorff space and X_0 a closed subset. We use the Alexander-Kolmogoroff cohomology theory as developed by Spanier (Ann. of Math. (1948)). A set A in X is p-connected (mod X_0) if the image of $H^p(A, A \cap X_0)$ in $H^p(B, B \cap X_0)$ is 0 for every compact subset B of A. A set is p-trivial (mod X_0) if each of its compact subsets is p-connected (mod X_0) (G. T. Whyburn). A point is a p-endpoint (mod X_0) if each of its nbhds contains a nbhd (of it) whose boundary is p-trivial (mod X_0). A set is totally p-disconnected (mod X_0) if each of its compact subsets which is p-connected (mod X_0) is also p-trivial (mod X_0). Theorem: If X is p-connected (mod X_0) and E is a set of its p-endpoints (mod X_0) then $X - E$ is p-connected (mod X_0) and E is totally p-disconnected (mod X_0). (Received January 3, 1950.)

238. G. S. Young: *Smooth interior transformations on 2-manifolds.*

Let $f: A \rightarrow B$ be an interior transformation of a compact 2-manifold A. If the set F of all points x of B such that some component of $f^{-1}(x)$ is not locally connected is an F_σ, then either f is light (and B is a 2-manifold), or B is a set obtained from a dendrite by a finite number of identifications. If F is empty, and f is not light, then B is an arc or a simple closed curve. Partial extensions of these to non-compact manifolds are given. (Received January 28, 1950.)

J. W. T. YOUNGS,
Associate Secretary

THE FEBRUARY MEETING IN NEW YORK

The four hundred fifty-fifth meeting of the American Mathematical Society was held at Columbia University on Saturday, February 25, 1950. The attendance was about one hundred forty, including the following one hundred thirty members of the Society:

M. I. Aissen, R. G. Archibald, P. T. Bateman, F. P. Beer, Stefan Bergman, Lipman Bers, Garrett Birkhoff, S. G. Bourne, A. B. Brown, F. H. Brownell, Jewell H. Bushey, P. A. Caris, P. G. Carlson, Alan Cathcart, Jeremiah Certaine, K. T. Chen, Y. W. Chen, W. L. Chow, Alonzo Church, A. B. Clarke, F. M. Clarke, J. A. Clarkson, L. W. Cohen, Philip Cooperman, Richard Courant, L. M. Court, H. B. Curry, C. W. Curtis, Martin Davis, Philip Davis, J. B. Diaz, C. H. Dowker, Y. N. Dowker, Aryeh Dvoretzky, Samuel Eilenberg, W. H. Fagerstrom, J. M. Feld, F. A. Ficken, Emanuel Fischer, A. D. Fleshler, J. B. Freier, Gerald Freilich, Bernard Friedman, K. O. Friedrichs, Orrin Frink, G. N. Garrison, R. E. Gilman, G. H. Gleissner, Sidney Glusman, W. H. Gottschalk, Emil Grosswald, V. H. Haefeli, Henry Helson, H. L. Herrick, Einar Hille, Banesh Hoffmann, T. R. Hollcroft, E. M. Hull, L. C. Hutchinson, Wenceslas Jardetzky, Bjarni Jónsson, Shizuo Kakutani, Edward Kasner, M. E. Kellar, J. R. Kline, Kunihiko Kodaira, Paul Koditschek, E. R. Kolchin, R. B. Leipnik, Benjamin Lepson, Howard Levi, M. A. Lipschutz, E. R. Lorch, Lee Lorch, L. A. MacColl, H. M. MacNeille, A. J. Maria, W. T. Martin, A. E. Meder, Jr., A. N. Milgram, K. S. Miller, W. H. Mills, Don Mittleman, D. J. Newman, P. B. Norman, I. L. Novak, A. F. O'Neill, J. C. Oxtoby, L. G. Peck, E. L. Post, Hans Rademacher, John Raleigh, K. G. Ramanathan, H. E. Rauch, M. S. Rees, K. W. Reidemeister, R. W. Rempfer, Moses Richardson, J. F. Ritt, M. S. Robertson, Robin Robinson, S. L. Robinson, C. A. Rogers, H. D. Ruderman, C. W. Saalfrank, Arthur Sard, A. H. Sarno, Samuel Schecter, Abraham Schwartz, Seymour Sherman, James Singer, P. A. Smith, Ernst Snapper, J. J. Sopka, E. J. Specht, C. F. Stephens, F. M. Stewart, R. C. Stewart, J. J. Stoker, D. L. Thomsen, C. A. Truesdell, Abraham Wald, H. V. Waldinger, J. L. Walsh, Alan Wayne, Alexander Weinstein, Louis Weisner, E. T. Welmers, P. M. Whitman, Albert Wilansky.

At 2:00 P.M. Professor C. L. Siegel of the Institute for Advanced Study gave an address on *Classes of analytic transformations*. President J. L. Walsh presided.

At 3:30 P.M. there were two sections for contributed papers, in which Professors Orrin Frink and R. E. Gilman presided.

Abstracts of the papers presented follow below. Abstracts whose numbers are followed by the letter "*t*" were presented by title.

ALGEBRA AND THEORY OF NUMBERS

239*t*. A. S. Amitsur and J. Levitzki: *Remarks on minimal identities for algebras.*

Let A_n denote the matric algebra of all n by n matrices over a field F, and $d(A_n)$ the degree of the minimal identities satisfied by A_n. Consider the standard identity $\sum \pm x_{i_1}x_{i_2} \cdots x_{i_{2n}}=0$, where $i_1, i_2, \cdots, i_{2n}$ is a permutation of $2n$ letters $1, 2, \cdots,$

$2n$, and the sign is + for even and − for odd permutations. It has been shown (J. Levitzki, *A theorem on polynomial identities*, Bull. Amer. Math. Soc. Abstract 55-5-233) that $d(A_n) \geqq 2n$. Later the present authors have obtained the following results (*Minimal identities for algebras*, to appear in Proceedings of the American Mathematical Society): (1) $d(A_n) = 2n$. (2) If either $n > 2$ for $F \neq P_2$ where P_2 is the prime field of characteristic 2, then each minimal identity of A_n is a linear combination of standard identities of degree $2n$. This does not hold for $n \leqq 2$ and $F = P_2$. (3) Conversely, in all cases the standard identity of degree $2n$ and all linear combinations of such identities are minimal identities for A_n. In the present note the authors show that above theorems hold also for semi-simple algebras over F. They further determine all minimal identities also in the exceptional case ($n \leqq 2$, $F = P_2$). Finally an upper and a lower bound for the degree of the minimal identities of an algebra with radical are given, and it is shown by examples that these estimates are in a certain respect the best possible ones. (Received December 30, 1949.)

240t. Bailey Brown: *An extension of the Jacobson radical.*

A part of N. Jacobson's theory of the radical of a ring (Amer. J. Math. vol. 67 (1945) pp. 300–320) is extended to an arbitrary cluster (R. A. Good, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 482–513). An element a of a cluster R is called *quasi-regular* if and only if a is in the right ideal generated by the set of all elements of the form $ax - x$, x in R. The *radical* N of R is defined to be the set of all elements a of R such that each element of the ideal generated by a is quasi-regular. If M is a right ideal in R, let M' denote the greatest ideal in R contained in M. A right ideal M is *modular* if and only if $ex - x$ is in M for some e and all x in R. Then $N = \bigcap M'$, where M ranges over all modular maximal right ideals in R. Call R *primitive* if and only if $M' = 0$ for some modular maximal right ideal M in R. Then a cluster $R \neq 0$ is isomorphic to a subdirect sum of primitive clusters if and only if $N = 0$. Under analogous definitions for *left-radical* N_1 and *left-primitive*, an example is given of an algebra, necessarily non-associative, for which $N \neq N_1$ and which is primitive but not left-primitive. (Received January 20, 1950.)

241. Emil Grosswald: *On the structure of some subgroups of the modular group.*

Let the square matrix with first row a, b and second row c, d, $ad - bc = 1$, be an element of the modular group Γ, and let p be any prime. The subgroups $\Gamma_0^0(p)$ of Γ are defined by the conditions $b \equiv c \equiv 0 \pmod{p}$ and the "principal subgroups" $\Gamma(p)$ by the conditions: $a \equiv d \equiv 1 \pmod{p}$ and $b \equiv c \equiv 0 \pmod{p}$. Using the methods of H. Rademacher (Abh. Math. Sem. Hamburgischen Univ. vol. 7 (1929) pp. 134–148) and results of K. Reidemeister (ibid. vol. 5 (1926) pp. 7–23) and O. Schreier (ibid. vol. 5 (1926) pp. 161–183) the structure of the subgroups $\Gamma_0^0(p)$ is completely determined. The number of independent generators, their defining relations, and the genus of the corresponding fundamental region are found as functions of p; also the explicit form of the independent generators as functions of the generators of Γ is found. By the same methods also the structure of the principal subgroups $\Gamma(p)$ is studied, but the results are less complete. (Received January 18, 1950.)

242t. Jakob Levitzki: *Prime ideals and the lower radical.*

It has been shown by McCoy (Amer. J. Math. vol. 71 (1949) pp. 823–833) that the intersection of all the prime ideals containing an ideal A of a ring R is an ideal B

such that B/A is a radical ideal of R/A. He raised the problem as to the connection of B/A to other radicals of R/A. In the present note this problem is solved by showing that B/A is actually the lower radical of R/A. McCoy's results combined with this theorem yield, for example, the following corollary: The intersection of all the prime ideals of a ring is equal to 0 if and only if the ring does not contain nilpotent ideals other than zero. Use is made of the concept of an m-sequence (+) $a_0, a_1, \cdots$, where $a_{n+1}=a_n b_n a_n$ with $b_n \in R$, which is a special type of McCoy's m-systems. This concept yields the following characterization of the lower radical L: For $a \in R$ we have $a \in L$ if and only if each m-sequence (+) with $a=a_0$ has at most a finite number of nonzero terms. (Received January 5, 1950.)

ANALYSIS

243. Michael Aissen: *A class of superadditive functions.*

Functionals are superadditive in the ordinary sense if their value for the sum of disjoint sets is not less than the sum of their values over the separate sets. They are superadditive S if for a finite number of sets E_i (not necessarily disjoint), included in a single set E, the value of the functional for E is not less than the Sylvester sum of the functional values for the sets E_i. A simple example shows that ordinary superadditivity does not imply superadditivity S. Admissible sets are defined as bounded open plane sets, whose components are simply connected. A generalized stress function of such an admissible set is defined and is shown to be superadditive S. The proof is by induction, with the case $n=1$ corresponding to the ordinary superadditivity of the generalized stress functions, which is well known. As special cases the St. Venant stress function and the torsional rigidity are superadditive S. (Received January 14, 1950.)

244. F. H. Brownell: *On the spectrum of the second order differential operator over n-space.* Preliminary report.

The continuity requirements on the potential function, $V(x)$, for the Schrödinger equation over all euclidean n-space, E_n, are replaced by a single weak integrability condition. This still allows the construction of a modified Green function, which defines a bounded linear operator on $L_2(E_n)$. By taking Fourier transforms the spectrum of this operator can be analyzed, the equation becoming the formal Fourier transform of the Schrodinger equation. This analysis generalizes the result of Putnam, Amer. J. Math. (1949) p. 109, from the half-line to E_n, repeats the result of Friedrichs, Math. Ann. (1934) Theorem 7, p. 705, now without continuity requirements, and finally removes the point spectrum from $\lambda > l$ if $|V(x)-l|$ has a very strong exponential bound at ∞, which seems new for E_n although much more is known for the half-line. (Received February 6, 1950.)

245t. N. J. Fine: *Proof of a theorem of Jacobi.*

A short, self-contained proof is given of the following theorem: *If* $G(z)$ *is defined in* $[-1, 1]$, *then* $\int_0^\pi G(\cos x) \cos nx dx = K_n^{-1} \int_0^\pi G^{(n)} (\cos x) \sin^{2n} x dx$, *where* $K_n = 1 \cdot 3 \cdot 5 \cdots (2n-1)$. (Received January 16, 1950.)

246t. N. J. Fine: *The generalized Walsh functions.*

In his dissertation [*On the Walsh functions*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 372–414] the author discussed various properties of the Walsh functions $\{\psi_n(x)\}$

and attempted to exhibit the close analogy between them and the exponentials $\{\exp 2\pi inx\}$. It is natural to ask whether the analogy can be extended to the system $\{\exp 2\pi iyx\}$, that is, whether the Walsh functions can be imbedded in a larger class $\{\psi_y(x)\}$ so as to preserve most of the properties of the exponential which are desirable and useful in analysis. This question is answered in the affirmative here, and again group-theoretic considerations are important. The generalized Walsh functions are defined (essentially) as the characters of the additive group of a certain topological field, and are proved to be the only nontrivial measurable solutions of a functional equation similar to $f(x+y)=f(x)f(y)$. It is shown that $\psi_y(x)$ is periodic if and only if y is a *dyadic* rational. Finally, the analogues of the Riemann-Lebesgue theorem, Fourier integral theorem, and Poisson formula are given. (Received January 16, 1950.)

247. Benjamin Lepson: *Series of analytic functions and polynomial Dirichlet series.*

In the first part of this paper, some results are obtained concerning sequences and series of analytic functions. One of these is the theorem that the set of boundary points of the region of convergence which belongs to this region forms a set of first category in the plane. An analog of Osgood's theorem is also obtained, which states that given a series of analytic functions which converges absolutely at every point of a domain, there is a subdomain for which the least upper bounds of the functions in that subdomain form a convergent series. These results are applied, in the second part, to "polynomial Dirichlet series," that is, series of the form $\sum_{n=0} P_n(s)e^{-\lambda_n s}$, where $P_n(s)$ is a polynomial of degree μ_n and $0<\lambda_0<\lambda_1< \cdots \rightarrow +\infty$. With the single assumption $\mu_n/\lambda_n \rightarrow 0$, properties of convergence and analyticity are found similar to those for Dirichlet series. Certain differences are observed, however, notably the fact that the Abel limit theorem may fail. These series occur naturally in the study of linear differential equations of infinite order. (Received January 10, 1950.)

248. Lee Lorch: *On the Lebesgue constants for* (*E*, 1) *summation of Fourier series.*

K. Prachar and L. Schmetterer have shown that the Lebesgue constants arising from the application of the Euler summation methods (E, 1) and (E, 2) to Fourier series diverge logarithmically (Anzeiger der Akademie der Wissenschaften in Wien. Mathematisch-Naturwissenschaftliche Klasse vol. 85 (1948)). For (*E*, 1) this was noticed by C. N. Moore, who discussed the effectiveness of Borel's method (which is equivalent to the Euler methods in the study of Fourier series), pointing out, among other things, that the Borel-Lebesgue constants are of logarithmic order (Proc. Nat. Acad. Sci. U.S.A. vol. 11 (1925)). The purpose of the author is to prove that the Lebesgue constants for (*E*, 1) have the same asymptotic representation as that obtained for the Borel-Lebesgue constants (L. Lorch, Duke Math. J. vol. 11 (1944); Bulletin of the Calcutta Mathematical Society vol. 37 (1945)). The method is essentially the one used in the Duke Journal reference. (Received January 3, 1950.)

249*t*. M. S. Robertson: *Applications of a lemma of Fejér to typically-real functions.*

If $f(z)=z+\sum_2^\infty a_n z^n$, $g(z)=z+\sum_2^\infty b_n z^n$ are regular and typically-real for $|z|<1$, then $F(z)=z+\sum_2^\infty a_n b_n z^n$ is regular for $|z|<1$ and typically-real for $|z|\leqq 2-3^{1/2}$, while $\int_0^z (F(z)/z)dz$ is typically-real for $|z|<1$. If, in addition to being typically-real, $f(z)$ and $g(z)$ are univalent and convex in the direction of the imaginary axis for $|z|<1$, then so is $F(z)$. If $f(z)$ is typically-real for $|z|<1$, a specially weighted average of the cosine of the angle subtended at the origin by the points $f(z), f(ze^{i\theta})$ is found to be non-negative for an explicit range of values of θ which depends on $|z|<1$. (Received January 16, 1950.)

APPLIED MATHEMATICS

250. C. A. Truesdell: *Bernoulli's theorem for viscous fluids.*

By a Bernoulli theorem is meant a statement that a certain finite expression one of whose terms is the specific kinetic energy $v^2/2$, a second contains one or more dynamical variables, and none involves an integral, remains constant or a function of time only upon each member of a certain family of curves. In any motion of a viscous compressible fluid in which the vorticity is steady such a finite expression is given explicitly, and a construction for the curves along which it is a function of time only is indicated. In the special case of an incompressible fluid subject to conservative extraneous force the finite expression reduces to that of the classical Bernoulli theorem, and the curves are defined in terms of the velocity field only, so that just as in classical hydrodynamics the Bernoulli theorem yields a formula for the pressure in terms of purely kinematic observables. The non-uniform character of the limit $\mu\rightarrow 0$ appears strikingly in the fact that these curves are independent of μ so long as $\mu\neq 0$, but in the limit $\mu=0$ they spread out into Lamb's Bernoullian surfaces. (Received January 9, 1950.)

GEOMETRY

251. L. M. Court: *Envelopes of plane curves.*

The envelope e of a plane family of curves touches each member c. Defining the envelope as the locus of intersections of "consecutive" members of the family, the author gives a purely geometric proof of this familiar proposition (the usual proof is analytic). Setting up a direction of advance along e and assuming, among other things, that the point P in which c and e meet is not a multiple point of c, he defines (locally) rear and forward ends for c with respect to P and e's direction of advance. He then shows that two members meet locally in *un*like ends, this property characterizing the envelope (as against other curves) vis-a-vis the family. From here on, it is a simple matter to show that c and e touch. (Received February 16, 1950.)

252. J. M. Feld: *On the geometry of lineal elements on a sphere, euclidean kinematics and elliptic geometry.*

Slides and turns of oriented lineal elements on the plane were first studied by Kasner (Amer. J. Math. vol. 33 (1911)). The author has for his object the investigation of the geometry of slides and turns of oriented lineal elements on a sphere. Slides and turns on a sphere generate the three-parameter group of spherical whirls W_3, which combined with the group of sphere rotations M_3 yield a six-parameter group of whirl-rotations G_6. The author determines invariants of lineal elements, spherical turbines, and flat fields of lineal elements under W_3, M_3, and G_6; also sets of fundamental differential invariants of series of lineal elements on a sphere. Finally he shows how the geometry of these groups of transformations on a sphere can be interpreted (1) as a model for the geometry of elliptic three-space and (2) as the kinematic geometry of a unit sphere sliding over a concentric unit sphere. These interpretations are analogous to those given by the author (Bull. Amer. Math. Soc. vol. 48 (1942); Amer. J. Math. vol. 70 (1948)) of the geometry of whirl-motions on the plane (1) as the geometry of quasi-elliptic space and (2) as the kinematic geometry of one plane sliding over another. (Received January 13, 1940.)

253. Edward Kasner: *Conformal symmetry and satellites.*

Conformal symmetry with respect to a curve C is also called Swarzian reflexion. If the curve C is algebraic of degree n, the symmetry is multivalued of degree n^2 (or lower). The image of C with respect to C consists of C and another algebraic curve S called the satellite. The satellite of a conic is another definite conic which sometimes degenerates. (The satellite of a cubic is usually of sixth degree.) If the conic is a circle, symmetry is merely inversion and no satellite exists. If the conic passes through one of the absolute points, symmetry is one-to-two. For the general conic, symmetry is one-to-four. Multivalued groups are considered. (Received January 13, 1950.)

254. Don Mittleman: *Areal-elements and areal-transformations.*

Let O be the pole of a polar coordinate system (ρ, θ), C a curve, and α a differentiable function defined for all points of C. At each point p of C there is defined a vector $\mathbf{r}$ whose initial point is p, positively directed toward O, and whose length r is defined by the equation $2d\alpha = (2\rho r - r^2)d\theta$. The point p on C and the value of the function α at p together define an *areal-element*. The ∞^1 areal-elements determined by C and α is called a *series*. The special series for which the terminal points of the vectors $\mathbf{r}$ is O is called a *union*. The necessary and sufficient condition that a series be a union is $\alpha(q) - \alpha(p) = \int_p^q 2^{-1}\rho^2 d\theta$ for all points p and q on C. The transformation (1) $P = P(\rho, \theta, \alpha)$, $\Theta = \Theta(\rho, \theta, \alpha)$, $A = A(\rho, \theta, \alpha)$ is called an *areal-transformation*. An areal-transformation which carries unions into unions is called an *area-transformation*. The necessary and sufficient conditions that the transformation (1) be an area-transformation are: $2A_\rho = P^2\Theta_\rho$ and $(2A_\theta + \rho^2 A_\alpha)\Theta_\rho = (2\Theta_\theta + \rho^2\Theta_\alpha)A_\rho$. (Received January 3, 1950.)

LOGIC AND FOUNDATIONS

255*t*. A. R. Schweitzer: *Selected topics from Sir A. S. Eddington's "Fundamental theory."* I.

The subjects selected are classified into two interdependent parts pertaining respectively to A. Mathematics, B. Cosmology. Under A are placed: I. Chirality (from χειρ, cheir, hand), II. Algebras: II_1. The quaternion calculus or ζ algebra, II_2. The "E" calculus or the square of the ζ algebra, II_3. The "EF" calculus or the fourth power of the ζ algebra, III. The vector calculus; the complete momentum vector; the wave vector, IV. The tensor calculus; the Riemann-Christoffel tensor; the energy tensor; the wave tensor calculus, V. The wave equation, VI. The electromagnetic field. (Received January 9, 1950.)

256*t*. A. R. Schweitzer: *Selected topics from Sir A. S. Eddington's "Fundamental theory."* II.

Continuing the preceding abstract, the author considers under B: I. Concept of physics. II. Space, time, space-time. III. Wave, particle, and wave-particle theories. IV. Energy, including radiant energy (radiation). V. Concepts of universe. VI. Special, intermediate and general relativity. VII. Fundamental theory as a combination of intermediate relativity and quantum theory. VIII. The constants of nature; the cosmical number. The author characterizes Eddington's *Fundamental theory* as an attempt to unify relativity and quantum theory from a mathematical point of view as distinguished from a unification envisaged by L. L. Whyte from the standpoint of logic (*Critique of physics*, New York, 1931) and a unification suggested by the philosophy of A. N. Whitehead (*Process and reality*, New York, 1929; see in particular, p. 364, last paragraph, and p. 365). (Received January 19, 1950.)

TOPOLOGY

257*t*. R. H. Bing: *Chained hereditarily indecomposable continua are homeomorphic.*

A continuum is hereditarily indecomposable if each of its subcontinua is indecomposable. Knaster gave an example of such a continuum and more recently such continua have been studied by Moise, Bing, and Anderson. It is shown that the compact nondegenerate hereditarily indecomposable continua M and M' are homeomorphic if for each positive number ϵ, each of M and M' can be covered by an ϵ-chain. In fact, if p and q are points of different composants of M while p' and q' are points of different composants of M', there is a homeomorphism carrying M into M', p into p', and q into q'. Most compact continua in a Euclidean n-space ($n>1$) E are pseudo-arcs; that is, if G is the space of bounded continua in E, the set of points of G which are topologically equivalent (in E) to M is a dense inner limiting (G_δ) subset of G. (Received January 13, 1950.)

258*t*. David Gale: *Separating algebras of continuous functions.*

It has been shown by Stone that if B is an abstract Boolean ring, then B has a perfect representation as the family of all open and closed subsets of a topological space S whose points are the maximal ideals of B. On the other hand starting with a perfect ring of subsets $B(X)$ on a space X and then going to the abstract ring B, the space S of maximal ideals will not in general be homeomorphic with X. It is shown here, however, that if $B(X)$ is considered as a set of continuous functions from X to the two element field, and given the compact-open topology, a perfect representation of the topological ring B may be constructed as continuous functions on a space S' whose elements are the *closed* maximal ideals of B. In this case the function ring $B(S')$ will be equivalent to the original ring $B(X)$. The analogue of the theorem is also proved for algebras $A(X)$ of continuous real-valued functions on a space X, provided that $A(X)$ is "separating," that is, contains enough functions to separate points. Use is made of the Stone-Weierstrass approximation theorem. (Received January 13, 1950.)

259*t*. S. T. Hu: *On generalizing the notion of fibre spaces to include the fibre bundles.*

The notion of fibre spaces is generalized by localizing the slicing function as follows. A topological space X is called a generalized fibre space over a topological space B relative to a map $\pi: X\rightarrow B$ of X onto B if, for each point $b\in B$, there exist an open set U of B containing b and a map ϕ_U: $U\times\pi^{-1}(U)\rightarrow X$ such that: (i) $\pi\phi_U(b, x)=b$ $(b\in U,\ x\in\pi^{-1}(U))$; (ii) $\phi_U(\pi(x), x)=x$ $(x\in\pi^{-1}(U))$. It is proved that every fibre bundle is a generalized fibre space. Nearly all familiar properties of fibre spaces are generalized to the generalized fibre spaces; in particular, the covering homotopy theorem is proved for the generalized fibre spaces in the usual form for fibre bundles. (Received January 11, 1950.)

T. R. HOLLCROFT,
Associate Secretary

R. G. D. RICHARDSON

1878–1949

Roland George Dwight Richardson served the American Mathematical Society for more than twenty-nine consecutive years: as vice-president, 1920; as the Society's third secretary, 1921–40; and as trustee, 1924–49. On 17 July 1949 he died, as the result of a coronary thrombosis, at South River Lake, Nova Scotia, his native province in Canada. He was born at Dartmouth on 14 May 1878, the elder son of George Josiah and Rebecca Archibald (Newcomb) Richardson. His mother was a direct descendant of the seventeenth century Simon Newcomb, from whom also sprang Simon Newcomb, the Nova Scotia-born astronomer, fourth president of the American Mathematical Society.

Entering Acadia College, Wolfville, N. S., in 1896 he graduated A.B. two years later. During 1895–96 and 1898–99 he was teacher of a school at the fishing village, Margaretsville, N. S., and then principal of the high school at Westport, N. S. from 1899 to 1902, when he started on his brilliant course at Yale University, A.B. 1903, A.M. 1904; instructor in mathematics 1904–07; Ph.D. 1906.

In 1907 Richardson accepted a call to Brown University as assistant professor of mathematics, with the understanding that he might spend the following year in study and research at the University of Göttingen, where Hilbert and Klein were still active and attracting many disciples. Thus in June, 1908, on the day after his marriage in Montreal to Louise Janet MacHattie, a former pupil at Westport, he set sail for Europe. In 1912 he was promoted to an associate professorship at Brown, and in 1915, to a professorship and the chairmanship of his department. Before indicating Richardson's contributions to the intellectual life of Brown University, we shall pause to survey his scientific research, and the American Mathematical Society when he was secretary.

Scientific Research. Richardson's first publications [1] and [3],[1] appeared shortly after his arrival at Brown University, and were developments of his doctoral dissertation at Yale, where Pierpont was the chief source of his mathematical inspiration. Connected with the theory of integration these papers supplemented and extended work done by Pierpont in his *Theory of functions of a real variable* and in his published papers. In consequence the point of view is essentially that of Riemann (content) rather than that of Lebesgue (measure). In [1] Richardson defines an improper integral on an n-dimensional set A having content after the manner of de la Vallée-Poussin in which the unbounded function $f(x)$ is approximated by the truncated functions $f(x, M_1, M_2) = f(x)$, M_1, or $-M_2$ according as $-M_2 \leqq f(x) \leqq M_1$, $M_1 \leqq f(x)$, or $f(x) \leqq -M_2$, so that $\int_A f = \lim_{M_1, M_2 \to \infty} \int_A f(x, M_1, M_2)dx$ whenever this limit exists, and he proves the equivalence of this integral to one defined by Pierpont.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

The first half of paper [3] is concerned with a sequence of functions f_n converging to f on A, and convergence of $\int_A f_n$ to $\int_A f$ is proved when f_n is a monotone sequence and the set of functions f_n is uniformly bounded on A. In the second half of the paper a multiple integral and corresponding iterated integrals are considered. If $x_1, \cdots, x_n$ are the coordinates of the space containing A and the positive integer p is $<n$, C is the section of A obtained by holding $x_1, \cdots, x_p$ fixed and varying $x_{p+1}, \cdots, x_n$; B is the set over which $x_1, \cdots, x_p$ ranges; thus A is the product set $B \times C$. All sets involved being assumed to have content, the principal result is that if $\int_A f$ exists, the iterated integral $\int_B \int_C f$ exists and has the same value. The integrand f may be unbounded, and improper integrals are defined as in [1]. The results obtained are about as far as one can go without the notions of measure and Lebesgue integral.

The contact which Richardson had with Hilbert at Göttingen exerted a deep influence on all of his subsequent mathematical research. As the major part of this work dealt with oscillation properties of solutions of ordinary linear second order differential equations, it is in order to comment briefly on the development of certain aspects of this subject that had occurred in the preceding decade, and to which Richardson's work was most closely related. First, Bôcher[2] had given rigorous and accessible form to the work of Sturm and Klein on the real solutions of such differential equations. Secondly, in the development of his theory of integral equations, and in the application of this theory to self-adjoint boundary problems, Hilbert[3] had pointed out the interrelations with variational principles. In particular, in 1906 Hilbert[4] had conjectured that for Sturm-Liouville boundary problems the oscillation properties of the proper functions (characteristic functions, or eigenfunctions) were consequences of the Jacobi condition imposed on these functions by the fact that they afforded a minimum for certain associated problems of isoperimetric type in the calculus of variations. This question had been considered by Robert König in his Göttingen dissertation, 1907, but König's treatment contained errors. Hilbert proposed to Richardson the reconsideration of this matter, and in [5] Richardson established the validity of Hilbert's conjecture for the particular boundary problem

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[2] *The theorem of oscillation of Sturm and Klein*, Bull. Amer. Math. Soc. vol. 4 (1897–98) pp. 295–313, 365–376; vol. 5 (1898–99) pp. 22–43.

[3] *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Nachr. Ges. Wiss. Göttingen, 1. Mitt. (1904) pp. 49–91; 2. Mitt. (1904) pp. 213–259; 3. Mitt. (1905) pp. 307–338; 4. Mitt. (1906) pp. 157–227; 5. Mitt. (1906) pp. 439–480; 6. Mitt. (1910) pp. 355–417.

[4] See Jber. Deutschen Math. Verein. vol. 16 (1907) p. 77.

$$\tag{1} \begin{gathered} \frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u + \lambda k(x)u = 0, \qquad 0 \leqq x \leqq 1, \\ u(0) = u(1) = 0, \end{gathered}$$

where $p(x)$, $q(x)$, $k(x)$ are real-valued, analytic, and $p(x)>0$, $q(x)\leqq 0$, $k(x)\not\equiv 0$ on $0\leqq x\leqq 1$.

Richardson next turned his attention to oscillation theorems for solutions of linear second order differential equations containing two or more parameters. His paper [6] on the saddle point in the theory of maxima and minima in the calculus of variations was prefatory to [7], in which the methods of [5] were extended to obtain certain oscillation theorems for the solutions of two equations involving two parameters. From the standpoint of a Klein oscillation theorem, however, the results of this paper were incomplete. A brief statement on how the full Klein oscillation theorem might be derived by the methods of the paper, together with a reference to his forthcoming paper [8], appeared in a remark added in proof at the end of the paper. In [8] Richardson treated the Klein oscillation theorem for the system

$$\tag{2} \begin{gathered} \frac{d}{dx}\left(p_1(x)\frac{du}{dx}\right) + q_1(x)u + [\lambda A_{11}(x) + A_{12}(x)]u = 0, \quad p_1(x) > 0, \\ \frac{d}{dy}\left(p_2(y)\frac{dv}{dy}\right) + q_2(y)v + [\lambda A_{21}(y) + A_{22}(y)]v = 0, \quad p_2(y) > 0, \\ u(a_1) = u(b_1) = 0, \qquad v(a_2) = v(b_2) = 0, \end{gathered}$$

where the coefficients are analytic, real-valued functions on the respective intervals $a_1\leqq x\leqq b_1$, $a_2\leqq y\leqq b_2$. In [9] the methods introduced in [8] were extended to the corresponding problem for three equations involving three parameters. In [8] and [9] Richardson was concerned with the study of both necessary and sufficient conditions for the validity of a Klein oscillation theorem; in particular, the results of these papers were valuable additions to earlier work on such problems by Bôcher,[5] Hilbert[6] and Yoshikawa.[7]

In connection with papers [8] and [9], two relevant comments seem worthy of mention. First, in contrast to that of [5] and [7], the discussion of these papers was quite independent of methods of the calculus of variations, although in the introduction to [8] he stated that the results of that paper were first obtained by such methods. This shift in his method of attacking oscillation theorems was due to a suggestion of G. D. Birkhoff, as acknowledged in a footnote on p. 32 of [8]. Secondly, in [8] the discussion of the oscillation theorem for a system of the form (1) involving the single parameter λ was extended to a case not treated previously, namely, the case of $q(x)$ positive in at least a part of the interval and $k(x)$ taking on both signs.

Richardson's work on oscillation theorems culminated in paper [12], a comprehensive investigation of the conditions to be imposed on $G(x, \lambda)$, regarded as a function of λ, so that one may determine definite oscillation theorems for the solutions of the differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + G(x, \lambda)y = 0, \qquad 0 \leqq x \leqq 1. \tag{3}$$

In this paper he established very general theorems for the existence of a unique solution of (3) vanishing at the end points of the interval and possessing a prescribed number of zeros, and also for the existence of two, and exactly two solutions of this nature. In addition, there were obtained general results on the oscillation of the proper functions of the boundary problem consisting of the differential equation (3) and arbitrary self-adjoint two-point boundary conditions.

Several years later, in paper [23], Richardson returned to the study of boundary problems of the form (1). The treatment therein given of the Jacobi condition for certain variational problems involving a finite number of isoperimetric conditions provided significant addenda to the results of [5], although the prime purpose of the paper was the study of properties enjoyed by the individual proper functions as extrema for variational problems involving an infinite number of isoperimetric conditions. The discussion of this latter problem was highly formal, and, in the words of the author, "omitting much in the way of justification of infinite processes." This deficiency was due to the author's attempt to carry over the machinery that sufficed for ordinary problems of the calculus of variations involving a finite number of isoperimetric conditions, instead of considering the problem in its proper setting as one in Hilbert space theory.

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[5] Loc. cit.

[6] See footnote 2, 6. Mitt.

[7] *Ein zweiparametriges Oscillationstheorem*, Nachr. Ges. Wiss. Göttingen (1910) pp. 586–594; *Dreiparametrige Randwertaufgaben*, ibid. pp. 563–585.

Paper [11] was the only one published by Richardson on partial differential equations. Using the classical method of approximating difference equations, he established in particular the existence of an infinite number of proper values for the first boundary problem over a square domain for a special type of self-adjoint linear second order partial differential equation of elliptic type. The paper also contained some partial results on the application of the method to a related boundary problem involving hyperbolic equations. A few years later the full value of the method of approximating difference equations was demonstrated by Courant and some of his associates.

The remainder of Richardson's research papers may be considered as secondary or auxiliary to his main interests. Paper [4], written jointly with W. A. Hurwitz at Göttingen in 1909, had points of contact with Richardson's paper [5]. In papers [17], [19], and [20] he presented an analytic discussion of certain algebraic properties of pairs of bilinear forms, and a related treatment of relative extrema for pairs of quadratic and hermitean forms.

In paper [14] Richardson gave a comprehensive and elegant review of Bôcher's *Leçons sur les méthodes de Sturm*. Although Richardson published no book on the subject, his mimeographed lecture notes on linear differential equations of the second order, listed as [24] of the bibliography, presented a well-formed introduction to the Sturmian theory and the significance of variational principles for the study of self-adjoint boundary problems.

Papers [11] and [12] were mainly the product of sabbatic leave spent at Harvard University during 1916–17. Scientific research was brought to an end in 1928, not only by demands made upon his time as secretary of the American Mathematical Society, but also by mounting obligations at Brown University, to which we shall presently refer.

The American Mathematical Society and its third secretary. The first period of the Society's history may be considered as terminating at about the time of its thirty-second birthday in December, 1920, when Cole withdrew from service as secretary. There were then 770 members; the total receipts for 1920 were $7167.24 and as yet there were no special funds, except that for Life Memberships, $4595.66.

By the time of Richardson's retirement from the secretaryship twenty years later enormous development of the Society's organization, resources, and activities had taken place. Some suggestions as to these may now be noted.

The membership had increased to 2314, and the total cash assets in December 1940 amounted to $189,226.65 in addition to productive assets from the sale of earlier publications. Besides the Life Membership fund, there were six other special funds (Bôcher, Frank Nelson Cole, Eliakim Hastings Moore, Marion Reilly, Ernest William Brown, Mathematical Reviews) and an Endowment Fund of $71,000. Reorganization of the Society's business with a New York office under the general direction of the secretary, incorporation, appointment of trustees to deal with financial matters, and life memberships based on actuarial principles, had been arranged long before. Greatly expanded activities including regular meetings of the Society in the East and Midwest, and on the Pacific Coast, necessitated the addition of four associate secretaries. In 1921–27 began Invited Addresses, Gibbs' Lectures, Visiting Lecturers, Reciprocity Memberships with Great Britain and Germany, and the Revolving Book Fund arrangement with the National Research Council. An elaborate Semi-Centennial Celebration was held in 1938. The Society collaborated in editorial representation on the Annals of Mathematics and the Duke Mathematical Journal, acquired editorial control of the American Journal of Mathematics, and established the new journal Mathematical Reviews (in part made possible by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society). Successful campaigns for an Endowment Fund, for Institutional and Contributing Memberships, for Patrons, and for special grants had made possible large increases in publication in spite of mounting costs.

For the attainment of the Society's 1940 status—financial (how remarkable the achievement of Coolidge, Veblen, and Ingraham!) and scientific—very many people were responsible. By my attendance at Council meetings throughout the whole of Richardson's term of office as secretary, and through constant personal contacts, I became convinced that his contributions and influences towards this status were indeed great. He had a thorough grasp of every topic which he presented, in excellent form, for the Council's consideration; he was endowed with infinite capacity for taking pains, and was always forward-looking, receptive to new ideas, while constantly mindful of the Society's finances. In personal relations he was remarkably kindly and cooperative, and inspired confidence and affection. No American mathematician was more widely known among his colleagues and the careers of scores of them were notably promoted by his time-consuming activities in their behalf. And such help was being freely rendered up to the time of his death.

Richardson was deeply appreciative of the Council's action not only in presenting to him a beautifully hand-illuminated copy of the resolution of appreciation adopted by the Council upon his retirement as secretary, together with a silver coffee set, and for Mrs. Richardson a handbag, but also in dedicating to him the 1941 volume of this Bulletin.

Achievements at Brown. Richardson arrived at Brown shortly after a "Graduate Department," with its own Dean, had been organized, and when only one member of the mathematics department was offering advanced courses. The number of such courses had been appreciably increased when he became chairman of the department. Soon he was a member of more than one important administrative committee including the Graduate Council. In 1926 he was appointed the successor to the Dean of the Graduate Department which became the Graduate School in the following year. He held this position until 1948, when he became listed among "Officers Emeriti." Both within and without the University the School became recognized as a really vitalizing intellectual force. This development, for which the Dean was almost wholly responsible, led to Brown's election in 1933 as an institutional member of the Association of American Universities.

Even though by 1928 Richardson's duties as dean and secretary, in addition to demands upon him as chairman of a department, had forced him to give up his own scientific research, he had through the years been laying sure foundations for the establishment of a strong department, for attracting not only candidates for the doctorate in mathematical analysis, but also those seeking post-doctoral inspiration. From 1929 until his retirement as chairman in 1942, 28 Ph.D. degrees in mathematics had been awarded at Brown.

In 1938 (after more than a decade of earlier consideration) when the American Mathematical Society was active in the establishment of Mathematical Reviews, Richardson was mainly instrumental in arranging not only that Brown should provide housing for the project, but also that its chief editor should be brought from Europe as a professor, adding strength to his department.

As early as 1930 the Council of the American Mathematical Society had given earnest consideration to the important problem of providing a means of publication for research memoirs in the field of applied mathematics. The seed thus sown in the secretary's thought, fertilized and greatly expanded by needs connected with World War II, led to a splendidly conceived pioneer undertaking. The following descriptive extract is taken from the Brown University Faculty Meeting records for October, 1949: "In the summer of 1941, through his [Richardson's] efforts and with financial aid from foundations and government, there was established at Brown a Program of Advanced Instruction and Research in Applied Mechanics. Able men, including many who had come to this country from abroad, were brought here to teach, and substantial numbers of students (at one time exceeding a hundred) were recruited. Under war conditions the original educational aim of the Program, to train a succession of able young doctors in applied mathematics, especially in the various fields of mechanics, was of necessity modified. The demand by government and industry for the services of young men trained even for the short space of a year provided sufficient evidence of the usefulness of this Program for war purposes. At the same time substantial amounts of investigation connected with the war were carried on by the staff. To supplement the Program, Brown University in 1943 founded the *Quarterly of Applied Mathematics*, a journal which renders international service in its field. In 1946 the original educational aim of this Program was restored and the Program reorganized under the name Graduate Division of Applied Mathematics." During 1943–46 the dean was also a member of the applied mathematics panel of the National Defense Research Committee.

Richardson was acting vice-president of Brown during the first semester of 1928–29, and, at the conclusion of his 41 years of service in the University, the degree of LL.D. was conferred upon him. A similar degree had been granted by Lehigh University in 1941, and in 1931, he received a D.C.L. from his alma mater, Acadia University. In 1919 he was elected a vice-president of the Mathematical Association of America, and in 1932–34 became a member of the Council of the American Academy of Arts and Sciences, and again in 1945–49 when he was a vice-president. He was a lecturer at the University of Chicago in 1918 and at the University of California, Berkeley, in 1920. Having become a naturalized citizen of the United States in May 1932, he was eligible for membership in the Division of Physical Sciences of the National Research Council, 1933–36. By election of the policy holders, he served as a trustee of the Teachers Insurance and Annuity Association, 1926–30, and 1931–35. Long a member of the First Baptist Church in Providence, from time to time he served on some of its most important committees.

The range of our friend's interests, apart from his recreations of golf and fishing, have thus been indicated, and his record of enduring worthy achievements is varied and rich. In connection with every project that he undertook he found no rest until the goal sought was won, and ever was his motto: "Whatsoever thy hand findeth to do, do it with thy might." He had insight and remarkable organizing abilities, associated with exceptional mental powers. His complete honesty of character, natural wisdom, and selfless devotion to high educational ideals, commanded the admiration of his associates. He was endowed with a spirit of unaffected friendliness, the desire to play a part in helping and encouraging others to achieve the best of which they were capable, and as a result he was blest with a host of friends. There was never-ending hospitality at his home. Indirectly Mrs. Richardson contributed in no small measure to the happy solution of many of her distinguished husband's problems.

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[8] It may be noted that this volume 9 of the *Transactions* also contains the doctoral dissertations of G. D. Birkhoff, A. Dresden, F. L. Griffin and R. L. Moore.

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26. *The present need of a constructive record of graduate schools*, Association of American Universities, Jn. Proc. and Addresses vol. 37 (1935) pp. 119–134.

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28. *The Ph.D. degree and mathematical research*, Amer. Math. Monthly vol. 43 (1936) pp. 199–215.

29. *New mathematical center*, Scripta Mathematica vol. 8 (1941) pp. 57–59.

30. *Advanced instruction and research in mathematics*, American Journal of Physics vol. 11 (1943) pp. 67–73.

31. *Applied mathematics and the present crisis*, Amer. Math. Monthly vol. 50 (1943) pp. 415–423.

32. *Mathematics and the war*, American Scholar vol. 12 (1943) pp. 503–505.

33. Brown University Bulletin, Editor of *Program of Advanced Instruction and Research in Mechanics*, vol. 40, no. 8, November 1943, 32 pp.; vol. 42, no. 4, April 1945, 31 pp.; vol. 42, no. 9, December 1945, 32 pp.

34. *Advanced instruction and research in mechanics at Brown University*, Science n.s. vol. 99 (1944) pp. 215–216.

35. *Administrative problems in graduate schools*, Bostonia, Boston University, vol. 17 (1944) pp. 39, 40, 64.

RAYMOND CLARE ARCHIBALD[9]

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[9] In the preceding Memorial the comments on Richardson's scientific research were made by specialists in the fields, namely: Professor T. H. Hildebrandt on nos. [1] and [3]; Professor W. T. Reid on nos. [4]–[9], [11], [12], [14], [19], [20], [23], [24].

BOOK REVIEWS

Foundations of economic analysis. By P. A. Samuelson. Cambridge, Harvard University Press, 1947. 11+447 pp. $7.50.

Professor Samuelson has used mathematics expertly to produce this unified treatment of economic theory. His *Foundations* provides a synthesis of statical economic theory, a survey of welfare economics and other important special topics, and an exploratory discussion of the problem of stability of equilibrium for dynamical economic systems.

The exposition is thoroughly mathematical. It is likely that most economists will find it very hard reading even though, as is noted in the Introduction: " . . . The pure mathematician will recognize all too readily the essentially elementary character of the tools used." Nevertheless, the book will undoubtedly lead to a greatly increased use of mathematics by economists in their future work. It sets a new high standard for the mathematical development of unified economic theory.

The pure mathematical aspects of the work are discussed separately in two appendices. The first appendix is concerned with the problem of maximum conditions for functions, and stresses maxima problems for quadratic forms treated from the matrix point-of-view. The second appendix is a sixty page self-contained treatment of the elements of the theory of systems of difference equations, and is included because of the essential use made of theorems in this field for the development and study of dynamical economic systems. Although the interest centers in systems of linear difference equations, there are some theorems concerning systems of differential equations and mention of mixed difference-differential equations.

The choice of material for the appendices was made on the basis of need for support of the main text. For this purpose, it was necessary to stress the topics that are not generally known to economists. As a result, the selection seems poorly balanced when gauged on the basis of general mathematical importance. I suspect also that most economists would like to have more in explanation of the elementary properties of matrices, and a brief discussion of the solutions of systems of linear difference equations in terms of ordinary trigonometric functions, even at the expense of additional pages.

Professor Samuelson set out to formulate a general theory of economics that would unify various particular fields with respect to analogous central features. He remarks: "Only after laborious work in each of these fields did the realization dawn upon me that essentially the same inequalities and theorems appeared again and again, and that I was simply proving the same theorems a wasteful number of times." He insists also that the theorems he proves be "operationally meaningful" in order that they can be treated as hypotheses about empirical data that could conceivably be refuted.

The unified treatment of the theory of consumer's behaviour (Chap. V) is a convincing sample of success with synthesis in operationally meaningful terms. It is the general notion that a consumer has an ordinal preference field $U=F[\phi(X)]$ where ϕ is some one cardinal index of utility such that preferences between combinations of goods X and Y are in accordance with $\phi(X)>\phi(Y)$, $\phi(X)=\phi(Y)$, or $\phi(X)<\phi(Y)$, where $\phi(X)>\phi(Y)$ implies that X is preferred to Y. (It is assumed that ϕ is continuous and differentiable and that $F'(\phi)>0$.) Now if total income is I and if the price of good i is p_i, the main problem is to derive the demand function $X_i(p_1, p_2, \cdots, p_n, I)$ for the quantity x_i of good i that the consumer would purchase, subject to the restriction $I=\sum p_i x_i$, if he wished to maximize U. It is shown that all restrictions on the demand function can be derived from the single condition that the form $K_{ij}=\delta x_i/\delta p_j+x_j\delta x_i/\delta I$ be symmetric and negative semi-definite. A constructive proof is sketched also to show that when this condition is satisfied there exists a $\phi(X)$ that satisfies the properties of a preference field. Professor Samuelson concludes that: "Despite its lofty beginnings, the pure theory of consumer's behavior, when its empirical meaning is finally distilled from it, turns out to be one simple hypothesis on price and quantity behavior." Here, then, is a concise hypothesis that provides the basis for an imposing theoretical economic structure that should soon be put to trial by some clever experimental economist.

The pure mathematician who wishes to sample an important segment of economic theory, written competently in his favorite language, will find the *Foundations* pleasant reading. He may well also be stimulated, especially by the discussion of dynamic economic theory in Part II, to extend some of the results reported by Professor Samuelson and this would constitute a well-deserved widening of the sphere of influence of this splendid book.

MERRILL M. FLOOD

Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability. Ed. by J. Neyman. Berkeley, University of California Press, 1947. 12+447 pp. $7.50.

The thirty papers which are collected in the volume cover such a wide range of subjects that a comprehensive review is impossible. The topics range from *Philosophical foundations of probability* (Reichenbach) to *Biological association of insects* (Holloway) and *Wheat-bunt field trials* (Baker and Briggs). Of particular interest to pure mathematicians are survey papers by Doob, *Time series and harmonic analysis*, and by Feller, *On the theory of stochastic processes, with particular reference to applications.*

Although the preponderance of papers is devoted to statistics (general, descriptive, and mathematical) mention should be made of a paper on *Statistical mechanics and its applications to physics* (Lenzen) and on *Statistical study of the galactic star system* (Trumpler).

There is even a paper on *The place of statistics in the university* (Hotelling) followed by a discussion by five other authors.

In spite of the somewhat chaotic arrangement and an overwhelming battery of topics this volume is a tribute to the great vitality of statistics and statistical methods. It should prove a valuable addition to the rapidly growing statistical literature.

M. KAC

An essay toward a unified theory of special functions. By C. Truesdell. (Annals of Mathematics Studies, no. 18.) Princeton University Press, 1948. 4+182 pp. $3.00.

It is very difficult to draw the line between mathematical physics and applied mathematics but this book shows that there does exist a definite and important difference between them. In mathematical physics many special functions such as the Legendre, Hermite, or Laguerre polynomials, Bessel or hypergeometric functions are used as tools to solve particular problems. As a consequence many properties of these functions and connections between them have been established. The author, as an applied mathematician, has posed the question of finding a unified approach to these different special functions so that from it most of the known properties could be found directly. The monograph under review gives an answer to this question.

The author found that many of the special functions, not only those previously mentioned but also such as the generalized Riemann zeta function, the incomplete gamma function, or the Poisson-Charlier polynomials, can be transformed into solutions of the equation

$$\frac{\partial F(z, \alpha)}{\partial z} = F(z, \alpha + 1). \tag{1}$$

A study of this equation shows that by simple techniques any solution of (1) can be represented as a power series, a contour integral, or as a differential operator. For example, from a formal use of Taylor's theorem it follows that

$$F(z + y, \alpha) = \sum_{0}^{\infty} y^{n} F(z, \alpha + n)/n!. \tag{2}$$

This formula gives directly a generating function for the sequence $F(z, \alpha+n)/n!$, $n=0, 1, 2, \cdots$, while for $z=0$ it gives a power series expansion of $F(y, \alpha)$.

As a result of these techniques the author is able to handle problems which could be treated only with difficulty by classical methods. For example, express the Laguerre polynomials in terms of the Legendre polynomials. The scope of the method and the ingenuity of the author are illustrated by the derivation of new, complicated formulas involving the special functions.

An essential point in the study of equation (1) is the proof of the following theorem: Given a bounded sequence $\phi(\alpha+n)$, $n=0, 1, 2, \cdots$, there exists one and only one solution of (1) such that

$$F(z_0, \alpha + n) = \phi(\alpha + n), \qquad n = 0, 1, 2, \cdots .$$

This theorem is obtained as a special case of an existence theorem for a general vector difference-differential equation which the author proves. Once the uniqueness is known, it is easy to justify the formal applications such as those in (2).

The author concludes his monograph with some still unsolved questions. One such question is this:

What are the conditions to ensure the existence of a unique solution of the equation

$$\frac{\partial F(z, \alpha)}{\partial z} = F(z, \alpha - 1) \tag{3}$$

when $\alpha=\alpha_0+n$, $n=0, 1, 2, \cdots$? If the answer to this were known, some important identities involving number-theoretic functions would be immediate consequences of (3).

BERNARD FRIEDMAN

Non-linear problems in mechanics of continua. (Proceedings of Symposia in Applied Mathematics, vol. 1.) New York, American Mathematical Society, 1949. 8+219 pp. $5.25.

This volume contains papers presented at the First Symposium in Applied Mathematics, held at Brown University, August 2–4, 1947.

The subject of the symposium was *Non-linear problems in mechanics of continua*. The content of the volume falls into two groups: the first (157 pages, sixteen papers) is concerned with fluid mechanics, the second (60 pages, eight papers) with elasticity and plasticity. The articles differ widely in both length and character.

The papers of the first group offer an instructive sample of the great variety of problems considered today in the above described direction. Several articles deal with the problems of so-called free boundaries. The first article in the book is a comprehensive and interesting paper by A. Weinstein entitled *Non-linear problems in the theory of fluid motion*. There is a brief abstract of work completed and in progress on recent developments in free boundary theory by G. Birkhoff. Non-linear free boundary problems arise likewise in the consideration of wave motion in water with a free surface. J. J. Stoker presents a study, *The breaking of waves in shallow water*, with emphasis on the mathematical analogy with shocks in gas dynamics. Shock problems form the subject matter of a few other articles. An extensive paper which reviews some less accessible material while also presenting new results and viewpoints has the title *On shock phenomena: interaction of shock waves in gases* written by H. Polachek and R. J. Seeger. A paper *Theory of the propagation of shock waves from cylindrical charges of explosives* is presented by S. R. Brinkley, Jr. and J. G. Kirkwood. The general problem of the integration of the equations of gas dynamics is studied in a number of papers under different aspects. S. Bergman presents a condensed review of his important and extensive work on *Operator methods in the theory of compressible fluid flow*. L. Bers, in an article *An existence theorem in two-dimensional gas dynamics* gives a complete existence theorem for the flow around a profile under the assumption of Chaplygin's well known linear approximation to the adiabatic relation. If this relation is approximated by a polygon rather than by a single straight line, a possibly efficient method for the numerical solution of flow problems is obtained in H. Poritsky's paper *Polygonal approximation methods in the hodograph plane*. The interesting article of L. Opatowski extends some results of Volterra and Levi-Civita on incompressible flow to compressible flow problems. A. H. Taub's paper on a variational principle for perfect compressible fluids may likewise be mentioned in this connection. A contribution to the integration theory in case of three-dimensional flow problems is offered in the article of N. Coburn and C. L. Dolph, *The method of characteristics in the three-dimensional stationary supersonic flow of a compressible gas*. Problems of boundary layer theory (incompressible flow) are considered in the papers by

L. Lees, *Stability of the laminar boundary layer in a compressible fluid*, and by R. Sears, *The boundary layer of yawed cylinders*. H. W. Emmons proposes to study turbulent flow by directly attacking the Navier-Stokes equations through a method of finite-difference-integration in a paper *The numerical solution of the turbulence problem*. A very interesting contribution to the statistical theory of turbulence, so much discussed today, is offered in C. C. Lin's article *Remark on the spectrum of turbulence*. M. Kuo comments briefly on the stability of a flow in the neighborhood of the sonic point ($M=1$).

It is of course impossible to do justice in a brief review to numerous articles, widely different in scope and character. What is termed "interesting" or "important" depends rather more on the reviewer's interests and training than on objective standards. In fact no evaluation can be attempted. Rather the aim is to assist the reader in forming some opinion as to whether the volume as a whole or one or more of the articles will be of interest to him.

The second group starts with F. D. Murnaghan's basic contribution *Foundations of the theory of elasticity* which studies the problem of non-infinitesimal displacements. A group of papers may be singled out that deal with large deflection of plates while small strains are still considered. There is E. Reissner's very competent paper *On finite deflections of circular plates*, S. Levy's paper *Large deflection theory for rectangular plates*, and the suggestive paper by K. Friedrichs on the problem of buckling, *The edge effect in bending and buckling with large deflections*. A non-linear strain-stress relation is assumed by W. Kaplan and dealt with by a numerical approach in the article *Numerical methods in the solution of problems of non-linear elasticity*. G. F. Carrier in the article *On dynamic structural stability* attempts to extend the stability theory of structures to the case of dynamic loads. There are only two papers on plasticity, one an abstract of W. Prager's stimulating paper on discontinuous solutions in plasticity theory, which appeared in full in the Courant Anniversary volume, and the other D. C. Drucker's paper *Stress-strain relations for strain hardening materials: discussion and proposed experiments* where the so-called "deformation theories" of plasticity versus "plastic flow" theories are examined, the latter appearing preferable.

On the whole, the volume offers an important cross section of certain aspects of work in applied mathematics.

HILDA GEIRINGER

NOTES

A grant of $10,000 has been made to the 1950 International Congress of Mathematicians by UNESCO to be used for transportation expenses of foreign mathematicians attending the Congress. A Conference which it is hoped will lead to the formation of an International Mathematical Union will be held just before the Congress. UNESCO has made an additional grant of $10,000 for transportation of delegates to this Conference.

The second summer seminar and the second quadrennial congress of the Canadian Mathematical Congress were held at the University of British Columbia from August 16 to September 10, 1949. The seminar consisted of four lecture series of eight lectures each and four expository courses of fifteen lectures each. In addition there were subseminars in pure and applied mathematics. The second congress included two lecture series of three lectures each, eight additional lectures, and a number of short research papers. A full report of these meetings will appear in the Proceedings of the Canadian Mathematical Congress of 1949.

The Institute for Teachers of Mathematics sponsored by the Association of Teachers of Mathematics in New England will be held at Tufts College August 22–29, 1950.

The Council of the Royal Society of Edinburgh has announced the award of the Keith Prize for the period 1947–1949 to Professor A. G. Walker of the University of Sheffield, for his papers on the general theory of relativity published in the Proceedings of the Society within the period and in recognition of his distinguished contributions to Riemannian geometry.

Professor J. E. Littlewood of Cambridge University is retiring and will be succeeded by Professor A. S. Bessicovitch.

Dr. Shmuel Agmon of the National Research Center in Paris has been appointed as visiting lecturer in mathematics at the Rice Institute.

Assistant Professor L. A. Aroian of Hunter College has accepted a position with Hughes Aircraft Company, Culver City, California.

Dr. Emilio Baiada of the University of Pisa is on leave of absence and has been appointed to a visiting professorship at the University of Cincinnati.

Associate Professor Joseph Barnett of Oklahoma Agricultural and Mechanical College has retired with the title associate professor emeritus.

Professor M. A. Biot of Brown University is now a member of the firm of consultants, Biot and Arnold, New York, New York.

Professor D. H. Blackwell of Howard University is on leave and has been appointed to a visiting professorship at Stanford University.

Mr. Joseph Blum of the Los Alamos Scientific Laboratories has accepted a position with the National Bureau of Standards, Washington, D. C.

Dr. R. P. Boas, Executive Editor of Mathematical Reviews, has been appointed to a professorship at Northwestern University.

Associate Professor J. W. Calkin of Rice Institute is on leave of absence at the Los Alamos Scientific Laboratories.

Dr. Ion Carstoiu of the University of Paris has been appointed to an assistant professorship at Johns Hopkins University.

Professor Lamberto Cesari of the University of Bologna is on leave and has been appointed to a visiting professorship at the University of Wisconsin.

Professor Patrick Du Val of the University of Istanbul, has been appointed to a professorship at the University of Georgia.

Mr. G. W. Evans of New York University has accepted a position as associate mathematician with the Argonne National Laboratory, Chicago, Illinois.

Mr. H. H. Goode of the Office of Naval Research has accepted a position as supervisor, Aeronautical Research Center, at the University of Michigan.

Professor M. R. Hestenes of the University of California at Los Angeles is on leave and has been appointed a visiting staff member at the National Bureau of Standards.

Mr. H. A. Hill has accepted a position as mathematician with the National Bureau of Standards, Washington, D. C.

Dr. S. T. Hu of the Academia Sinica has been appointed lecturer at Tulane University.

Associate Professor F. B. Jones of the University of Texas has been appointed to a professorship at the University of North Carolina.

Assistant Professor William Karush of the University of Chicago has accepted a position as mathematician with the Institute for Numerical Analysis, National Bureau of Standards.

Professor E. S. Kennedy of the American University of Beirut, Lebanon, is on leave and has accepted a Rockefeller Fellowship at the Institute for Advanced Study.

Dr. Jacob Korevaar of the Mathematical Center in Amsterdam has been appointed visiting lecturer at Purdue University.

Mr. R. M. Lakness has been appointed lecturer at the University of California.

Dr. Nathan Lazar of Teachers College, Columbia University, has been appointed to an associate professorship of education at Ohio State University.

Associate Professor Eugene Leimanis of the University of Riga, Latvia, has been appointed to an assistant professorship at the University of British Columbia.

Professor Emeritus W. R. Longley of Yale University has been appointed to a visiting professorship at Haverford College.

Professor J. N. Michie of Texas Technological College has retired.

Mr. J. L. Miller is now associated with the Mathematical Computing Service, Brooklyn, New York.

Mr. C. N. Mooers is now owner and director of research of the Zator Company in Boston.

Dr. G. K. Overholtzer of the University of Kansas has been appointed to an assistant professorship at Purdue University.

Professor F. W. Owens of Pennsylvania State College has retired with the title emeritus.

Assistant Professor Helen B. Owens of Pennsylvania State College has retired.

Mr. T. K. Pan has been appointed lecturer at the University of California.

Professor W. V. Parker of the University of Georgia has been appointed to a professorship at Alabama Polytechnic Institute.

Mr. G. W. Petrie of Bethlehem Steel Company has accepted a position with International Business Machines, Philadelphia, Pennsylvania.

Dr. H. E. Rauch of Rutgers University is now a member of the Institute for Advanced Study.

Dr. T. D. Reynolds of Duke University has accepted a position as Director of Instruction for schools in Buncombe County, North Carolina.

Assistant Professor Fritz Rothberger of Acadia University has been appointed to an assistant professorship at the University of New Brunswick.

Mrs. Mollie A. Rubenstein has accepted a position as statistician with Michigan State College.

Dr. W. C. Sangren of the University of Michigan has been appointed to an assistant professorship at Miami University.

Professor A. C. Schaeffer of Purdue University is on leave and has been appointed to a visiting professorship at Carnegie Institute of Technology.

Miss Esther Seiden has been appointed lecturer at the University of California.

Mr. D. J. Smith of the University of Notre Dame has accepted a position with the Royal-Liverpool Groups of Insurance Companies, New York, New York.

Assistant Professor W. H. Smith of the University of Chicago has been appointed to an associate professorship at Bluefield State College, Bluefield, West Virginia.

Dr. R. G. Stanton of the University of Michigan has been appointed lecturer at the University of Toronto.

Dr. N. P. Yeardley of the University of Cincinnati has been appointed to an assistant professorship at Purdue University.

The following promotions have been announced:

F. S. Beale, Lehigh University, to an associate professorship.

I. W. Burr, Purdue University, to a professorship.

Eckford Cohen, Syracuse University, to an assistant professorship.

S. P. Diliberto, University of California, to an assistant professorship.

Bernard Epstein, University of Pennsylvania, to an assistant professorship.

Caspar Goffman, University of Oklahoma, to a professorship.

Mary E. Haller, University of Washington, to an associate professorship.

J. L. Hodges, University of California, to an assistant professorship.

Nathan Jacobson, Yale University, to a professorship.

J. W. Lawson, University of Manitoba, to Acting Dean of Arts and Sciences.

P. E. Lewis, North Carolina State College, to an associate professorship.

Sophia L. McDonald, University of California, to a professorship.

W. H. McEwen, University of Manitoba, to Dean of the Faculty of Graduate Studies and Research.

A. P. Morse, University of California, to a professorship.

A. E. Trabant, Purdue University, to an assistant professorship.

M. L. Vest, West Virginia University, to an associate professorship.

Fumio Yagi, University of Washington, to an assistant professorship.

The following appointments to instructorships are announced: University of British Columbia: Dr. T. E. Hull; University of California: Dr. H. B. Ribeiro; Catholic University of America: Dr. Meyer Jerison; Iowa State College of Agriculture and Mechanic

Arts: Mr. R. N. Goss; Iowa State Teachers College: Miss Dorothy DeWitt; Johns Hopkins University: Mr. A. C. Smith; Lehigh University: Mr. B. C. Kenny, Mr. Michael Tikson; University of North Carolina: Mr. Leo Moser; Oberlin College: Mr. M. E. White; Oklahoma Agricultural and Mechanical College: Dr. C. J. Clark, Dr. Arthur Zeichner; Pennsylvania State College: Mr. J. D. E. Konhauser, Mr. E. F. Ormsby; Purdue University: Dr. C. C. Goldman; Rensselaer Polytechnic Institute: Mr. J. C. Gibson, Dr. M. R. Spiegel; University of Southern California: Dr. Harold Shniad; University of Utah: Mr. R. E. Janssen; University of Washington: Mr. G. E. Uhrich; Yale University: Dr. D. T. Finkbeiner.

Professor Constantin Carathéodory of the University of Munich died February 2, 1950 at the age of seventy-six years.

Professor Emeritus J. A. Eiesland of West Virginia University died March 11, 1950 at the age of eighty-three years. He had been a member of the Society for fifty years.

Dr. E. D. Hellinger of Illinois Institute of Technology died March 28, 1950 at the age of sixty-six years. He had been a member of the Society for eleven years.

Mr. Alfred Korzybski of the Institute of General Semantics died the week of March 6, 1950 at the age of seventy years. He had been a member of the Society for twenty-four years.

Professor C. E. Wilder of Dartmouth College died on February 9, 1950 at the age of sixty-one years. He had been a member of the Society for thirty-four years.

The following one hundred and thirty-nine doctorates, with mathematics, mathematical physics, or statistics as a major subject, were conferred during 1949 in universities in the United States and Canada. The university, month in which degree was conferred, minor subject (other than mathematics) and the title of the dissertation are given in each case if available.

F. S. Acton, Carnegie Institute of Technology, June, minor in chemical engineering, *The coupling of two cylindrical acoustical ducts.*

R. V. Andree, Wisconsin, June, *A development of the g-adic numbers of Hensel.*

Julius Aronofsky, Pittsburgh, June, *Investigation of the necked region for flat bars of ductile metals.*

Helmut Aulbach, Syracuse, June, *Some geometrical inequalities for sets in Hilbert space.*

W. E. Barnes, Cornell, June, minor in electronics, *Abel transforms and partial sums of Tauberian series.*

Isaac Battin, New York, February, *On the use of triangular nets* *and generalized differences in the solution of the first boundary value problem for the Laplace equation in the plane.*

Howard Bergmann, New York, February, *The boundary layer problem for certain nonlinear ordinary differential equations.*

R. R. Bernard, Virginia, June, *Probability in dynamical transformation groups.*

W. A. Blankinship, Princeton, June, *Generalization of a construction of Antoine.*

H. D. Block, Iowa State, June, minor in physics, *Explicit solutions of certain singular integral equations.*

W. M. Boothby, Michigan, June, *A topological study of the level curves of harmonic functions.*

F. E. Bortle, Iowa State, December, minors in electrical engineering and physics, *Analytical study of dynamic loads on elastically supported slabs.*

R. H. Bott, Carnegie Institute of Technology, June, minor in electrical engineering, *Electrical network theory.*

Evelyn Boyd, Yale, June, *On Laguerre series in the complex domain.*

R. A. Bradley, North Carolina, June, *Sampling distributions for non-normal universes.*

F. H. Brownell, Princeton, June, *Non-linear difference differential equations.*

H. E. Campbell, Wisconsin, June, *An extension of the "principal theorem" of Wedderburn.*

O. S. Carpenter, Iowa State, June, minor in economics, *Sequential tests of the linear hypothesis.*

E. D. Cashwell, Wisconsin, February, *The asymptotic solutions of an ordinary differential equation in which the coefficient of the parameter is singular.*

Uttam Chand, North Carolina, June, *On certain composite hypotheses concerning regression coefficients and means.*

D. G. Chapman, California, June, I. *Applications of the hypergeometric distribution to sample census problems.* II. *A class of lower bounds for the variance of an estimate.*

M. T. Cheng, Princeton, June, *On the uniqueness theorem of multiple trigonometrical series.*

Peter Chiarulli, Brown, June, *Stability of two-dimensional velocity distributions of the half-jet type.*

T. T. Chung, Cornell, June, *On the sums of independent random variables.*

R. A. Clark, Massachusetts Institute of Technology, June, minor in civil engineering, *On the theory of thin elastic toroidal shells.*

P. A. Clement, California at Los Angeles, June, *Generalized convexity and inequalities im grossen of functions on surfaces of negative curvature.*

H. J. Cohen, Wisconsin, June, *Concerning homogeneous plane continua.*

Jane S. Cronin, Michigan, February, *Branch points of solutions of equations in Banach space.*

J. M. Danskin, California, September, *On the existence of minimizing surfaces in parametric problems in the calculus of variations.*

Robert Davies, Wisconsin, June, *Vibration of a rod weighted at one end.*

A. S. Day, Yale, June, *On an operational calculus for unbounded operators.*

M. D. Donsker, Minnesota, August, minor in physics, *The invariance principle for Wiener functionals.*

Avron Douglis, New York, February, *Some existence theorems for hyperbolic systems of partial differential equations in two independent variables.*

R. F. Dressler, New York, February, *Mathematical solution of the problem of roll-waves in inclined open channels.*

J. C. Eaves, North Carolina, August, *On quasi k^1 commutative matrices.*

R. H. Edwards, Illinois Institute of Technology, June, *Stress concentrations due to spheroidal inclusions.*

F. E. Ehlers, Brown, October, *The transonic flow through a de Laval nozzle.*

H. E. Ellingson, Iowa, February, *Problems involving a third order differential equation and irregular boundary conditions.*

J. H. Engel, Wisconsin, June, *Some contributions to the solution of the word problem for groups (canonical forms in hypoabelian groups).*

A. E. Engelbrecht, Iowa State, August, minor in aeronautical engineering, *Coupled bending and torsional free vibration of a swept wing.*

Mary E. Estill, Texas, May, minor in philosophy, *Concerning abstract spaces.*

M. W. Eudey, California, June, I. *On the treatment of discontinuous random variables.* II. *Statistical model for comparing two methods of diagnosis.*

R. M. Exner, Syracuse, August, *Surfaces invariant under motions.*

D. T. Finkbeiner, California Institute of Technology, June, minor in physics, *A general dependence relation and its application to lattice imbeddings.*

Morton Finston, Brown, June, *Thermal effects in calendering viscous fluids.*

H. M. Flanders, Chicago, September, *Unification of class field theory.*

Walter Fleming, Minnesota, July, minor in physics, *The double Wiener integral of a function of a certain quadratic functional.*

J. R. Foote, Massachusetts Institute of Technology, June, minor in aeronautical engineering, *On the stability of certain unsymmetric two-dimensional parallel flows.*

Abraham Franck, Minnesota, June, minor in physics, *Analytic functions of bounded type.*

D. A. S. Fraser, Princeton, June, *Generalized hit probabilities with a Gaussian target.*

Gerald Freilich, Brown, June, *On the measure of cartesian product sets.*

E. J. Frey, Massachusetts Institute of Technology, September, minor in physics, *On the flow of heat through a slotted wall.*

W. B. Fulks, Minnesota, June, *On integral representations and uniqueness of solutions of the heat equation.*

David Gale, Princeton, June, *Solution of finite two-person games.*

I. C. Gentry, Duke, June, minor in physics, *On extending a homeomorphism defined over subsets of 2-dimensional orientable manifolds.*

C. C. Goldman, Cincinnati, June, *On multiple Fourier series.*

S. R. Goldner, New York, February, *Existence and uniqueness theorems for two systems of non-linear hyperbolic differential equations for functions of two independent variables.*

D. B. Goodner, Illinois, October, minor in education, *Projections in normed linear spaces.*

F. G. Graff, Pittsburgh, January, *Limits to the characteristic roots of a matrix.*

F. E. Grubbs, Michigan, June, *Sample criteria for testing outlying observations.*

H. L. Harter, Purdue, August, minor in physics, *On the distribution of Wald's classification statistic.*

N. S. Hawley, Princeton, June, *Homogeneous and complex manifolds.*

J. L. van Heijenoort, New York, June, *On locally convex surfaces.*

G. P. Henderson, Toronto, November, *Curves in Euclidean space of four dimensions.*

P. G. Hodge, Brown, June, I. *On torsion of plastic bars.* II. *Approximate solutions of problems of plane plastic flow.*

L. Aileen Hostinsky, Illinois, June, minor in education, *Endomorphisms and direct decompositions in lattices.*

T. E. Hull, Toronto, June, *The factorization method and problems in quantum mechanics.*

Eugene Isaacson, New York, February, *Water waves over a sloping bottom.*

M. P. Jarnagin, Duke, June, minor in physics, *Open non-alternating at most two-to-one transformations of connected linear graphs.*

J. G. Kemeny, Princeton, June, *Type theory vs. set theory.*

V. L. Klee, Virginia, June, *Convex sets in linear spaces.*

George Laush, Cornell, September, minor in physics, *Relations among the Weierstrass methods of summability.*

Peter Lax, New York, June, *Non-linear system of hyperbolic partial differential equations in two independent variables.*

F. C. Leone, Purdue, June, minor in psychology, *Some extensions of Bayes' theorem.*

B. W. Lindgren, Minnesota, June, minor in physics, *An integral on the space of functions which are real and continuous but otherwise unrestricted on the unit interval.*

H. D. Lipsich, Cincinnati, June, *On hypergeometric summability.*

S. C. Lowell, New York, June, *Propagation of waves in shallow water.*

Myles McConnon, Pittsburgh, September, *The effect of the anharmonic potential terms on the energy of a one dimensional lattice.*

L. E. Malvern, Brown, October, *The propagation of longitudinal waves of plastic deformation in a bar of material exhibiting a strain-rate effect.*

J. S. Mamelak, Pittsburgh, September, *A singular integral equation.*

W. P. Manger, Massachusetts Institute of Technology, June, minor in aeronautical engineering, *On the foundations of the theory of thin elastic shells.*

W. R. Mann, California, June, *Researches in the mathematical theory of heat transfer between solids and gases under non-linear boundary conditions.*

Imanuel Marx, New York, February, *Type numbers of unstable minimal surfaces spanned in polygonal contours.*

W. S. Massey, Princeton, January, *Classification of mappings of an* $n+1$*-dimensional space into an* n*-sphere.*

R. C. Meacham, Brown, October, *Theory of creep in pin-jointed trusses.*

B. C. Meyer, Stanford, June, minor in chemistry, *On spherical harmonics with a given crystallographic symmetry.*

E. P. Miles, Duke, June, minor in physics, *A minimal problem for harmonic functions in space.*

W. H. Mills, Princeton, June, *Prime power reciprocity in algebraic number fields.*

Yeh Mo, Washington, December, *Foci of plane and spherical curves.*

H. H. Mostafa, Syracuse, December, *On flows governed by simplified gas dynamical equations.*

C. E. Mullan, Pittsburgh, September, *A method of approximating the probability distribution of generating capacity forced outages in electric light and power systems.*

T. D. Newton, Princeton, June, *Localized states for elementary systems.*

Helen K. Nickerson, Harvard, June, *Studies of overconvergence.*

Louis Nirenberg, New York, June, *The determination of a closed convex surface having given line element.*

D. A. Norton, Wisconsin, August, *Hamiltonian loops.*

Bernard Ostle, Iowa State, August, minor in economics, *On certain criteria for optimum estimation.*

T. K. Pan, California, June, *Hypergeodesics and dual hypergeodesics.*

C. E. Pearson, Brown, October, *Bifurcation criterion and plastic buckling of plates and columns.*

Leslie Peck, New York, February, *Diophantine equations in algebraic number fields.*

C. L. Perry, Michigan, June, *The bending of thin elliptic plates.*

G. W. Petrie, Lehigh, October, *Axially symmetric stress distribution in a solid of revolution.*

F. J. Polansky, Minnesota, July, minor in physics, *Some approximation theorems in conformal mapping.*

J. F. Price, Oregon State, June, minor in physics, *Systems of rational differential equations.*

R. C. Prim, Princeton, June, *Steady rotational flow of ideal gases.*

Sister Miriam Frances Quinn, Catholic University, June, minors in physics and philosophy, *An investigation of the number of in- and circumscribed quadrilaterals in the plane quartic curve with one or more nodes.*

Gordon Raisbeck, Massachusetts Institute of Technology, June, minor in physics, *Summability: Diophantine approximations.*

L. T. Ratner, California at Los Angeles, June, *On multi-valued transformations.*

L. L. Rauch, Princeton, June, *Oscillation of a third order non-linear autonomous system.*

J. H. Raymond, Washington, August, *An extension of Chebyshev's inequality.*

J. K. Reckzeh, Kentucky, August, *Modification of the first necessary condition in the calculus of variations for the parametric non-homogeneous case.*

I. S. Reed, California Institute of Technology, June, minor in physics, *Quadratic differential equations in Banach spaces and analytic functionals.*

C. L. Riggs, Kentucky, June, *On the scedastic curve of a compound surface.*

L. G. Riggs, Northwestern, August, minor in physics, *Partially bounded J-fractions.*

E. K. Ritter, Virginia, June, *Second differentials of functions in exterior ballistics.*

R. C. Roberts, Brown, October, *Unsteady one-dimensional flow of a compressible fluid through a porous medium.*

Murray Rosenblatt, Cornell, June, minor in physics, *On distributions of certain Wiener functionals.*

J. A. Rottenberg, Toronto, June, minor in physics, *Stellar exteriors—a study of radiation problems in mathematical physics.*

Walter Rudin, Duke, June, minor in physics, *Uniqueness theory for Laplace series.*

Charles Saltzer, Brown, October, *On the numerical determination of the conformal mapping function of a nearly circular region.*

Esther Seiden, California, September, I. *On a problem of confounding in symmetrical factorial design.* II. *Contribution to the theory of tests of composite hypotheses.*

M. C. Sholander, Brown, October, I. *On the existence of the inverse operation in alternation groupoids.* II. *On certain minimum problems in the theory of convex curves.*

Annette Sinclair, Illinois, June, minor in education, *Generalization of Runge's theorem to approximation by analytic functions.*

M. L. Slater, Harvard, February, *Hausdorff measure in abstract metric spaces.*

E. V. Somers, Pittsburgh, September, *Mathematical investigation of cooperative phenomena.*

E. J. Specht, Minnesota, March, minor in physics, *Estimates on the mapping function and its derivatives in conformal mapping of nearly circular regions.*

M. R. Spiegel, Cornell, June, *On the random vibrations of harmonically bound particles in a viscous medium.*

George Springer, Harvard, June, *The coefficient problem for univalent mappings of the exterior of the unit circle.*

R. G. D. Steel, Iowa State, June, minor in genetics, *Minimum generalized variance for a group of linear functions.*

A. F. Strehler, Wisconsin, June, *The generalization of the Moebius mu-function to a partially ordered set.*

C. M. Tyler, Pittsburgh, June, minor in engineering, *Solution of the lifting line and lifting surface integral equations by the relaxation method.*

J. L. Ullman, Stanford, September, *Studies on Faber polynomials.*

C. E. Vogeley, Pittsburgh, January, *An investigation of an analysis of a simple waveguide discontinuity.*

D. D. Wall, California, September, *Normal numbers.*

Shianghaw Wang, Princeton, January, *On Grundwald's theorem.*

Maria A. Weber, Cornell, February, minor in physics, *The solution of a differential equation of parabolic type.*

Anne M. Whitney, Pennsylvania, June, *On the positivity of translation determinants for Pólya frequency functions with an application to the interpolation problem by spline curves.*

C. W. Williams, Virginia, June, *Transformation groups and recurrence.*

V. M. Wolontis, Harvard, June, *Properties of conformal invariants.*

N. P. Yeardley, Cincinnati, June, *The representation of an analytic function by a general Laguerre series.*

The following four doctorates were conferred in 1948, but were not included in the list in the preceding volume of this Bulletin (vol. 55, pp. 536–542).

Leon Henkin, Princeton, February, *The completeness of formal systems.*

E. D. Nering, Princeton, October, *Reduction of an algebraic function field modulo a prime in the constant field.*

H. E. Rauch, Princeton, February, *Generalizations of some classic theorems to the case of functions of several variables.*

J. E. Walsh, Princeton, February, *Some significance tests for the median which are valid under very general conditions.*

NEW PUBLICATIONS

Adams, D. P. *An index of nomograms.* New York, Wiley, 1950. 10+174 pp. $4.00.

Archibald, R. C. See Steiner, J.

Bagemihl, F. See Kamke, E.

Berzolari, L. *Enciclopedia delle matematiche elementari.* Vol. 3, part 2. Milan, Hoepli, 1950. 19+1037 pp. 3800 L.

Bouligand, G. *Les principes de l'analyse géométrique.* Vol. 2. Paris, Vuibert, 1950. 22+207 pp. 1500 fr.

Cochran, W., and Cox, G. *Experimental designs.* New York, Wiley, 1950. 10+454 pp. $5.75.

Cox, G. See Cochran, W.

Delachet, A. *La géométrie contemporaine.* Paris, Presses Universitaires de France, 1950. 128 pp.

Duarte, F. J. *Monografía sobre los números* π *y* e. Caracas, Tipografia Americana, 1949. 246 pp.

Hofmann, J. E. *Die Entwicklungsgeschichte der Leibnizschen Mathematik während des Aufenthaltes in Paris* (1672–1676). Munich, Leibniz, 1949. 7+252 pp.

Kamke, E. *Theory of sets.* Trans. by F. Bagemihl from 2d German ed. New York, Dover, 1950. 8+152 pp. $2.45.

Labin, É. *Calcul opérationnel.* Paris, Masson, 1949. 150 pp. 780 fr.

Lefschetz, S. *Contributions to the theory of nonlinear oscillations.* (Annals of Mathematics Studies, no. 20.) Princeton University Press, 1950. 10+350 pp. $4.00.

Lefschetz, S. *L'analysis situs et la géométrie algébrique.* Paris, Gauthier-Villars, 1950. 6+154 pp. 650 fr.

Levinson, H. C. *The science of chance.* New York, Rinehart, 1950. 8+348 pp. $2.00.

Lichnerowicz, A. *Éléments de calcul tensoriel.* Paris, Colin, 1950. 216 pp. 180 fr.

Littlewood, D. E. *The skeleton key of mathematics: a simple account of complex algebraic theories.* London, Hutchinson, 1949. 138 pp. 7s, 6d.

Mann, H. B. *Analysis and design of experiments.* New York, Dover, 1949. 10+198 pp. $2.95.

Peterson, T. S. *Elements of calculus.* New York, Harper, 1950. 10+369 pp. $3.50.

Ritt, J. F. *Differential algebra.* (American Mathematical Society Colloquium Publications, vol. 33.) New York, American Mathematical Society, 1950. 8+181 pp. $4.40.

Roy, L. *Cours de mécanique rationnelle.* Vol. 4. *Problèmes et exercises.* Paris, Gauthier-Villars, 1950. 12+276 pp. 1250 fr.

Schläfli, L. *Gesammelte mathematische Abhandlungen.* Ed. by Steiner-Schläfli Committee of the Swiss Scientific Society. Vol. 1. Basel, Birkhauser, 1950. 392 pp. 54 Swiss fr.

Shannon, C., and Weaver, W. *The mathematical theory of communication.* Urbana, University of Illinois Press, 1949. 117 pp. $2.50.

Stark, M. E. See Steiner, J.

Steiner, J. *Geometrical constructions with a ruler given a fixed circle with its center.* Trans. by M. E. Stark; ed. with an introduction and notes by R. C. Archibald. (The Scripta Mathematica Studies, no. 4.) New York, Scripta, 1950. 3+88 pp. $2.00.

Stoker, J. J. *Nonlinear vibrations in mechanical and electrical systems.* New York, Interscience, 1950. 20+273 pp. $5.00.

Toeplitz, O. *Die Entwicklung der Infinitesimalrechnung.* Berlin, Springer, 1949. 10+180 pp. 22.60 DM.

Weaver, W. See Shannon, C.

MATHEMATICS AND THE NATIONAL SCIENCE FOUNDATION

(A statement by the policy Committee for Mathematics based upon a report of the Committee on the Needs of Mathematics which can be met by a National Science Foundation.)

It is essential that the greatest flexibility be maintained in the adminstration of the funds which are made available for mathematics by the National Science Foundation. We feel that the charge of the Mathematics Division of the National Science Foundation, and of any advisory committee which is established should be to advance the welfare of mathematics. The means of doing this should remain a constantly evolving process.

With this in mind we make the following recommendations.

I. Immediately mathematicians should, through the Policy Committee, take steps to insure:

A. That persons are appointed to Foundation posts—both on the Foundation itself and in the responsible Civil Service positions—who are competent and persons of good judgment. We recommend that the Policy Committee inform the proper authorities in Washington, that the Policy Committee holds itself ready to consult with them on matters affecting mathematics connected with the establishment of a Foundation, is concerned to see that competent persons are appointed to the posts in the two roles mentioned above, and in particular wishes to suggest suitable nominations for Membership on the Foundation.

B. A Division of Mathematics of the National Research Council is established whose principal jobs with respect to the Foundation would be to:

1. take whatever steps are appropriate to secure a proper budget for mathematics. Here an assessment of the need should be made carefully, taking account of available talent, and of fields which lie outside the principal interests of organized mathematics, but will not receive support elsewhere;

2. provide for the appointment *in consultation with* the Policy Committee of an Advisory Committee for Mathematics consisting of active mathematicians;

3. maintain proper contacts with other divisions of the National Research Council and with the Social Science Research Council, in order to secure an alert awareness of their needs for mathematics.

II. One of the most acute problems facing mathematics today is the publication problem. We strongly urge that explicit provision be made for offering subsidies to existing journals and for the support of new journals when the establishment of such journals is recommended by the Advisory Committee. The amount of subsidy should be determined on the basis of advice from the Advisory Committee. One possible scheme would be for the Foundation to allocate funds for subsidization of publications. The total sum might be a fixed percentage of the total grants made to individuals, universities, and so forth. In addition to this there are two other matters relating to publication which the Foundation should be requested to aid. These are:

A. The publication of historical or comprehensive surveys of various fields.

B. The regular publication of translations of articles in Russian and other unfamiliar languages.

III. A. General remarks—While some of the funds will be for specific projects on a contractual basis, in general it seems preferable to make grants to suitable individuals or universities. These grants would usually be made with the knowledge of an informal proposal, but such a proposal should not be intended to restrict studies too narrowly to programs laid down in advance. Annual reports by recipients of grants should be made to the Advisory Committee. This body should also advise on the renewal of grants. Grants for full time research should, in general, be for one year, renewable for a second year, and not normally renewable beyond that.

B. Assistance to libraries should be made to universities, and the funds should be administered by them.

C. Fellowships should be awarded for graduate and post-doctoral study and research on a national basis on recommendation of the Advisory Committee.

D. Grants to individuals should be made solely on the basis of individual merit without regard to field of research or age, and should be awarded on a national basis.

E. The Foundation should take as a major charge carefully selected attempts to assist an institution to develop its mathematics department according to plans proposed by the institution and approved by the Foundation. It may be desirable to have regional boards to advise on this feature of the problem. Contributions to sabbatical year salaries would be decided here.

F. Grants should be available for projects if group undertakings seem warranted.

IV. The Foundation should seek additional means of discovering and fostering promising mathematical talent among young students, particularly those who, for geographic or other reasons, may have only poor opportunities for discovering this talent for themselves. In particular, the Foundation should explore the desirability of publishing, or assisting the Mathematical Association to publish, a periodical for the benefit of secondary school students in order to stimulate and direct younger students showing definite mathematical ability and interest.

V. The Mathematics Division of the National Research Council, with financial assistance where needed from the National Science Foundation, should sponsor studies of the quality of teaching and the curriculum in mathematics at the secondary school level throughout the country, and take steps to bring about whatever changes are indicated as necessary by the study. Among other things, study should be made of the desirability of the development of visual and auditory aids to teaching. The writing of sound mathematics books at the secondary school level, and of good texts for undergraduates should be encouraged.

VI. The Foundation should consider from time to time additional activities, such as conferences on research fields of current interest.

PARAMETRIC SURFACES

A. S. BESICOVITCH

I want first to express my thanks to the Presidium of the American Mathematical Society for the invitation to deliver an address to the Society. I feel very much honored by this invitation and I appreciate it highly.

My address will be devoted to parametric surfaces, to that part of the subject which is connected with the definition of the area, with the expression of the area by an integral, and with the minimum area problem. It is an exceptional privilege to lecture on this subject to the American Mathematical Society, whose members have been responsible for such an impressive progress of the subject during the last two decades. I shall start with the definition of the area.

In 1914 Carathéodory defined the m-dimensional measure in the n-dimensional space, for integral values of m.[1] He considered the one-dimensional measure as a generalization of the length and he proved that the length of a rectifiable curve coincides with its one-dimensional measure.

In 1919, Hausdorff,[2] developing Carathéodory's ideas, constructed a continuous scale of measures. After the work of Carathéodory and Hausdorff the obvious problem was to treat the area as the two-dimensional measure and to establish the well known integral formulae. But a considerable amount of work on establishing the integral formulae for the area had been done before that, with a variety of definitions of the area, due no doubt to the difficulty of the problem. Later the Lebesgue definition, somewhat modified by Frechet, of the area as the lower limit of areas of approximating polyhedra became the dominant one. A particularly valuable feature of the Lebesgue-Frechet (L.-F.) definition is that it provides the area with the property of the lower semi-continuity.

When applied to plane figures bounded by a Jordan curve the L.-F. definition gives in fact the value of the interior area of the figure, and thus in cases when the boundary is of positive L_2-measure, the L.-F. area differs from the value of the closed area in the usual

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An address delivered before the Annual Meeting of the Society at Columbus, Ohio, December 29, 1948, by invitation of the Committee to Select Hour Speakers for Summer and Annual Meetings; received by the editors April 14, 1949.

[1] *Über das lineare Mass von Punktmengen—eine Verallgemeinerung des Längenbegriffs*, Nachr. Ges. Wiss. Göttingen (1914) pp. 404–426.

[2] *Dimension und äusseres Mass*, Math. Ann. vol. 79 (1919) pp. 157–179.

Lebesgue sense. This of course need not be considered in any sense a serious fault with the definition, and of course there is nothing astonishing then if the area of the projection measured as a *closed area* happens to be greater than the area of the projective figure measured as an open area. This is the so-called projected conflict,[3] conflict only in name. Being entirely based on an obvious confusion of two measures, it could not have shaken any confidence in the L.-F. definition. As a matter of fact there is no conflict whatsoever, and the L.-F. area of the orthogonal projection of a parametric surface, which (the projection) is itself a parametric surface, is always less than or equal to the L.-F. area of the projected figure. This is obvious for polyhedral surfaces, and then it follows at once for the general case.

In 1913, Geöcze[4] gave an example of a parametric surface that fills up a cube and has at the same time L.-F. area equal to zero. But the Geöcze parametric surface is no surface in any geometric sense and the notion of its area is entirely artificial. Whatever value might have been obtained for its area, it could not have been a test for the definition. Thus no fault was suspected in the L.-F. definition until recently, and a large amount of work was done on the basis of the L.-F. definition, culminating in the proof of the integral formula for parametric surfaces under the most general conditions, and in the solution of the Plateau problem. But the study of the area cannot be restricted to the very narrow class of the parametric surfaces, that is, to the class of continuous images of a disk. A notion of area is attached to a much wider class of surfaces, such as continuous images of other figures than a disk, many cases of discontinuous images and to surfaces given as point sets. With the experience of more than fifty years of modern theory of functions of a real variable, any attempt of extending the definition of the area to more and more general classes of surfaces would inevitably lead to treating the area as a measure, just as the length and the volume are treated.

A few years ago I set myself the problem of treating the area as the Hausdorff Λ^2-measure and of proving the identity of the L.-F. area of parametric surfaces and of their Λ^2-measure (excluding the boundaries). However I came to results very different from ones I was hoping for. In my paper, *On the definition and the value of the area of a surface*,[5] I constructed a *topological disk* and a *topological*

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[3] T. Radó, *On the problem of Plateau*, Ergebnisse der Mathematik und ihre Grenzgebiete, vol. 2, 1933.

[4] Z. Geocze, *Sur l'exemple d'une surface dont l'aire est egale a zero et qui remplit un cube*, Bull. Soc. Math. France vol. 41 (1913) pp. 29–31.

[5] Quart. J. Math. Oxford Ser. vol. 16 (1945) pp. 86–102.

sphere whose surface, on the one hand, has L.-F. area as small as we please, and, on the other hand, has a positive Lebesgue three-dimensional measure. In order to avoid too complicated diagrams, I shall first construct a plane curve to which the required surface is very similar. Let $S_0=A$ be a unit square (Fig. 1). Divide A into four equal squares and let A_1, A_2, A_3, A_4 be four smaller equal squares concentric with the above squares and of total area greater than 1/2. Write $S_1=\sum_1^4 A_{i_1}$, meaning by S_1 the set of interior and boundary points of the squares $A_1, \cdots, A_4$. Similarly a set S_2

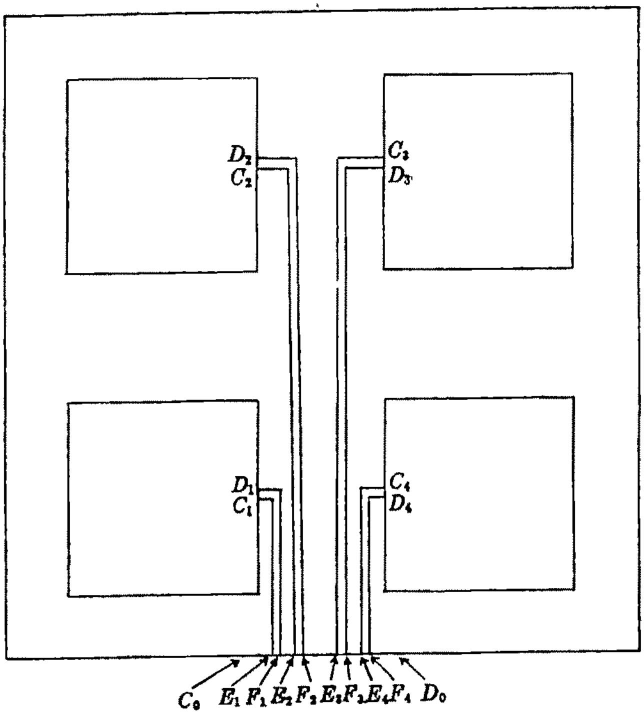

FIG. 1.

$=\sum_{i_1=1,\cdots,4,i_2=1,\cdots,4} A_{i_1 i_2}$ of 16 equal squares of total area greater than 1/2 and less than the total area of S_1 are constructed, four squares in each square of S_1, and so on. In this way a decreasing sequence $\{S_n\}$ is determined, where S_n is a set of 4^n equal squares of total area greater than 1/2. Writing $S=\lim S_n$, we have $m^2S \geqq 1/2$ (m^2 being the Lebesgue two-dimensional measure). Take now a segment C_0D_0 on a side of the square A and segments $C_1D_1, \cdots, C_4D_4$ on sides of $A_1, \cdots, A_4$ and construct a polygonal line $\Pi_1=P_0 = C_0E_1C_1D_1F_1E_2C_2D_2F_2 \cdots D_4F_4D_0$ of four branches. Similarly on each of the segments $C_1D_1, \cdots, C_4D_4$ construct polygonal lines $P_1, \cdots, P_4$, each of four branches. Replacing the sides $C_1D_1, \cdots, C_4D_4$ of Π_1 by the polygonal lines $P_1, \cdots, P_4$, we shall expand the line Π_1 to a line Π_2 of four branches of the first order and 16 branches of the second order. Similarly Π_2 is expanded to Π_3, Π_3 to Π_4, and so on. The sequence of polygonal lines $\{\Pi_n\}$ defines a limit curve Π, which is obviously a simple curve. The line Π_n has a segment in common with every square of S_n, and consequently every point of S_n, and in particular every point of S is within $2^{-n+1/2}$ from Π_n. Hence every point of S is on Π. *Thus* Π *is a simple curve of infinite length and it contains the set* S, *whose two-dimensional measure is not less than* $1/2$.

Now we carry out a similar construction in the three-dimensional space. Let $S_0=A$ be a unit-cube, and $S_1=\sum_{i=1}^{8} A_{i_1}$ a set of 8 equal cubes of total volume less than 1 and greater than $1/2$, *symmetrically situated in* A, by which we mean that the cubes A_{i_1}, $i_1=1, \cdots, 8$, are concentric with the eight cubes into which A is divided by the three mid-planes. Let further $S_2=\sum_{i_1=1,2,\cdots,8, i_2=1,2,\cdots,8} A_{i_1i_2}$ be 64 equal cubes of total volume less than that of S_1 and greater than $1/2$, the cubes $A_{i_11}, \cdots, A_{i_18}$ being symmetrically situated in A_{i_1}, and so on. In this way a decreasing sequence of sets S_n is determined. Writing $S=\lim S_n$, we have $m^3S\geqq 1/2$ (m^3 being the Lebesgue 3-dimensional measure) since $m^3S_n>1/2$ for every n.

On the bottom side of A take a small square C_0D_0 (C_0, D_0 are a pair of opposite vertices) and on C_0D_0 eight smaller disjoint squares $E_1F_1, \cdots, E_8F_8$, then take squares $C_1D_1, \cdots, C_8D_8$ on the bottom sides of the cubes $A_1, \cdots, A_8$ and join them to $E_1F_1, \cdots, E_8F_8$ respectively by rectangular tubes, that do not meet one another. The part $C_0D_0-\sum_1^8 E_iF_i$ of the square C_0D_0, the eight tubes constructed above, and the squares $C_1D_1, \cdots, C_8D_8$ form a polyhedral surface P_0 of eight branches. Call it Π_1. Π_1 is a topological disk. Similarly polyhedral surfaces $P_1, \cdots, P_8$, of eight branches each, will be constructed on the squares $C_1D_1, \cdots, C_8D_8$ respectively. Replacing the squares $C_1D_1, \cdots, C_8D_8$ of Π_1 by the polyhedral surfaces, $P_1, \cdots, P_8$ respectively, we shall get a polyhedral surface Π_2 of eight branches of the first order and 64 branches of the second order. Π_2 is also a topological disk. Similarly Π_2 will be "expanded" to Π_3, Π_3 to Π_4, and so on. The sequence of surfaces Π_n converges in the Frechet sense to a surface Π. Π *is a topological disk.* Just as in the case of the plane construction Π contains all the points of S, so that $m^3\Pi\geqq 1/2$. Now comes a fundamental difference between the plane figure and the three-dimensional one. While in the case of the plane figure the length of the polygonal line Π_n tends to infinity, in the case of the three-dimensional figure the areas of the polyhedral surfaces $\Pi_1, \Pi_2, \cdots$ may be as small as we please, so that given $\epsilon>0$ we may assume

$$\lim \Lambda^2\Pi_n < \epsilon$$

and hence L.-F. area of Π less than ϵ.

(i) *Thus we have constructed a topological disk, whose area, in the L.-F. definition, is less than* ϵ, *and whose three-dimensional Lebesgue measure is not less than* 1/2.

(ii) If we seal the surface Π by a surface of small area subtending the boundary of the square C_0D_0 and lying below it, *we shall get a topological sphere* Π'. The volume of Π' (that is, m^3-measure) is greater than 1/2 and the area of the surface of Π' may be as small as we please. The solid bounded by the surface Π' has different values for the upper volume and for the lower volume. Therefore at pre-Lebesgue time it would have been considered that the volume was not defined for this solid, and the surface Π' would have had no bearing on the classical isoperimetric problem of finding the solid of maximum volume, bounded by a surface of a given area. However, since the development of methods of measure the notions of the upper and lower volumes have lost their relevance and the volume is defined as the m^3-measure. Therefore the result presented by Π' is in conflict with the existing solution of the above problem. The result (i) is in conflict with all geometrical ideas of the area:

(a) It shows that the L.-F. area cannot be fitted into any scale of measures.

(b) It has no property of additivity, even in the most clear cases. A construction similar to one employed above leads to a parametric surface that includes all points of a unit-square, and whose area is as small as we please.

(c) It has not been, and it cannot be, extended to surfaces given as point-sets. For these reasons, clinging to the L.-F. definition of area would imply treating it as a functional defined in an artificially restricted domain and completely detached from the general study of the area.

The general study of area can be brought to the standards of perfection of the modern theory of functions of a real variable only by treating the area as a measure. I define the area as the Hausdorff two-dimensional measure (with a corrective for the case of multiple points), and I solve anew the fundamental problems on area: that of integral expression of the area and the Plateau problem. The right principle brings a generous reward: tremendous topological complications for establishing the integral formulae are avoided completely, the integral formulae are valid not only for the whole surface, but also for sets of points on the surface; in the case of the Plateau problem all restrictions on the contour are dropped. In the solution of the Plateau problem I was helped by having discovered conditions of compactness and of semi-continuity of the area in the class of parametric surfaces.

Before going further into the problems of my lecture, we have to introduce strict definitions concerning parametric surfaces.

We shall consider continuous functions $P=\Phi(M)$ whose domain of definition is a disk H and whose values are points in the three-dimensional space. For any point P, the set $\Phi^{-1}(P)$ is closed. Any component of $\Phi^{-1}(P)$ is called a Φ-element of H. Any Φ-element is either a point or a saturated continuum in $\Phi^{-1}(P)$. Thus the equation $P=\Phi(M)$ defines a representation $H=\sum Q$ of H as the sum of Φ-elements. Any pair of Φ-elements are either coincident or have no points in common. We form pairs (P, Q) where Q are Φ-elements, and, for every Q, P is the value of $\Phi(M)$ on Q. The set $\Pi=\sum(P, Q)$ of all such pairs is called the ***parametric surface*** defined by the equation $P=\Phi(M)$ (or by the function $\Phi(M)$) and the pairs (P, Q) the points of the parametric surface.

If a set $E\subset H$ is given, the symbol $\Phi(E)$ will be used in two different senses: it is the set of values of $\Phi(M)$ at all points of E, and also it is the set $\sum(P, Q)$ of points of Π for all Φ-elements Q such that $Q\cdot E\neq 0$. In either case it is "the image of E."

Given two points $X_i=(P_i, Q_i)$ $(i=1, 2)$, let c be any continuum in H joining Q_1 and Q_2. The lower bound of the diameter of $\Phi(c)$ for all possible c is called the Π-distance between X_1 and X_2, and is denoted by $D_\Pi(X_1, X_2)$. If $Q_1\neq Q_2$, $D_\Pi(X_1, X_2)>0$ whether P_1 is different from P_2 or not.

If there are exactly k points of Π, $(P, Q_1), \cdots, (P, Q_k)$, with the same first element, then P is said to be of multiplicity k. The k points are distinct and are at a positive Π-distance from one another.

Given two parametric surfaces Π_1 and Π_2 defined by the functions $\Phi_1(M)$ and $\Phi_2(M)$, let $\Psi(M)$ be any homeomorphism on H. Let $D(\Psi)$ be the maximum of the Euclidean distance between the points $\Phi_1(M)$ and $\Phi_2\{\Psi(M)\}$ as M varies in H. The lower bound of $D(\Psi)$ for all $\Psi(M)$ is called the F-distance (Frechet distance) between Π_1 and Π_2 and is denoted $D_F(\Pi_1, \Pi_2)$. If $D_F(\Pi_1, \Pi_2)=0$, we say that Π_1 and Π_2 are identical parametric surfaces. Thus the surfaces defined by $\Phi(M)$ and by $\Phi\{\Psi(M)\}$ are identical. F-distance can be defined in a different way, in terms of Φ-elements, but we shall not dwell on that. Once the distance between parametric surfaces is defined, convergent sequences of parametric surfaces and limit parametric surfaces are defined too. We have a simple theorem:

THEOREM. A *necessary and sufficient condition for a sequence* $\{\Pi_n\}$ *to be convergent is that it should be representable by a uniformly convergent sequence* $\{\Phi_n(\mathcal{U})\}$ *of functions.*

A set $E(\delta, \Pi)$ of points of Π such that no pairs of them are within δ from each other and that any point of Π is within δ of at least one point of E is called a δ-set of Π. *A δ-set is always finite.*

We shall be considering aggregates of parametric surfaces in a bounded part of the space. We shall assume them bounded. Given an aggregate A, if there exists a finite function $N(\delta)$ of a positive variable δ such that, for any $\Pi \in A$, $\mathcal{N}E(\delta, \Pi) < N(\delta)$ ($\mathcal{N}$ stands for the number of points of E) we say that A is *uniformly bounded with respect to neighborhoods.*

A plane domain G whose boundary consists of a finite number of disjoint Jordan curves is called *a regular domain.* A representation $H = \sum_1^n \bar{G}_i$ where $G_1, \cdots, G_n$ are disjoint regular domains is called *a regular dissection of H.* Given a regular domain G in H, $\Phi(G)$ will be called *a regular domain in* Π. Similarly, to a regular dissection $H = \sum_1^n \bar{G}_i$ corresponds *a regular dissection* $\Pi = \sum \Phi(\bar{G}_i)$ of Π.

We can now state the main results concerning our problems. We shall start with compactness. *An aggregate of parametric surfaces is said to be compact if any sequence of parametric surfaces of the aggregate contains a convergent subsequence.* A complete solution of the compactness problem is given by the following theorem.

THEOREM. *A necessary and sufficient condition for an aggregate A to be compact is that it should be uniformly bounded with respect to neighborhoods.*

The proof of the theorem depends partly on the following problem.

PROBLEM. *To divide* Π *into regular domains of small diameters and to estimate the number of domains.*

This can be done in terms of the number n of points of the set $\mathcal{E}(\delta, \Pi)$. The problem is more difficult than it looks. It seems that if we take δ-neighborhoods of points of $E(\delta, \Pi)$, they would give a solution. But these neighborhoods are overlapping, and if we try to reduce them for eliminating the overlapping, then this, in general, leads to dividing some of the neighborhoods into many disjoint ones. The actual answer to the problem is given by the following lemma.

LEMMA 1. Π *can be represented as the sum of not more than n regular domains each of diameter less than* 50δ.

This lemma becomes effective when coupled with the following lemma.

LEMMA 2. *The number of topologically different dissections of H into m regular domains is finite for every m.*

The next problem is that of lower semi-continuity of area. Let U be the set of the first elements of the points of the parametric surface Π defined by the equation $P=\Phi(M)$. Write $U=\sum_1^\infty U_k$, where U_k is the set of points of Π of multiplicity k. Measure of Π is defined by the sum

$$\Lambda^2\Pi = \sum_{k=1}^{\infty} k\Lambda^2 U_k,$$

where the term corresponding to $k=\infty$ is 0 if $\Lambda^2 U_k=0$, and ∞ if $\Lambda^2 U_k>0$. Similarly $\Lambda^2\Phi(E)$ is defined for any $E\subset H$. The measure defined in this way obviously has the property of additivity.

A parametric surface is called smooth if any part of Π (if any) whose boundary lies in a plane lies itself in a plane.

The lower semi-continuity theorem is established in the following form.

If $\Pi=\lim \Pi_n$, *then* $\Lambda^2\Pi^0 \leqq \liminf \Lambda^2\Pi_n^0$ (Π^0, Π_n^0 *are the sets of interior points of* Π, Π_n) *in two cases*:

(i) *when* Π *is a smooth surface*;

(ii) *when* Π *has a tangential plane at almost all points in the sense of* Λ^2*-measure.*

Thus it is established *only for interior areas and it is not true for the closed areas.*

Having properties of compactness and of lower semi-continuity, we come in an obvious way to a proof of existence of a surface of minimum area subtending an arbitrary parametric contour Γ.

Let $a=\text{l.bd}\ \Lambda^2\Pi$, where Π is any surface subtending Γ, and let $\{\Pi_n\}$ be a sequence of surfaces subtending Γ and such that $\Lambda^2\Pi_n\to a$. By smoothing operations, which may only reduce the value of the area, every Π_n may be replaced by a smooth surface of area not greater than $\Lambda^2\Pi_n$. Thus we may simply assume all the surfaces Π_n to be smooth. In this case the aggregate $\{\Pi_n\}$ is easily seen to be uniformly bounded with respect to neighborhoods. Consequently, the compactness theorem is applicable. Thus we can assume the sequence $\{\Pi_n\}$ convergent. Let $\Pi=\lim \Pi_n$; Π is obviously smooth and thus the lower semi-continuity theorem is applicable:

$$\Lambda^2\Pi^0 \leqq \liminf \Lambda^2\Pi_n^0.$$

We have to establish a similar relation for the boundaries, which, one may think, is obvious as the boundary is the same. This is true in the case when all the points of the boundary are simple. If the boundary has multiple points, then their multiplicity may depend on the surface subtending them and thus measure of the boundary may vary from surface to surface. Still the required relation can be easily established and the proof of the formula $\Lambda^2\Pi = a$ completed.

The second part of the Plateau problem, that of showing that Π is a harmonic surface, follows in the usual way since according to Cesari for a smooth surface Π, $\Lambda^2\Pi$ coincides with the L.-F. area of Π.

Now only a few words about the last problem, that of expressing the value of the area in the form of an integral.

In the case of a non-parametric surface $z=f(x, y)$, the formula is established under the Tonelli conditions.

In the case of a parametric surface $P=\Phi(M)$, the integral formula holds if $\Phi(M)$ is approximately differentiable at almost all points of H and if it is absolutely continuous in the sense that for any $E\subset H$ of m^2-measure zero, $\Lambda^2\Phi(E)=0$.

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COMPLEXES AND HOMOTOPY CHAINS

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The theory I have to speak about is a chapter of the algebraic topology of complexes. Its definition parallels the classical homology theory.

Let A be a complex with the oriented cells a_i^k, where k is the dimension number. Then the homology theory starts with the free Abelian groups of chains

$$c^k = \sum \xi a^k, \qquad \xi \in N,\ a^k \in A,$$

generated by the a_i^k (N is the set of the integers) and the boundary homomorphism of chains

$$(c^k)^{\cdot} = \sum \xi (a^k)^{\cdot}$$

where

$$(a_i^k)^{\cdot} = \sum \rho_{ij}^k a_j^{k-1}$$

is the boundary chain of the oriented cell a_i^k, the ρ_{ij}^k being the incidence numbers of the cells a_i^k, a_j^{k-1}. These chains and boundary matrices change by subdivision of the complex A, and the homology groups are the invariants with regard to this process.

The chains and boundary matrices which I introduce are defined for complexes U with an adjoined group G of mappings γ of U in itself, that is, of mappings

$$\gamma u^k = \bar{u}^k$$

of the cells u^k of U preserving the dimension, the orientation, and the incidence relations of cells. The subdivision of the euclidean plane in squares, which is mapped in itself by the group of translations with integer coefficients, is an example for a complex U.

The mappings γ of the cells induce automorphisms γ of the chains of U,

$$\gamma c^k = \gamma \sum \xi u^k = \sum \xi (\gamma u^k),$$

and these automorphisms commute with the boundary homomorphism; for

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$$(\gamma c^k)^{\cdot} = (\sum \xi \gamma u^k)^{\cdot} = \sum \xi(\gamma u^k)^{\cdot},$$
$$\gamma(c^k)^{\cdot} = \gamma(\sum \xi(u^k)^{\cdot}) = \sum \xi\gamma(u^k)^{\cdot},$$

and the equation $(\gamma u^k)^{\cdot} = \gamma(u^k)^{\cdot}$ holds, because γ preserves the incidence relations.

So we can describe these chains with operators and their boundary homomorphisms by taking the elements

$$x = \sum \xi\gamma, \qquad \xi \in N, \gamma \in G,$$

of the group ring GN of G as coefficients of chains. Because

$$xc^k = (\sum \xi\gamma)c^k = \sum \xi(\gamma c^k)$$

is defined, we can speak of the chains

$$\sum x_i c_i^k$$

generated by the chains c_i^k and of relations

$$\sum x_i c_i^k = 0$$

between the c_i^k. And because the equation

$$(xc^k)^{\cdot} = x(c^k)^{\cdot}$$

holds, we can compute the boundary chains of the generated chains by the boundary chains of the generators.

If the mappings $\gamma \neq 1$ (the identity) have no fixed elements, then there exists a fundamental region F of U modulo G, namely a subset F of cells u^k, so that

$$\gamma u_i^k \qquad \text{with } \gamma \in G, u_i^k \in F$$

represent uniquely the cells of U. And in this case we see the groups of chains of U with operators are free groups with a set F of free generators u_i^k. Indeed the

$$c^k = \sum x_i u_i^k = \sum \xi_{if}\gamma_f u_i^k$$

represent uniquely the chains of U because $\gamma_f u_i^k$ represent uniquely the cells of U. And the boundary homomorphism is described by a set of matrices r_{ij}^k with coefficients in GN, namely the coefficients of the boundary chains of the generators expressed in these generators

$$(u_i^k)^{\cdot} = \sum r_{ij}^k u_j^{k-1}, \qquad r_{ij}^k \in GN.$$

For the subdivision of the euclidean plane in squares we have in a fundamental region one vertex u^0, one square u^2, and two edges u^1_1, u^1_2, which represent two vectors of unit length, orthogonal one to the other. The group G is generated by the translations σ_1, σ_2 determined by these vectors and we can take F so that we get

$$(\overset{1}{u}_1)^\cdot = -\overset{0}{u} + \sigma_1\overset{0}{u},$$
$$(\overset{1}{u}_2)^\cdot = -\overset{0}{u} + \sigma_2\overset{0}{u},$$
$$(\overset{2}{u})^\cdot = \overset{1}{u}_1 + \sigma_1\overset{1}{u}_2 - \sigma_2\overset{1}{u}_1 - \overset{1}{u}_2.$$

Now remember a classical theorem. To every connected complex A there exists a universal covering complex U with a group G of mappings γ of U in itself, the structure of which is uniquely determined by A. The group G of U is isomorphic to the fundamental group G^1 of A defined by the homotopy classes of the paths of A. The mappings γ are the deck transformations of the covering complex. That means: if π is the projection of U in A, which gives for every cell u^k of U the cell a^k of A which is covered by u^k, then we have: $\gamma u = \bar{u}$ implies $\pi(\bar{u}) = \pi(u)$ and $\pi(\bar{u}) = \pi(u)$ implies the existence of a γ with $\bar{u} = \gamma u$. And because π has to be faithful in the small, it follows that $\gamma u = u$ implies $\gamma = 1$.

So we can define: the system of homotopy chains of the complex A is the system of the chains of the universal covering complex U of A with the operators induced by the deck transformations of U. And we have the theorem: *If the u^k_i are a fundamental region of U modulo G, then the u^k_i are free generators of the system of homotopy chains and there exists a uniquely determined set of matrices r^k_{ij} with elements in the group ring GN defining the boundary homomorphism of the chains in the chosen generators.*

The square net of the euclidean plane is the universal covering complex of a torus A, the edges a^1_1, a^1_2 of which are singular ones beginning and ending at the same point a^0. The chains of the square net are the homotopy chains of A.

To finish the exposition of the theory we have to characterize algebraically the topological invariants of the matrices r^k_{ij}.

We can change the fundamental region F in $\bar{F}$ by

$$\bar{\overset{k}{u}}_i = \gamma_{ik}\overset{k}{u}_i, \qquad \gamma \in G,$$

and A may change by subdivision. Then U is also changed by a set of induced subdivisions. This step is equivalent to a set of unimodular transformations of the generators and to the introduction of new pairs of generators u^k, u^{k-1} with the boundary relation

$$(u^k)^\cdot = u^{k-1}, \quad (u^{k-1})^\cdot = 0.$$

So we have three ways to change the set of generators. The first and the second type preserve the number of generators and by iteration of these steps we get a group of linear transformations T, the basis transformations, with coefficients in the group ring GN which is determined in the same way for all group rings and is uniquely determined by GN and the number of generators.

And the theorem holds:

If $\bar{A}$ *is a complex, which is equivalent to* A *by subdivision, then the set of the homotopy boundary matrices* $\bar{r}^k_{ij}$ *of* $\bar{A}$ *is equivalent to* r^k_{ij} *of* A *by basis transformations and introduction of new pairs of generators* u^k, u^{k-1} *with*

$$(u^k)^\cdot = u^{k-1}, \quad (u^{k-1})^\cdot = 0.$$

Now we have to ask how to solve the equivalence problem for homotopy chain systems, and clearly the parallelism to the homology theory, which is so obvious by the definition of the equivalence, does not hold for its solution at all. There is little hope to get a method to decide if the homotopy systems of two complexes A, $\bar{A}$ are equivalent or not. For, in order to compute the boundary matrices r^k_{ij}, we need to know the fundamental group G^1; G^1 is given by generators and defining relations and there exists no method to decide if two systems of generators and relations define isomorphic groups. But there exists a method to get invariants.

To be clear I define this method at first in a purely algebraic way.

Let M be a ring with elements y which is homomorphic to GN and let

$$\phi x = y$$

be a homomorphism of GN onto M. Then we get by

$$\phi \overset{k}{r}_{ij} = \overset{k}{\tilde{r}}_{ij}$$

a set of matrices with elements in M. We define the M-chains of A by

$$\overset{k}{\tilde{c}} = \sum y_i \overset{k}{v}_i,$$

the number of the v_i^k being the same as the number of the u_i^k, and the boundary homomorphism by

$$
\begin{aligned}
\left(\sum y\overset{k}{v}\right)^{\cdot} &= \sum y\left(\overset{k}{v}\right)^{\cdot}, \\
\left(\overset{k}{v_i}\right)^{\cdot} &= \sum \tilde{r}_{ij}^{k} v_j^{k-1}.
\end{aligned}
$$

This M-chain system with its boundary operator is homomorphic to the homotopy chain system. Indeed the projection

$$\phi c^k = \phi \sum x u^k = \sum \phi x \cdot v^k$$

is a homomorphism of the free homotopy chain groups in the free M-chain groups and ϕ commutes with the boundary operator, because the equation

$$(\phi u^k)^{\cdot} = \phi(u^k)^{\cdot}$$

holds.

By projection of the basis transformation T we get the transformation ϕT for M-chain basis and an equivalence relation for M-chain systems and we see that the equivalence classes of M-chain systems correspond to those of the homotopy chains.

Now we may look for invariants of the M-chain system.

Following the classical way we define the homology group $\tilde{H}^k$. Let $\tilde{Z}^k$ be the groups of the cycles, that is, of the chains $\tilde{z}^k$ with $(\tilde{z}^k)^{\cdot} = 0$ and $\tilde{B}^k$ the group of the bounding chains $\tilde{b}^k$, that is, of the chains for which exist a chain $\tilde{b}^{k+1}$ with $(\tilde{b}^{k+1})^{\cdot} = \tilde{b}^k$. Then $(\tilde{b}^k)^{\cdot} = 0$, $\tilde{B}^k \subset \tilde{Z}^k$. And the $\tilde{H}^k$ are defined as the factor group $\tilde{Z}^k/\tilde{B}^k$, all these groups being groups with operators in M.

Observe further that the projection ϕc^k defines a characteristic subgroup in $\tilde{H}^k$. Namely if z^k is a homotopy chain with $(z^k)^{\cdot} = 0$, then ϕz^k is an M-chain with $(\phi z^k)^{\cdot} = 0$ because ϕ commutes with the boundary operator, and thus we get by projection of the group Z^k of the homotopy cycles a subgroup ϕZ^k of $\tilde{Z}^k$ and, because $\phi B^k = \tilde{B}^k$, a subgroup of $\tilde{H}^k$. The $\tilde{H}^k$ with these characteristic subgroups $\phi Z^k/\tilde{B}^k$ are invariants of the homotopy system.

The geometrical meaning of this algebraic construction is easy to realize for special rings M.

If $M = GN$, the H^k are the homology groups of the universal covering complex with operators induced by the deck transformations. If $M = N$, and $\phi \sum \xi\gamma = \sum \xi$, the $\tilde{H}^k$ are the homology groups of A. The example of the torus shows that the homology groups of the complex U ($M = GN$) may vanish without the homology groups of a homomorphic chain system doing the same. If M is a group ring of a factor group G^* of G and ϕ is defined by a homomorphism $\phi G = G^*$, then $\bar{H}^k$ are the homology groups of a regular covering complex U^* of A with operators induced by the deck transformations of U^*. All these groups $\bar{H}^k$ are well known invariants of A, invariant also under continuous mappings of the point set complex A.

The characteristic subgroup ϕZ^k is of algebraic and geometric interest for $M=N$, $k=2$. The existence of this group gives the argument for the remarkable theorem of Hopf that the second homology group $\bar{H}^2$ of a complex is homomorphic to a group $H^2(G^1)$, which is determined by the fundamental group G^1 of A. We have seen that $\bar{H}^2 = \bar{Z}^2/\bar{B}^2$ is homomorphic to $\bar{Z}^2/\phi Z^2$ and the theorem will be proved if we show that $\bar{Z}^2/\phi Z^2$ is determined only by G^1. Now $\bar{Z}^2$, Z^2 are determined by the 2-dimensional subcomplex U^2 of U, and the U^2 is essentially determined by the generators and defining relations of the fundamental group G^1.

Without loss of generality we may assume that A has only one vertex a^0. Our torus complex gives an example for such a complex A. Then the 1-dimensional cells a_k^1 represent closed paths σ_k, which generate the free group S of paths.

Every 2-dimensional cell a_j^2 has a boundary path $\rho_j(\sigma)$, and the set $\rho_j(\sigma)$ is the set of the defining relations of G^1 in the generators σ_k.

Otherwise if we have these generators and defining relations we can construct A^2 and with help of A^2 in the usual way the universal covering complex U^2 of A^2. The complex U^2 is the Dehn group image of G^1 with the given generators and defining relations. And we see that $\bar{Z}^2/\phi Z^2$ is defined for every Dehn group image of G^1. It is easy to realize that $\bar{Z}^2/\phi Z^2$ does not change if we change the generators or the defining relations and so the theorem of Hopf is proved.

The cycles z^2 may be characterized as chains induced by spheres, as Hurewicz has shown, and as chains induced by identities of defining relations, that is, products of relations $\lambda\rho\lambda^{-1}$, which equal 1 in the free group S, as R. Peiffer has shown, and so we may understand the formal definition of the group $H^2(G^1)$ both geometrically and algebraically.

We may compute the boundary matrices of the 2-dimensional homotopy chain system of a group with generators and defining relations directly, without constructing U^2 and R. Fox has shown that a process of partial derivation defined for group rings of free groups is helpful for this task. He gets the boundary matrices r_{ij}^2 as the "Jacobian"

$$\overset{2}{r}_{ij} = \phi \frac{\partial \rho_i(\sigma)}{\partial \sigma_j},$$

ϕ being the projection of the free ring in the group ring of G^1.

Following Hopf, we can extend the construction of homotopy chain systems of a group without using a complex U in higher dimension. Let C^3 be the group of the

$$\overset{3}{c} = \sum x_i \overset{3}{v}_i, \qquad x_i \in GN,$$

where the

$$(\overset{3}{v}_i)^{\cdot} = \overset{2}{z}_i, \qquad \overset{2}{z}_i \in \overset{2}{Z},$$

are a set of generators of Z^2. Then there exists a subgroup Z^3 of relations z^3 in the generators v^3 or of cycles with $(z^3)^{\cdot} = 0$ and the process can be iterated.

The homotopy chain systems of dimension n of group G^1, which we get in that way, choosing successively sets of generators v^k, are equivalent up to the dimension $n-1$. So the groups

$$\bar{Z}^k/\phi Z^k$$

are the same for all homotopy chain systems of G^1 and define homology groups $H^k(G^1)$ of higher dimension adjoined to G^1.

The homotopy chain systems of G^1 are characterized as acyclic systems up to the dimension $n-1$, that means systems in which every cycle bounds or in which the bounding cycles generate the groups of cycles. And so we get the theorem of Hurewicz, that the homology groups of a complex A are determined by its fundamental group G^1 up to the dimension $n-1$, if the universal covering complex U is acyclic up to the dimension $n-1$.

Hurewicz has proved this theorem with the methods of his homotopy theory. Another interesting approach in a combinatorial way is made by Eilenberg and MacLane and a little later by Eckmann, who adjoin directly to a group a combinatorial complex U with a group G of mappings of U in itself, namely the complexes of ordered simplices the vertices of which are the group elements.

About M-chain systems of a group which are defined by a homomorphism $\phi G = G^*$, I mention that they give a natural approach to the metabelian groups and to the argument for the Haupt ideal theorem of the number theory; although this result of Schumann is not a new one, it may be of interest to see this connection between the chain systems and the groups themselves.

Now let us go back to the M-chain systems of a complex U.

If M is not a group ring we get a new type of chains, which we may define otherwise using locally groups of cells. Steenrod has extended this definition to singular chains using locally groups of points and has proved that the groups $\bar{H}^k$ also for this kind of chain are invariant under topological mappings.

But the homology groups do not characterize the equivalence classes of M-chain systems. There exists an invariant of certain M-chain systems, the torsion τ, which is not determined by its homology groups. We do not know if the torsion is invariant by topological mappings although we know that it is not a property of the homotopy classes of complexes. But you see the answer will be of interest in any case. And the classical question, if two complexes, which are topologically equivalent, are also equivalent by subdivision, will be answered in a negative sense, if the torsion is not a topological invariant. The torsion is defined, if M is a subring of a field. We get such rings if we map the element γ by ϕ in roots of the unit

$$\phi\gamma = e^{2\pi i k/\mu};$$

$\phi\gamma$ is a so-called character of G and M a subring of a cyclotomic number field. The basis transformations ϕT are linear transformations the determinant of which also equals a root unity, and we may ask for invariants of the boundary matrices if we admit all linear transformations of the field the determinant of which equals a root unity. Then we have to admit also new chains, namely chains with coefficients in the field and a boundary operator which commutes with the multiplication in the field. The homology groups of these chains "with division" are free groups and the torsion equals zero if one of these groups is not the identity.

If all homology groups equal the identity, then there exists a certain set of subdeterminants d^k of the boundary matrices which are not equal to zero, and the torsion is defined by

$$\tau = \prod \frac{d^{2l+1}}{d^{2l}}$$

and we have the theorem of Franz: Two systems of boundary matrices of a field are equivalent if and only if the torsion of the two systems is the same modulo a factor of a root unity.

These algebraic calculi get geometrical interest by the fact that the torsion gives a classification of the so-called lens spaces, which are useful to show that the classical topological invariants do not give a full classification of complexes. The lens spaces are defined as fundamental regions of a rotation γ of finite order μ which maps a sphere of dimension $2\nu+1$ in itself.

Let

$$y = \xi + i\eta, \qquad \bar{y} = \xi - i\eta$$

be complex numbers, so we get the $(2\nu+1)$-sphere by the equation

$$\sum_{k=0}^{\nu} y_k \bar{y}_k = 1$$

and a rotation γ of order μ by

$$y_k' = e^{2\pi i \lambda_k/\mu} y_k, \qquad (\lambda_k, \mu) = 1.$$

Then there exists a subdivision U of the sphere, which is mapped by γ in itself, and U defines by projection a complex A for which U is the universal covering complex. Then A is a complex of a lens space.

The fundamental group $G^1 = G$ of A is the cyclic group of order μ and we get a homomorphism of the group ring GN in a ring M of a number field by

$$\phi\gamma = e^{2\pi i/\mu_0}, \qquad \mu_0 \mid \mu, \ \mu_0 > 2.$$

The torsion of this M-chain system of A is

$$\tau(A, \mu_0) = \prod (1 - e^{2\pi i \lambda_k/\mu_0}) e^{2\pi i \lambda/\mu_0}$$

and we get a purely arithmetical problem, namely to consider the multiplication groups of numbers generated by

$$1 - e^{2\pi i \lambda/\mu_0}, \pm e^{2\pi i \lambda/\mu_0}, \qquad (\lambda, \mu_0) = 1,$$

and to determine the representations of the unit in these generators.

If μ is a prime number, this question is answered by the theorem that the circle units of Hilbert generate a free group. If μ is not a prime number we have to argue (following Franz) a generalization of this theorem using the L-series $L(s, \phi\gamma)$ adjoined to the characters $\phi\gamma$. In both cases we realize that if the equation

$$\tau(A, \mu_0) = \tau(A', \mu_0)$$

holds, the two sets of numbers $(\pm\lambda_k)$, $(\pm\lambda_k')$ are the same.

But then we go from A' to A, changing the enumerating of the λ (or the y) and changing some λ in $-\lambda$ (or some y in $\bar{y}$), and so it follows that lens spaces with the same torsion are equivalent under subdivision.

The fact that τ is not a property of the homotopy classes follows from the homology classification of the lens spaces, which L. W. Franz, J. H. C. Whitehead and Olum have given, the latter using a homotopy invariant defined by Eilenberg and MacLane.

You may foresee that it is possible to combine the notions of intersection, of cohomology and products of chains and cochains with the operator process γ. The development of the operator calculus in this direction given by Burger, Eckmann, Eilenberg, MacLane, Steenrod, I can only mention. Computing the duality properties of homotopy chain systems, we get for 3-dimensional manifolds a condition for the fundamental group G^1; if G^1 is abelian, G^1 is either cyclic or the free abelian group with 3 generators. Last but not least we can adjoin to a mapping f: $A \rightarrow A$ a class of mappings f_u: $U \rightarrow U$ and then a set of linear transformations of the homotopy chains. Thus we get a generalized Lefschetz trace formula and Wecken has proved that this trace invariant gives the exact minimal number of fixed points of a deformation class of the mappings f: $A \rightarrow A$, if the dimension of A is greater than 2.

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SOME CLASSES OF FUNCTIONS DEFINED BY DIFFERENCE OR DIFFERENTIAL INEQUALITIES[1]

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1. **Introduction.** In the year 1934[2] I introduced classes of functions which I called monotonic of order n. The class corresponding to $n=1$ represents the functions which are monotonic in the ordinary sense. The monotonic functions of higher order are closely connected with some distinguished functions of matrices whose independent and dependent variables are real symmetric matrices of the same order.

In a recent investigation I found that the study of monotonic functions can be subordinated to the consideration of more general classes of functions which have distinguished *group theoretical* properties. I wish to give an account of this study.

All functions considered are supposed to be *real functions of a real variable.* To avoid unessential complications, I shall consider them defined *in open intervals and continuous and strictly monotonically increasing there.* A class S of such functions will be called a *transformation semigroup* if it has the following properties:

(a) If $f(x)\in S$ and is defined in an interval (a, b), then $f(x)$, considered only in a subinterval $(a', b')\subset(a, b)$, should also belong to S.

(b) If $f(x)\in S$ and $g(x)\in S$ and the composition $g[f(x)]$ can be performed, that is, if the range of $f(x)$ falls into the domain of $g(x)$, then $g[f(x)]\in S$.[3]

(c) If $f(x)$ is a limit of functions $f_n(x)\in S$ $(n=1, 2, 3, \cdots)$, all functions of the sequence being defined in the same interval (a, b) and the convergence being uniform in any closed bounded subinterval of (a, b), then $f(x)\in S$. Condition (c) represents a closure condition on S.

In the following the expression *transformation group* shall mean a class of functions which in addition to (a), (b), (c) satisfy condition

(b′) If $f(x)\in S$, then the inverse $f^{-1}(x)\in S$.

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[1] The results of §§3 and 4 were obtained while the author was under contract with the Office of Naval Research N60NR248.

[2] Löwner, Karl, *Über monotone Matrixfunctionen,* Math. Zeit. vol. 38 (1934).

[3] Because we do not assume that any two functions of S may be composed, we might speak also of a semigroupoid. We prefer the simpler expression semigroup.

Several semigroups have been studied in the past that have their origin in important problems of analysis. For example, the class of *absolutely monotonic* functions from which only the constant functions have been excluded form a transformation semigroup. A function is called absolutely monotonic if all its difference quotients and the function itself are non-negative. This class remains a transformation semigroup if the condition of non-negativity of the function itself is dropped. After this modification the semigroup contains the whole group of linear functions $ax+b$ $(a>0)$. More extensive semigroups are obtained if the condition of non-negativity is restricted to the first m difference quotients. These semigroups also contain all linear functions mentioned above.

It seems to be of great interest to study the transformation semigroups which contain completely the group of all proper projective transformations

$$\frac{ax+b}{cx+d}, \qquad \begin{vmatrix} a & b \\ c & d \end{vmatrix} > 0. \tag{1}$$

(A function of (1) may be considered in any open interval not containing its pole.)

The interest in these transformation semigroups stems from the well known fact that this group cannot be extended to a Lie group of higher dimension. We shall, therefore, restrict ourselves to the study of those transformation semigroups which satisfy, besides conditions (a), (b), (c), the further condition

(d) S contains the whole group of proper projective transformations.

The strictly monotonically increasing functions of a given order n form a semigroup satisfying condition (d). It turns out that these semigroups have a distinguished place among the extensions of the projective group to a transformation semigroup. We shall consider them more in detail in the following section.

2. **Functions monotonic of order n.** I would like to start with a rather general generating principle of transformation semigroups: Suppose we operate in a set Σ of elements in which a partial order is defined. That is, it can be said that of two elements A and B of Σ the element A is before B, in symbols $A \subset B$. We assume that the order relation is transitive and reflexive. That is, from $A \subset B$ and $B \subset C$ it follows that $A \subset C$ and $A \subset A$ is true for any element A of Σ. Consider now a mapping

$$X' = F(X), \tag{2}$$

where the independent and dependent variables range over some sets of Σ. A mapping (2) will be called monotonic if $A \subset B$ has $F(A) \subset F(B)$ as consequence for any two elements A and B of the domain of (2) which may be compared in the sense of the given order relation.

Suppose now that there is given a homomorphism of the group of all continuous monotonically increasing functions and a group of mappings in Σ. By this the following is meant:

(α) To every interval $I=(a, b)$ there corresponds a set $\sigma(I)$ in Σ such that

$$I' \subset I \quad \text{implies} \quad \sigma(I') \subset \sigma(I).$$

(β) To a function $f(x)$ defined in I there corresponds a mapping $X'=F_f(X)$ with the domain $\sigma(I)$ and the range $\sigma(J)$ where J is the interval which is the image of I by the function $f(x)$.

(γ) If two functions $f(x)$ and $g(x)$ can be composed then

$$F_{g[f(x)]} = F_g[F_f(X)]. \tag{3}$$

Consider now only those functions $f(x)$ which lead to *monotonic* mappings $F_f(X)$. Conditions (α), (β), (γ) imposed on the homomorphism insure that these functions form a class satisfying conditions (a) and (b) for transformation semigroups. By adding concepts and conditions of topological nature we could insure also that condition (c) is satisfied, but we shall omit these considerations since we shall now restrict ourselves to the special case leading to monotonic functions of higher order where (c) is almost obvious.

Take as Σ the set of all real quadratic forms A in n variables $x_1, x_2, \cdots x_n$,

$$A = \sum_{\rho,\sigma=1}^{n} a_{\rho\sigma} x_\rho x_\sigma \qquad (a_{\rho\sigma} = a_{\sigma\rho}). \tag{4}$$

We may also identify A with with the symmetric matrix $(a_{\rho\sigma})$ of the coefficients. The following natural partial order may be introduced in Σ: We say that $A \subset B$ if

$$\sum_{\rho,\sigma=1}^{n} a_{\rho\sigma} x_\rho x_\sigma \leqq \sum_{\rho,\sigma=1}^{n} b_{\rho\sigma} x_\rho x_\sigma \tag{5}$$

for all values of the variables x_ι. One may also say that the form $B-A$ is non-negative definite. Consider now a function $f(x)$ defined in an open interval (a, b) and a quadratic form X whose characteristic values λ_i all fall into (a, b). As usual we mean by $f(X)$ that quadratic form which has the same characteristic vectors as X but the new characteristic values $f(\lambda_i)$. If, for example, $f(x)$ is a polynomial $f(x)$

$=\sum_{\rho=0}^{m} a_\rho x^\rho$, then $f(X)=\sum_{\rho=0}^{m} a_\rho X^\rho$. The domain of the matrix function $f(X)$ consists of all the symmetric matrices whose characteristic values fall into (a, b). If we now set

$$F_f(X) = f(X), \tag{6}$$

a correspondence $f \rightarrow F$ is obtained satisfying all the conditions defining a homomorphism. We obtain a definite semigroup S_n if we select those functions $f(x)$ which lead to a monotonic matrix function in the space of symmetric matrices of degree n. The functions of S_n are called *montonic of order n*.

Monotonicity of order 1 means evidently monotonicity in the ordinary sense. Further it is easily seen by considering reduced matrices

$$A = \begin{pmatrix} B_m & 0 \\ 0 & C_{n-m} \end{pmatrix},$$

where B_m and C_{n-m} are symmetric matrices of degrees m and $n-m$ respectively, that

$$S_n \subset S_m \qquad (m < n). \tag{7}$$

In other words, the monotonicity of order n implies monotonicity of any lower order.

The fundamental properties of monotonic functions of higher order will now be formulated in several theorems.[4]

THEOREM 1. *In order that a function $f(x)$ be monotonic of order n in the interval (a, b) it is necessary and sufficient that for any $2m$ values*

$$a < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_m < \eta_m < b \qquad (m \leqq n) \tag{8}$$

the inequality

$$\det \left| K(\xi_i, \eta_k) \right|_{i,k=1}^{m} \geqq 0 \tag{9}$$

holds, where $K(\xi, \eta) = (f(\xi) - f(\eta))/(\xi - \eta)$ represents the difference quotient of $f(x)$.

For $n = 2$ these conditions have a simple meaning. They say that if $f(x)$ is not constant it is strictly monotonically increasing and that for any four values $x_1 < x_2 < x_3 < x_4$ from (a, b) and the corresponding values $f(x_1) < f(x_2) < f(x_3) < f(x_4)$ the inequality

$$(f(x_1), f(x_2), f(x_3), f(x_4)) \geqq (x_1, x_2, x_3, x_4) \tag{10}$$

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[4] See paper [1] and the doctoral thesis of O. Dobsch, *Matrixfunctionen beschränkter Schwankung*, Math. Zeit., vol. 43 (1937).

holds. The expression $(u_1, u_2, u_3, u_4) = (u_1 - u_3)(u_2 - u_4)/(u_1 - u_4)(u_2 - u_3)$ designates as usual the cross-ratio of the four values u_1, u_2, u_3, u_4.

A simple consequence of this inequality is the fact that a function which, along with its inverse, is monotonic of order 2 must be a projective one.

From the fundamental Theorem 1, differential properties of the monotonic functions can be derived. They are expressed in the following four theorems.

THEOREM 2. *A monotonic function $f(x)$ of order $n>1$ is of class C_{2n-3} (that is, $(2n-3)$ times continuously differentiable) and its $(2n-3)$rd derivative is a convex function. The derivatives $f^{(2n-2)}(x)$ and $f^{(2n-1)}(x)$ exist, therefore, almost everywhere. For nonconstant $f(x)$ the first derivative $f'(x)>0$.*

In the following, by convergence of a sequence of functions it is understood that the functions are defined in the same interval (a, b) and that the convergence is uniform in any closed bounded part of (a, b).[5] With this in mind we formulate the following theorems.

THEOREM 3. *A monotonic function of order $n>1$ is the limit of a sequence of monotonic functions of order n all belonging to class C_{2n-1}.*

THEOREM 4. *A function $f(x)$ of class C_{2n-1} in (a, b) is monotonic of order n there if and only if the quadratic form*

$$\sum_{i,k=1}^{n} \frac{f^{(i+k-1)}(x)}{(i+k-1)!} \xi_i \xi_k \geqq 0 \tag{11}$$

for all x in (a, b).

THEOREM 5. *If a_i $(i=1, \cdots, 2n-1)$ are values such that the quadratic form*

$$\sum_{i,k=1}^{n} a_{i+k-1} \xi_i \xi_k > 0, \tag{12}$$

then there exists a monotonic function $f(x)$ of order n and of class C_{2n-1} such that at some preassigned point x_0

$$\frac{1}{i!} f^{(i)}(x_0) = a_i \qquad (i = 1, \cdots, 2n-1).$$

It is of great interest to give a characterization of the functions which are monotonic of *arbitrarily high order*. They also form a semigroup, say S_∞, which is contained in every S_n. A complete answer to this problem can be given by combining the preceding theorems with some results of the theory of moments. From Theorem 2 we conclude that a function belonging to S_∞ is differentiable of arbitrarily high order and (11) holds, therefore, for any n. From this we can derive by using results from the theory of moments the following theorem.

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[5] This convergence is sometimes called "normal" convergence.

THEOREM 6. *A (nonconstant) function $f(x)$ defined in (a, b) belongs to S_∞ if and only if it is analytic in (a, b), can analytically be continued into the whole upper half plane, and represents there an analytic function $f(z)$ of $z=x+iy$ whose imaginary part is positive. (Examples are given by the functions x^μ $(0<\mu\leqq 1)$ and $\log x$ for $x>0$.)*

It is natural to ask whether a still more restricted class of functions forming a semigroup may be obtained by considering, instead of quadratic forms in finite dimensions, *bounded self-adjoint operators in Hilbert space.* The order definition can evidently be taken over from the definition in finite dimensions. It is also well known how to define $f(X)$ if the spectrum of X falls into the domain where $f(x)$ is defined. But it turns out that *the functions $f(x)$ leading to monotonic functions in the operator space form the full semigroup S_∞.*

3. **Transformation semigroups containing the proper projective group which may be defined by differential inequalities.** The contents of Theorems 3, 4, and 5 suggest a new description of the semigroup of nonconstant monotonic functions of order n which brings it into closer connection with a Lie group.

If two functions $f(x)$ and $g(x)$ both belonging to C_m are combined, $f[g(x)]=h(x)$, then the derivatives of $h(x)$ up to order m can be computed from the derivatives of the composing functions $f(x)$ and $g(x)$, up to the same order. If, for $u=g(x)$,

$$\frac{f^{(i)}(u)}{i!} = a_i, \quad \frac{g^{(i)}(x)}{i!} = b_i, \quad \text{and} \quad \frac{h^{(i)}(x)}{i!} = c_i \qquad (i = 1, 2, \cdots, m)$$

then

$$\begin{aligned} c_1 &= a_1b_1, \\ c_2 &= a_1b_2 + a_2b_1^2, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ , \\ c_m &= a_1b_m + a_2\Big[\sum_{\alpha_1+\alpha_2=m}(b_{\alpha_1}b_{\alpha_2})\Big] + a_3\Big[\sum_{\alpha_1+\alpha_2+\alpha_3=m}(b_{\alpha_1}b_{\alpha_2}b_{\alpha_3})\Big] \\ &\quad + \cdots + a_mb_1^m. \end{aligned} \tag{13}$$

Equations (13) can be interpreted as a composition rule for ordered pairs of points $(a_1, a_2, \cdots, a_m)$ and $(b_1, b_2, \cdots, b_m)$ of the m-dimensional Euclidean space E_m. As is well known, this composition rule defines a Lie group l_m if only points whose *first* coordinate is *positive* are admitted.

By Theorems 4 and 5 a subset of l_{2n-1} is distinguished; namely, the set of points $(a_1, a_2, \cdots, a_{2n-1})$ of l_{2n-1} for which (12) is satisfied. Call its closure s_{2n-1}. From the named theorems we immediately derive the fact that the composition rule (13) applied to any pair of points of s_{2n-1} leads again to a point of s_{2n-1}. To facilitate the description of s_{2n-1} we introduce the *definition*: A subset of a topological group will be called a *subsemigroup* of it if it represents a closed set and if it is closed under the composition rule of the group.

We now can say that s_{2n-1} represents a subsemigroup of l_{2n-1}.

The transformation semigroup of monotonic functions of order n can be derived from s_{2n-1} by the following procedure: Consider all functions $f(x)$ of class C_{2n-1} such that the point

$$\left(\frac{f'(x)}{1!}, \frac{f^{(2)}(x)}{2!}, \cdots, \frac{f^{(2n-1)}(x)}{(2n-1)!}\right)$$

falls into s_{2n-1} for all x where $f(x)$ is defined. The transformation semigroup S_n consists now of all functions thus defined and their limits.

This characterization of S_n leads to a more general principle of construction of transformation semigroups. Take an arbitrary subsemigroup s of l_m. In the same way as from s_{2n-1} a class of functions can be derived from s by considering those $f(x)$ of class C_m such that

$$\left(\frac{f^{(1)}(x)}{1!}, \frac{f^{(2)}(x)}{2!}, \cdots, \frac{f^{(m)}(x)}{m!}\right) \tag{14}$$

always belongs to s, and adding to these functions their limits. It is easily seen that they form a transformation semigroup. We thus arrive at a method of constructing a transformation semigroup from an arbitrary subsemigroup of l_m. If, furthermore, all elements (14) derived from a transformation of this semigroup of class C_m belong to s, we shall say that *the semigroup is characterized by or belongs to* s.

We are particularly interested in transformation semigroups which contain the proper projective group. In order that the transformation semigroup characterized by a subsemigroup s of l_m have this property, it is easily seen that s must contain the elements

$$a_i = p\alpha^{i-1} \qquad (i = 1, \cdots, m) \quad (p > 0,\ \alpha \text{ arbitrary}). \tag{15}$$

We shall call the elements (14) the *projective elements of* l_m. *We require* *therefore that s contain all projective elements of l_n.*

The investigation of subgroups of a Lie group is essentially facilitated by the introduction of the infinitesimal viewpoint. It is a well known fact that a closed connected subgroup of a Lie group g is completely determined by its infinitesimal elements. It is, therefore, appropriate to ask whether a *subsemigroup* s of g can be connected in a similar way with a system of infinitesimal elements and also how far the subsemigroup is determined by them.

The definition of an infinitesimal element of s *if the unit element e of g is an accumulation point of s* can be introduced exactly in the same way as in the case of a subgroup: Take an arbitrary sequence of elements a_n of s different from the unit element e of g but converging to e. Any vector α at e whose direction is a limiting direction of the segments ea_n will be called an infinitesimal element of s.

According to this definition the following lemma holds.

LEMMA 1. *Any multiple $c\alpha$ $(c\geqq 0)$ of an infinitesimal element α of s is also an infinitesimal element of s.*

The next two lemmas on infinitesimal elements of a subsemigroup can be proved in the same manner as in the case of a subgroup.

LEMMA 2. *The sum of two infinitesimal elements of s is also an infinitesimal element of s.*

LEMMA 3. *The infinitesimal elements of s form a closed set.*

For a sub*group* multiplication of infinitesimal elements with arbitrary *real* factors is allowed. The infinitesimal elements of a subgroup form therefore a vector space. (Its dimension coincides with the dimension of the group.) This is not the case for a semigroup. But, as a result of our lemmas the following theorem holds.

THEOREM 7. *The infinitesimal elements of s form a closed convex set of vectors invariant under multiplication with an arbitrary positive factor.*

We shall call the totality of the infinitesimal elements of s the *infinitesimal cone* of s and designate it with σ_s.

A fundamental means of investigation of Lie groups is the generation of elements from infinitesimal elements. As was stated before, each connected subgroup of a Lie group can be completely generated from its infinitesimal elements. By the same methods used in Lie group theory the following theorem regarding *subsemigroups* can be proved.

THEOREM 8. *Each element of a Lie group l generated from infinitesimal* *elements of a subsemigroup s of l also belongs to s. The elements obtained in this way together with their limiting elements form again a subsemigroup $\bar{s}$ of l.*

In general $\bar{s}$ will not coincide with s as will soon be shown by examples.

The following further slight generalization of a well known theorem of Lie groups will shortly be used.

THEOREM 9. *Let l be a Lie group, s a subsemigroup of l, and a an element of l which with its inverse belongs to s. Then the transform $a\alpha a^{-1}$ of any infinitesimal element α of s also represents an infinitesimal element of s.*

After stating these general theorems we return to the special group l_m. We are interested in finding all subsemigroups s of l_m satisfying condition (a): *s shall contain all projective elements of l_m.*

The cases $m=1$ and $m=2$ are of no interest since there evidently do not exist proper subsemigroups of this kind. We assume therefore from now on that $m\geqq 3$.

A slight restriction will further be added resulting from the fact that we are interested only in semigroups leading to transformation semigroups which are *proper* extensions of the proper projective group. This has as consequence that there must exist an element $a=(a_1, a_2, \cdots, a_m)$ of s such that (a_1, a_2, a_3) is not a projective element of l_3. Otherwise each function $f(x)$ of class C_3 belonging to the corresponding transformation semigroup would satisfy the differential equation $f'f'''/3!=(f''/2!)^2$ whose solutions are the projective transformations. In order to obtain *proper* extensions of the projective group we have therefore to introduce the condition (b): *s shall contain elements such that*

$$a_1a_3 - a_2^2 \neq 0. \tag{16}$$

A first remark about the structure of s is contained in the following easily proved theorem.

THEOREM 10. *A subsemigroup s of l_m satisfying conditions* (a) *and* (b) *represents a connected subset of l_m.*

Let us now consider the infinitesimal cone σ_s of s.

An infinitesimal element of l_m is represented by a vector $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_m)$ in m dimensions. In particular the infinitesimal elements of the group of the projective elements of l_m have the form

$$\pi = (\alpha_1, \alpha_2, 0, 0, \cdots, 0) \qquad (\alpha_1, \alpha_2 \text{ arbitrary}). \tag{17}$$

The following important theorem can now be derived.

THEOREM 11. *At least one of the elements*

$$\omega = (0, 0, 1, 0, \cdots), \qquad -\omega = (0, 0, -1, 0, \cdots) \tag{18}$$

belongs to the infinitesimal cone σ_s *of* s.

The proof is obtained by use of suitable transforms of an element satisfying condition (16) with projective elements.

Assume now, for example, that σ_s contains ω. New elements of σ_s can now be obtained by use of Theorem 9, which implies also that any transform $p^{-1}\omega p$ of ω with an arbitrary projective element p must belong to σ_s. With the aid of Lemmas 1 and 2 we may say that all sums of the form

$$\sum_{\rho=1}^{N} p_\rho^{-1} \omega p_\rho, \tag{19}$$

where the p_ρ ($\rho = 1, 2, \cdots, N$) represent arbitrary projective elements, are elements of σ_s.

If the computation is performed one finds that the first two components of (19) are zero and the following $(m-2)$ components are the first $(m-2)$ moments of N non-negative masses. According to an important result of the moment problem these systems of numbers and their limits can be characterized by the non-negativity of the quadratic form

$$\sum_{\rho,\sigma=1}^{[(m-1)/2]} \alpha_{\rho+\sigma+1}\xi_\rho\xi_\sigma. \tag{20}$$

To the elements thus obtained we can add an arbitrary projective element (17) without leaving σ_s. Call the totality of the elements obtained τ_m.

If we had started with $-\omega$ instead of ω, we would have obtained the result that the set $-\tau_m$ of elements making (20) nonpositive belongs completely to σ_s. We arrive therefore at the following theorem.

THEOREM 12. *The infinitesimal cone* σ_s *of a subgroup* s *of* l_m *satisfying conditions* (a) *and* (b) *must completely contain either* τ_m *or* $-\tau_m$.

We now make the important observation that τ_m and $-\tau_m$ are themselves the infinitesimal cones of subsemigroups of l_m satisfying conditions (a) and (b). To show this we go back to the inequalities (12) characterizing the monotonic functions of order n. It easily leads to:

THEOREM 13. *The infinitesimal cone of the semigroup defining the* *monotonic functions of order n consists of the infinitesimal elements of* l_{2n-1} *that make form* (20) *non-negative.*

Theorem (13) proves our assertion for odd values of m. For even dimension the proof is furnished by the following theorem.

Theorem 14. *If m is an even number, the elements* $(a_1, a_2, \cdots, a_m)$ *which can be extended to elements* $(a_1, a_2, \cdots, a_{m+1})$ *belonging to* s_{m+1} *form a semigroup themselves and the corresponding infinitesimal cone is* τ_m.

Theorems 12, 13, and 14 lead now to the fundamental theorem.

Theorem 15. *The infinitesimal cones* τ_m *and* $-\tau_m$ *play a distinguished role among the infinitesimal cones of subsemigroups of* l_m *satisfying conditions* (a) *and* (b). *This role is expressed by the minimum property: Each of these cones contains either* τ_m *or* $-\tau_m$.

Theorem 15 can be sharpened by:

Theorem 15'. *If an infinitesimal cone* σ *satisfying conditions* (a) *and* (b) *contains both cones* τ_m *and* $-\tau_m$, *then it consists of all infinitesimal elements of* l_m.

The proof follows almost immediately from the fact that τ_m contains m linearly independent elements. If both τ_m and $-\tau_m$ belong to σ, any linear combination with arbitrary *real* coefficients of these elements must belong to σ.

It is possible to describe a procedure by which all infinitesimal cones of subsemigroups of l_m satisfying conditions (a) and (b) can be constructed: Take an arbitrary infinitesimal element α such that $\alpha_3 \neq 0$ and form all transforms

$$p\alpha p^{-1} \tag{21}$$

with projective elements p of l_m. Next, form the least closed convex set containing all elements (21) and all infinitesimal projective elements. Call it $C(\alpha)$. Then the following theorem holds.

Theorem 16. *The sets* $C(\alpha)$ *are infinitesimal cones of subsemigroups of* l_m *satisfying conditions* (a) *and* (b). *One obtains the most general such cone belonging to a proper subsemigroup of* l_m *by forming the least convex closure of any union of cones* $C(\alpha)$ *belonging to elements* α *whose third components* α_3 *have the same sign.*

After the possible infinitesimal cones have been described, the problem remains to determine the subsemigroups to which they belong.

As was already mentioned before, the problem of constructing semigroups from their infinitesimal cones presents greater difficulties than the corresponding problem for *subgroups* of Lie groups. This is due to the fact that two different subsemigroups of a Lie group, even if they are represented by *connected* sets, may have the same *infinitesimal* cone. *This is already true of our* τ_m, *to which our primary interest is directed, for* $m \geqq 5$.

According to Theorem 8, there exists for each infinitesimal cone, σ_s, a least subsemigroup $\mathfrak{s}(\sigma_s)$. It consists of the elements of the Lie group under consideration that can be generated by the elements of σ_s, and their limiting elements. If we now compare the subsemigroups s_m of l_m derived from the monotonic functions of higher order with the least semigroup $\mathfrak{s}(\tau_m)$ having the same infinitesimal cone τ_m, then the result is obtained: For $m=1, 2, 3, 4$, $\mathfrak{s}(\tau_m)=s_m$. For $m \geqq 5$, $\mathfrak{s}(\tau_m)$ is a proper subsemigroup of s_m.

It can be shown that the transformation semigroup T_m generated by $\mathfrak{s}(\tau_m)$ is *characterized* by $\mathfrak{s}(\tau_m)$. This leads to the following theorem.

THEOREM 17. *Among the transformation semigroups which are proper extensions of the projective group and which are characterized by subsemigroups of* l_m, *there exist two distinguished ones,* T_m *and* T'_m, T'_m *consisting of the inverse transformations of* T_m, *having the minimum property: Each such semigroup contains either* T_m *or* T'_m *completely.*

4. Construction of minimal transformation semigroups including the proper projective group as proper subgroups. The discovery of the minimum property of the semigroups T_m leads us to the following conjecture:

Among the proper extensions of the proper projective group to a transformation semigroup there exist two distinguished ones (call them T and T′, T′ being formed by the inverses of the transformations of T) characterized by the following property: Each such transformation semigroup contains either T or T′ completely. If it contains nonprojective transformations from both T and T′, then it is the group of all continuous transformations.

I could verify this conjecture under the restrictive assumption that only transformation semigroups are considered that contain also nonprojective transformations of class C_3.

T can be characterized in the following way: It consists of those monotonic transformations of arbitrarily high order which can be analytically continued into the total upper half plane and yield schlicht mappings of the latter into itself.[6]

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[6] The proofs of the theorems of §§3 and 4 will be given in another paper.

THE APRIL MEETING IN OAK RIDGE

The four hundred fifty-sixth meeting of the American Mathematical Society was held at Oak Ridge, Tennessee on Friday and Saturday, April 21–22, 1950, in conjunction with a meeting of the Biometric Society. The total attendance at the meeting was one hundred fifty-three, including the following eighty-seven members of the American Mathematical Society:

G. E. Albert, R. E. Allan, Herman Betz, R. H. Bing, M. G. Boyce, A. T. Brauer, R. C. Bullock, P. B. Burcham, Robert Butz, C. L. Carroll, C. E. Clark, J. M. Clarkson, A. C. Cohen, J. A. Cooley, R. R. Coveyou, J. C. Currie, L. J. Derr, N. M. Dismuke, W. L. Duren, Patrick DuVal, E. L. Eagle, E. D. Eaves, M. E. Estill, G. M. Ewing, F. A. Ficken, H. A. Fisher, Tomlinson Fort, G. A. Garrett, J. R. Garrett, Wallace Givens, A. W. Goodman, E. H. Hadlock, J. C. Harden, O. G. Harrold, I. R. Hershner, A. K. Hinds, A. S. Householder, S. T. Hu, G. B. Huff, Ralph Hull, R. B. Johnson, A. M. Jones, V. L. Klee, F. W. Kokomoor, G. B. Lang, C. G. Latimer, H. L. Lee, R. J. Levit, F. A. Lewis, Z. L. Loflin, Nathaniel Macon, W. R. Mann, J. M. Marr, E. P. Miles, D. D. Miller, Lewis Nelson, O. M. Nikodým, W. V. Parker, C. R. Partington, C. G. Phipps, E. E. Posey, Anatol Rapoport, L. T. Ratner, E. K. Ritter, M. Ethyl Rivers, J. H. Roberts, L. V. Robinson, Walter Rudin, W. A. Rutledge, J. W. Sawyer, E. B. Shanks, Seymour Sherman, Annette Sinclair, M. B. Sledd, C. B. Smith, W. S. Snyder, C. T. Taam, W. B. Temple, J. M. Thomas, W. R. Utz, A. D. Wallace, J. A. Ward, W. M. Whyburn, Ernest Williams, R. L. Wilson, E. A. Winter.

At 8:00 P.M. on Friday evening, by invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings, addresses were delivered by Professor Alfred Brauer, University of North Carolina, and Professor W. V. Parker, Alabama Polytechnic Institute. The title of Professor Brauer's address was *Criteria for the irreducibility of algebraic equations*, and that of Professor Parker, *Characteristic roots and field of values of a matrix*. Professor Wallace Givens presided during these addresses.

Four sessions were held for contributed papers: Friday, 10:30 A.M., Dr. A. S. Householder presiding; Friday, 2:00 P.M., Professor O. G. Harrold presiding; Saturday, 9:30 A.M., Professor Ralph Hull presiding; and Saturday, 11:15 A.M., Professor C. G. Latimer presiding.

The Biometric Society held a session at 4:00 P.M. on Friday at which the following addresses were presented: *The thermodynamics and steady-state kinetics of some biological systems* by Professor J. Z. Hearon, *Probabilistic theory of neural nets* by Professor Anatol Rapoport, *Mathematical basis of the interpretation of isotope experiments* by Dr. C. W. Sheppard.

Abstracts of papers presented at the meeting are given below. Papers read by title are indicated by the letter "*t*." Paper number 278 was read by Mr. Hinds, and paper number 284 was read by Professor Roberts. Mr. Hopper was introduced by Professor Wallace Givens, Mr. Butz by Professor Tomlinson Fort, Messrs. Conner and Keesee by Mr. R. E. Allan, and Mr. Derr by Professor A. D. Wallace.

Algebra and Theory of Numbers

260. R. K. Butz: *On 2-commutative matrices.*

The methods used by W. V. Parker for writing the matrix X such that $AX=XB$ (Bull. Amer. Math. Soc. Abstract 55-7-287) are extended to solve the problem of finding a matrix X with respect to which a given matrix A, with elements in any field F, is 2-commutative. When A is in rational canonical form and X is partitioned properly, these methods display X explicitly in terms of parameters. (Received March 9, 1950.)

261*t*. Leonard Carlitz: *A theorem on higher congruences.*

This note contains a proof of the theorem that if the congruence $f(x)\equiv 0 \pmod{p^r}$ is solvable for $r=\delta+1$, where p^δ is the highest power of p dividing the discriminant of $f(x)$, then the congruence is solvable for all r. While the theorem is contained in a more general theorem of Hensel a direct proof seems of interest. (Received March 6, 1950.)

262*t*. Leonard Carlitz: *Some special functions connected with $GF(p^n, x)$.*

This paper is concerned with the properties of the function $g(u, t) = \sum_0^\infty \psi_i(u)\phi_i(t)/F_i$ defined in Duke Math. J. vol. 15 (1948) p. 1008, and some related functions. (Received March 6, 1950.)

263*t*. Leonard Carlitz: *Sums of kth powers in $GF[p^n, x]$.*

In this paper an asymptotic formula is obtained for the number of solutions of $M=\alpha_1X_1^k+\cdots+\alpha_sX_s^k$, where $\alpha_j\in GF(p^n)$; M, $X_j\in GF[p^n, x]$, deg $M\leqq mk$, deg $X_j=m$, X_j primary, $j\leqq l$; deg $X_i<m$, $i>l$, and $1\leqq l\leqq s$; finally $s\geqq s_0(k)$. The method is that developed in Duke Math. J. vol. 14 (1947) pp. 1121–1137; vol. 15 (1948) pp. 795–801. The case $l=0$ is excluded. (Received March 6, 1950.)

264. J. C. Currie: *Unitary-canonical matrices.* Preliminary report.

It is known (I. Schur, *Über die charakteristischen Wurzeln einer linearen Substitution*, Math. Ann. vol. 66 (1909) pp. 488–510) that for any matrix A there exists a unitary matrix U such that $UA\bar{U}'$ is triangular. However, the matrix U which performs this transformation is not unique, and the resulting triangular form is not unique. In this note it is established that from the set of matrices which are unitary transforms of a given matrix A a particular triangular matrix can be designated as the canonical representative of the set. Let the characteristic roots of A be ordered so that if m_j is the nullity of λ_jI-A, then $m_j\geqq m_{j+1}$, $j=1, \cdots, n-1$. Then A is unitarily transformed into a triangular matrix in which square submatrices of the form λ_jI_j (where I_j

is of order m_i) appear in the principal diagonal. Additional zeros are introduced above the principal diagonal in an orderly procedure such that the end result is a triangular matrix with a maximum number of zeros above the principal diagonal, the zeros being systematically placed so that the form is unique for a given A. (Received March 6, 1950.)

265. J. R. Garrett: *Normal equations and resolvents in fields of characteristic p.*

The general equation of degree not less than 5 with coefficients in a field F of characteristic p is reduced to the so-called *principal* and *normal* forms by means of Tschirnhaus transformations. In addition, a sextic resolvent is obtained for the normal quintic when $p=2$, 3, and 5. For corresponding results when F is of characteristic zero see Dickson's *Modern algebraic theories*, chap. 12. (Received March 6, 1950.)

266. Wallace Givens: *A Hadamard bound theorem for the determinant of a real quaternion matrix.*

The determinant of a regular matrix A of order n with elements in a division ring K is defined by Dieudonné [Bull. Soc. Math. France vol. 71 (1943) pp. 27–45] to be an element $\Delta(A)$ of the quotient group of the multiplicative group of nonzero elements of K with respect to its commutator subgroup. If one specializes to real quaternions, the group of determinantal values is isomorphic to the non-negative real numbers (with $\Delta(A)=0$ if and only if A has no inverse). Normalizing by requiring the determinant of a diagonal matrix to be the square root of the products of the norms of its elements (which gives the square root of Study's determinant) the Hadamard bound theorem is proved in the form: $\Delta(A) \leqq n^{n/2}\alpha^n$, where the elements of A are real quaternions a_{ij} such that Norm $(a_{ij}) \leqq \alpha$. The stated bound is shown to be attained by specializing the a_{ij} to be complex numbers. If the bound can also be attained by a matrix every element of which is one of the sixteen numbers $(\pm 1 \pm i \pm j \pm k)/2$, then the Hadamard bound for matrices of order $4n$ is also attained by a real matrix. (Received March 7, 1950.)

267. E. H. Hadlock: *A method of construction of a properly primitive form.*

A method is derived and stated, whereby, given the integral invariants Ω, Δ and possible sets of progressions involving 2 and the odd prime factors of Ω and Δ, the integral coefficients a, b, c, r, and S are determined in the properly primitive form $f=ax^2+by^2+cz^2+2ryz+2Sxz$ with determinant $d=\Omega^2\Delta\neq 0$ so that f will have associated with it the given invariants and sets of progressions. (Received March 6, 1950.)

268. E. H. Hopper: *The decision problem in the free modulator lattice.*

Let $FL[4]$, $FM[4]$, and $FD[4]$ be the free, the free modular, and the free distributive lattice, respectively, on four generators. Polynomials equal in $FL[4]$ are necessarily equal in $MF[4]$ and $FD[4]$ while inequality in $FD[4]$ implies inequality in the other two lattices. The unsolved decision problem in $FM[4]$ is thus reduced to a question of the equality in $FM[4]$ of pairs of polynomials having equal canonical forms in $FD[4]$ and unequal canonical forms in $FL[4]$. This method is used to obtain canonical forms in $FM[4]$ for polynomials of length not greater than 5. Inclusion relations between these polynomials are also determined. (Received March 6, 1950.)

269*t*. Ralph Hull: *An algebraic proof of a theorem on the full linear group.*

The theorem is: every rational integral representation of the full linear group of dimension n is either homogeneous or else is expressible as a sum of homogeneous rational integral representations. (Cf. Murnaghan, *The theory of group representations*, Johns Hopkins Press, 1938, p. 45.) (Received March 20, 1950.)

270. R. J. Levit: *Divided differences and some extremal properties of functions on finite sets.*

Let A_n be a finite set consisting of elements $a_0, \cdots, a_n$ of an ordered field Γ_ω, and consider functions $f(x)$ on A_n with values in Γ_ω. For any nonzero element $c \in \Gamma_\omega$ let Φ_{nk} be the class of all functions $f(x)$ such that $f(a_0, \cdots, a_n) = c$, where $f(a_0, \cdots, a_n)$ denotes the nth divided difference of $f(x)$ with respect to the indicated arguments. This paper has the dual purpose of deriving certain formal algebraic properties of divided differences and then applying them to the determination of the function of Φ_{nk} which deviates least from zero on A_n and also the functions whose divided differences of any given order $k \leqq n$ deviate least from zero on A_n. (Received March 8, 1950.)

271. Nathaniel Macon: *Some theorems on the approximation of irrational numbers by the convergents of their continued fractions.*

Let ξ be any positive irrational number. Denote by A_n/B_n the convergents of its expansion as a regular continued fraction, and set $|\xi - A_n/B_n| = 1/\lambda_n B_n^2$. Recently Brauer and Macon (Amer. J. Math. vol. 71 (1949) pp. 349–361 and vol. 72 (1950)) obtained the best possible lower bound for the sum of k consecutive λ_n, for every k. In this paper, the best possible lower bounds for the products of two, three, and four consecutive λ_n are obtained. For $k=3$, this generalizes a well known theorem of Borel (J. Math. Pures Appl. (5) vol. 9 (1903) pp. 329–375). Moreover, Brauer and Macon proved that in any sequence of five consecutive λ_n either at least two are greater than $5^{1/2}$ or one is greater than 3. They obtained similar results for every sequence of $3m+2$ consecutive λ_n. These results are now improved for eight and eleven consecutive λ's. (Received March 6, 1950.)

272*t*. Leo Moser: *On sets of integers which contain no three in arithmetical progression.*

Let $g(n)$ be the maximum number of integers under n, no three of which form an arithmetical progression. F. A. Behrend (Proc. Nat. Acad. Sci. U.S.A. (1946) pp. 331–333) showed that $g(n) > n^{1-c/(\log n)^{1/2}}$, where c is a fixed constant. In Behrend's method the existence of appropriate sets is proved, but these differ for different n. An infinite sequence is here constructed, which has no three elements in arithmetical progression, and which has under n at least $n^{1-c/(\log n)^{1/2}}$ integers. Upper bounds for $g(n)$ were given by P. Erdős and P. Turán (Proc. London Math. Soc. (1936) pp. 261–264). Their results are here improved to $g(n) < 4n/11+5$. (Received March 6, 1950.)

273. O. M. Nikodým: *Remarks on the cartesian product of Boolean measure algebras.*

Kakutani's theorem, stating the possibility of extension of the finitely additive measure of the cartesian product of denumerably additive σ-fields of sets provided with denumerably additive measures to a denumerably additive measure on the Borelian extension of this product, can be generalized to abstract Boolean measure algebras. Circumstances relating to this extension are discussed. (Received May 8, 1950.)

274. E. B. Shanks: *An elementary proof of the fundamental theorem of algebra.*

Let $N(t)$ be the net number of complete revolutions made by a line segment (of variable length) joining the origin to the point representing $f(z)=z^n+a_1z^{n-1}+\cdots+a_n$ (where n is a positive integer and $a_n\neq 0$) when z traces out the circle $|z|=t$ one time. It is shown (1) if $t=t_1$ is sufficiently small, then $N(t_1)=0$; (2) if $t=t_2$ is sufficiently large, then $N(t_2)=n$ (this is proved in *What is mathematics?*, Courant and Robbins, Oxford University Press, New York, 1941, p. 271); (3) if $N(t)$ is not a constant for all t such that $0\leqq T_1\leqq t\leqq T_2$, then $f(z)=0$ for some z in the annular region $0\leqq T_1\leqq |z| \leqq T_2$. It follows from these three lemmas that $f(z)=0$ for some z in the annular region $0\leqq t_1 \leqq |z| \leqq t_2$. (Received March 3, 1950.)

275. J. A. Ward: *A function of k constants.*

Let $C_1, C_2, \cdots, C_k$ be an arbitrary set of k constants over a field F, with $C_j=0$ for $j>k$. For $p\geqq 0$ and n integers the function $W_{p,n}$ is defined as follows: (1) For $n>0$, $W_{0,n}=\sum C_{i_1}C_{i_2}\cdots C_{i_t}$ where $\sum$ denotes the sum of products obtained by all permutations of all sets of i's $(i=1, 2, \cdots, k)$ such that $i_1+i_2+\cdots+i_t=n$; (2) for $n>0$, $W_{p,n}$ is defined to be the result obtained from $W_{0,n}$ by increasing by p the first subscript in each term of $W_{0,n}$; (3) if $n\leqq 0$, $W_{p,n}=\delta_{p,-n}$. It then follows for all i and n that $C_iW_{0,n}+W_{i,n}=W_{i-1,n+1}$; and other similiar relations are derived. Typical applications of the $W_{p,n}$ are: If θ satisfies the equation $X^k-\sum_{i=1}^{k}C_iX^{k-i}=0$, then for all $j\geqq 0$, $\theta^j=\sum_{t=0}^{k-1}W_{t,j-k+1}\theta^{k-1-t}$. Also if A is the companion matrix of the above equation, then for all $i\geqq 0$, $A^i=(W_{k-r,i+s-k})$, where r and s are the row and column indices, respectively. (Received March 8, 1950.)

ANALYSIS

276. W. L. Duren: *Extension of the Daniell-Stone integral to vector-valued functions.*

M. H. Stone in Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) has redeveloped the Daniell general integral arising from a given elementary integral. Stone's treatment, in common with other treatments of the Daniell integral, develops the theory for real-valued functions. The purpose of the present paper is to show that the Stone theory may be extended to functions whose values lie in a Banach space. The additional restrictions required for this extension are that, for functions f in the elementary class and their elementary integrals $E(f)$, $\|E(f)\|\leqq\int\|f(x)\|dx$, the latter integral being required to exist as a general integral of a real-valued function. (Received March 9, 1950.)

277. F. A. Ficken: *Severely singular integral operators and Hölder continuity.* Preliminary report.

In discussing second order partial differential equations of elliptic type, one may encounter integral operators of the form $u \rightarrow Qu = \int q(R, R')u(R')dv_{R'}$, where R and R' are position vectors of points in n-dimensional Euclidean space E_n, $dv_{R'}$ is the element of volume at R', and the integration is extended over E_n. The kernel $q(R, R')$ and the operator Q are said to be severely singular if $q(R, R') = q(R-R')$, where $q(R)$ is continuous for $R \neq 0$ and behaves like $|R|^{-n}$ as $R \rightarrow 0$ or $R \rightarrow \infty$. It is assumed further that: (i) if $|R| \leqq |R'|$, then, for some positive constant K, $|q(R')-q(R)| \leqq K|R|^{-n}|R'|^{-1}|R'-R|$ and (ii) $\int q(R)dv_R = 0$ if the integral is extended through any spherical shell $0 < a \leqq |R| \leqq b$. As a special case, $q(R-R')$ has these properties if it is the second derivative of $|R-R'|^{-n+2}$. With a view to iterative methods it is useful to know, under these circumstances, that if u is Hölder continuous, then Qu is Holder continuous. The object of the present note is to give for this fact a proof believed to be simpler than those now available. Simplification results mainly from the observation that $\int [q(R'-S)-q(R-S)]dv_S$, extended over a half-space $|X-R| > |X-R'|$, exists and has an upper bound that is independent of R and R'. (Received March 16, 1950.)

278. A. K. Hinds and W. M. Whyburn: *Characteristic sets for a non-self-adjoint second order differential system.*

The second order differential system $y' = K(x, \lambda)z$, $z' = G(x, \lambda)y$, $\psi(a, \lambda) = 0$, $\phi(a, \lambda) = \phi(b, \lambda)$, $\psi(x, \lambda) = \gamma(x, \lambda)z(x, \lambda) - \delta(x, \lambda)y(x, \lambda)$, $\phi(x, \lambda) = \alpha(x, \lambda)z(x, \lambda) - \beta(x, \lambda)y(x, \lambda)$, is examined for existence and and oscillation theorems. Under mild hypotheses on the coefficients, it is shown that there exist sets $S_1, S_2, \cdots, S_n$ of characteristic lambda values such that the given differential system is compatible for each lambda in these sets Furthermore, the sets are ordered in the sense that any number in S_i is less than any number in S_j when $i < j$. Oscillation theorems are established and separation theorems are developed for solution functions associated with two characteristic numbers that belong to the same S_i. (Received February 24, 1950.)

279*t*. J. D. Mancill: *Identically non-regular problems in the calculus of variations.*

This paper is concerned with simple problems of the calculus of variations whose integrand $f(x, y, y')$ satisfies the identity $f_{y'y'}(x, y, y') \equiv 0$ in a closed region R of the xy-plane and for all finite values of y', from which it follows that $f(x, y, y') \equiv P(x, y) + y'Q(x, y)$. Since Euler's condition for this problem has the form $P_y = Q_x$, either it is an identity in R and no proper minimum exists if the *end points are fixed*; or Euler's equation is not an identity and is either never satisfied or is satisfied only along certain paths. In the latter case, given two points of the region R there will not in general exist an *extremal* joining these points. For this reason one is likely to assume that this problem contains little of interest. That such is not the case follows, in the author's opinion, from the very definite properties of a curve which does furnish a minimum. Also, when the end points are on the boundary of R, the necessary conditions are more generally satisfied. Furthermore, this whole problem is an excellent example in contrast to the well known results for regular problems. A minimum cannot exist for the analogous problem in parametric representation unless $P_y = Q_x$ identically. (Received March 9, 1950.)

280t. K. S. Miller: *A Sturm-Liouville problem associated with iterative methods.*

The problem considered in this paper is that of estimating rates of convergence in solving linear ordinary differential equations by an iterative process. Consider a linear differential operator M of the $(n-2)$th order and associated two point boundary conditions $U_\alpha(u)$, $\alpha=1, 2, \cdots, n-2$, such that the totally homogeneous system $Mu=0$, $U_\alpha(u)=0$ is incompatible. Then a sequence of iterants $u_0(x)$, $u_1(x)$, $u_2(x)$, $\cdots$ is constructed such that $u_m(x)$ converges to a solution of the differential system as m tends to infinity. Let $Mu_m(x)=\epsilon_m(x)$. Then it is shown that there exists a transformation Q, whose inverse is a Sturm-Liouville system, such that $\epsilon_m(x)=Q\epsilon_{m-1}(x)$. This problem is solved yielding a complete orthonormal set of functions $\{\psi_n(x)\}$ and associated characteristic number $\{\Lambda_n\}$. It is then shown that $\epsilon_0(x)=\sum a_\alpha \psi_\alpha(x)$ where all the $\psi_\alpha(x)$ appearing in the sum have characteristic numbers Λ_α such that $|\Lambda_\alpha|<1$. Hence $\|\epsilon_m(x)\|^2=\sum a_\alpha^2 \Lambda_\alpha^{2m}$ and the rapidity of convergence can be determined explicitly without actually calculating the $u_\alpha(x)$. (Received February 27, 1950.)

281. L. V. Robinson: *Solution of linear second-order partial differential equations by operator methods.*

By making use of the differential operator, $\lambda\equiv p_1(x_1, x_2, \cdots, x_n)\partial/\partial x_1+p_2(x_1, x_2, \cdots, x_n)\partial/\partial x_2+\cdots+p_n(x_1, x_2, \cdots, x_n)\partial/\partial x_n$, it is shown how second-order linear partial differential equations can be solved. Conditions necessary for the solution of such equations are also derived. (Received March 9, 1950.)

282. W. S. Snyder: *A characterization of set functions with derivative zero a.e.*

A necessary and sufficient condition is derived for a set function to have derivative 0 a.e. Let f be real-valued and defined on an arbitrary class Γ of closed sets lying in a bounded region of the plane. $\mathcal{E}$ denotes generically a finite class of disjoint sets of Γ, and $\dot{\mathcal{E}}$ is the point set they cover. $\|\mathcal{E}\|=\max\{\text{diameter } T\}$ for $T\in\mathcal{E}$. The derivative in question $D(f, x)=\lim_n f(\mathcal{E}_n)/|\dot{\mathcal{E}}_n|$ where $f(\mathcal{E}_n)=\sum f(T)$ for $T\in\mathcal{E}_n$ and $\{\mathcal{E}_n\}$ is any sequence such that $\{\dot{\mathcal{E}}_n+x\}$ is regular, and diameter $\{\dot{\mathcal{E}}_n+x\}\to 0$. A necessary and sufficient condition that $D(f, x)=0$ a.e. is that given $\epsilon>0$ there exists a $\delta>0$ such that $\|\mathcal{E}\|<\delta$ implies the existence of a subclass $\mathcal{E}_1\subset\mathcal{E}$ for which $|\dot{\mathcal{E}}-\dot{\mathcal{E}}_1|<\epsilon$ and $|f|(\mathcal{E}_1)<\epsilon$. When $\overline{D}(f, x)$ can be defined a.e., a necessary and sufficient condition that $D(f, x)$ exist and be summable is that $f(T)=g(T)+h(T)$, where $g(T)=\int_T D(g, x)$ and $D(h, x)=0$ a.e. In particular, if f is B.V. and $D(f, x)$ exists a.e., then $D(f, x)$ is summable and the previous decomposition generalizes the Lebesque decomposition without additivity for f. (Received March 6, 1950.)

APPLIED MATHEMATICS

283. W. R. Mann: *On a class of nonlinear integral equations of the Volterra type.* Preliminary report.

Certain nonlinear boundary value problems arising in the study of heat transfer and diffusion phenomena can be expressed in terms of nonlinear integral equations of the form $y(t)=\int_0^t K(t, \tau)G[y(\tau)]d\tau$ where G is continuous and strictly decreasing in y, and $K(t, \tau)$ is positive and increasing in τ for all t. Using Schauder's generalization of the Brouwer fixpoint theorem to Banach spaces, it is easily shown that this equation has a solution, $y(t)$, for all $t \geqq 0$. From the proof of this existence theorem, important bounds for $y(t)$ can also be derived. If one assumes that G satisfies a Lipschitz condition and that the nth iterated integral of $K(t, \tau)$ with respect to τ from 0 to t tends to zero sufficiently rapidly for all t, as $n \to \infty$, a set of successive approximations $\{y_n(t)\}$ is defined which have the important property that for all $t > 0$, $y_1 > y_3 > y_5 > \cdots > y_{2n+1} > \cdots > y(t)$, and $y_0 < y_2 < y_4 < \cdots y_{2n} < \cdots y(t)$. From these inequalities, a good estimate for the error in the nth approximation is easily found. (Received February 24, 1950.)

284. W. R. Mann and J. H. Roberts: *Concerning a certain integral equation.*

Mann and Wolf considered the equation (1): $y(t) = \int_0^t \{G[y(\tau)]/[\pi(t-\tau)]^{1/2}\} dt$ and showed that if $G(y)$ is continuous and decreasing for $y \geqq 0$, and $G(1) = 0$, then (1) has at least one solution. Under the additional assumption that G satisfies a Lipschitz condition on the interval $[0, 1]$, they proved that the solution of (1) was unique and nondecreasing. In the present paper the same conclusion is obtained without the extra hypothesis on G. Furthermore the function $[\pi(t-\tau)]^{1/2}$ can be replaced by any function $K(t-\tau)$ which, for $x > 0$, is continuous, positive, decreasing, and convex, and such that $\lim \int_\epsilon^1 K(x)dx$, as $\epsilon \to 0_+$, is finite. For example, instead of the one-half power in the denominator of the integrand of (1), we can take any power k such that $0 < k < 1$, and the resulting equation has a unique solution $y(t)$, which is strictly increasing and such that $y(t) \to 1$ as $t \to \infty$. (Received March 9, 1950.)

285. E. K. Ritter: *A method for computation of normal trajectories of missiles.*

The normal equations of motion of the center of mass of a self-propelled missile referred to a pair of rectangular axes with origin on the earth's surface, can be transformed through the introduction of drag, lift, and thrust functions similar to the conventional drag function of artillery ballistics. The transformed equations may readily be solved by numerical techniques, applicable alike to "hand" computation and to computation by high-speed, automatic calculating machines. (Received April 11, 1950.)

286. C. B. Smith: *Effect of a change in moisture content on the elastic behavior of plywood plates.*

This investigation is concerned with the elastic behavior of a plywood plate subjected to a uniform change in moisture content. By means of a mathematical analysis the distribution of stress is found for infinite plates of two and three plies. This distribution of stress is fairly simple, for all the stress components vanish except the normal stress components parallel to the surfaces of the plate. These are found to be discontinuous at the planes of separation of adjacent plies in both types of plates studied. The distribution of stress is found to be linear in each ply in the two-ply plate, and constant in each ply of the three-ply plate. (Received March 3, 1950.)

GEOMETRY

287. J. M. Clarkson: *A vector representation of the regular polyhedrons.*

The five regular polyhedrons are related in the following manner: the tetrahedron is such that vectors from the center of the circumscribed sphere to the centroids of the faces meet the sphere, when they are multiplied by a suitable scalar, in the vertices of a second tetrahedron; the same relationship between the cube and the regular octahedron exists, and so does that between the dodecahedron and the icosahedron. This paper describes the vector representation of the vertices of the polyhedrons, and derives the scalar multipliers for the vectors to the centroids of the faces. As a by-product, the representation, by radicals, of the trigonometric functions of certain angles not multiples of 30° or 45° are obtained. (Received May 8, 1950.)

288. Patrick Du Val: *Regular surfaces of genus three.*

The main theorem is that the general surface of order $4n$ in $n+3$ dimensions defineable as the intersection of the intersection of the cone projecting a Veronese surface from an $n-3$ dimensional space, with a fivefold variety whose general surface section is the del Pezzo surface of order n, is the bicanonical model of a regular surface of superficial genus 3 and linear genus $n+1$. For $n \leqq 5$, this is the most general surface of its genera; and for any n, the bicanonical model of every surface of these genera lies on the Veronese cone, its canonical system being traced by the net of quadric cones on the latter. The canonical model is an n-ple plane, with branch curve of order $4n$, having $24(n-2)$ cusps and $8(n-2)(n-3)$ nodes. For $n \leqq 3$, a complete specification of the branch curve is already known (P. Du Val, J. London Math. Soc. vol. 8 (1933) p. 14). For $n=4$, it is the envelope of a rational family of quartics, of index 3, of which two reduce to repeated conics; and every such envelope is the branch curve of a canonical surface of these genera. (Received February 25, 1950.)

289. G. B. Huff: *Cremona's equations and the properness inequalities.* II. Preliminary report.

In part I (Bull. Amer. Math. Soc. Abstract 55-7-314) it was shown that all solutions of Cremona's equations and the properness inequalities are indeed proper for $p=0$, 1 and $d>0$. In the present paper, a substantial generalization of Noether's inequality is obtained, and it is shown that, for any particular p ($p \geqq 2$), the question as to whether the properness inequalities do imply that a solution of Cremona's equations is actually proper can be settled by examining a finite number of cases. In particular, a check is obtained on tables constructed by Du Val (Proc. Lond. Math. Soc. (2) vol. 35, Part 1) and the result of part I is found to be true for $p=2$. (Received March 6, 1950.)

290*t*. Jack Levine: *Collineations in generalized spaces.* I.

The general space of paths is characterized by its equations $d^2x^i/dt^2+H^i(x, dx/dt)=0$ defining the paths, where H^i are homogeneous of the second degree in dx^i/dt (M. S. Knebelman, Amer. J. Math. vol. 51 (1928) pp. 527–564). It is shown that the equations of affine collineations $\Delta\Gamma^i_{jk}=0$ obtained by Davies (J. London Math. Soc. vol. 18 (1943) pp. 100–107) can be simplified to $\Delta H^i=0$, where Δ represents the Lie derivative operator, and $2\Gamma^i_{jk}=\partial^2H^i/\partial p^j\partial p^k$ ($p^i=dx^i/dt$). The components ΔH^i are those of a contravariant vector. A complete solution of $\Delta H^i=0$ is obtained for $n=2$ using Lie's classification of all real continuous groups G_r in two variables. Nine types of generalized two-dimensional space are found admitting groups of affine collineations. The maximum number of parameters is three, there being six spaces admitting G_3's, two admitting G_2's and one admitting a G_1. These exclude spaces reducing to ordinary affine spaces A_2. These results are an extension of some previous results of a former paper (Bull. Amer. Math. Soc. Abstract 55-7-381*t*). (Received March 6, 1950.)

TOPOLOGY

291. R. H. Bing: *Hereditarily indecomposable continua.*

Examples are described showing that the plane contains as many topologically different hereditarily indecomposable compact continua as it does continua. The property of being a chained compact continuum is invariant under a monotone continuous transformation. Using this result and the fact that two nondegenerate hereditarily indecomposable compact continua are homeomorphic if each can be chained, we show that if N is a nondegenerate image of a hereditarily indecomposable compact continuum M under a monotone continuous transformation, then N is homeomorphic to M. (Received March 8, 1950.)

292. W. J. Conner: *On locally closed sets.*

A set A of a topological space X is defined to be locally closed at a point p if p has a neighborhood whose closure relative to A is closed in X (stronger than the definition given by Kuratowski). "Local closure" and "locally closed set" are defined and the relationship between closure and local closure is used to characterize regular spaces, normal spaces, locally compact subsets of locally compact spaces. (Received March 6, 1950.)

293. L. J. Derr: *A theorem on exact sequences.*

Let G_1, G_2 be subgroups of an abelian group G. Using the Alexander-Kolmogoroff cohomology groups there exists an exact sequence: $\rightarrow H^p(X;\ G_1)\times H^p(X;\ G_2) \rightarrow H^p(X;\ G_1+G_2)\rightarrow H^{p+1}(X;\ G_1\cap G_2)\rightarrow$ where the groups in the parentheses indicate the coefficient group used in each case. It is clear that $G_1\cap G_2=0$ implies $H^p(X;\ G_1) \times H^p(X;\ G_2)\cong H^p(X;\ G_1+G_2)$. (Received March 8, 1950.)

294*t*. W. H. Gottschalk: *Choice functions and Tychonoff's theorem.*

The purpose of this note is to point out that the following set-theoretic theorem [R. Rado, *Axiomatic treatment of rank in infinite sets*, Canadian Journal of Mathematics vol. 1 (1949) pp. 337–343] is an easy consequence of Tychonoff's theorem that the cartesian product of a family of compact spaces is compact. Theorem. Let $(X_\alpha \mid \alpha\in I)$ be a family of finite sets, let $\mathfrak{A}$ be the class of all finite subsets of I, and for each $A\in\mathfrak{A}$ let ϕ_A be a choice function of $(X_\alpha \mid \alpha\in A)$. Then there exists a choice function ϕ of $(X_\alpha \mid \alpha\in I)$ such that $A\in\mathfrak{A}$ implies the existence of $B\in\mathfrak{A}$ such that $B\supset A$ and $\alpha\phi=\alpha\phi_B$ $(\alpha\in A)$. Proof. For $A\in\mathfrak{A}$ let E_A be the set of all $\phi\in X=\times_{\alpha\in I}X_\alpha$ such that $\alpha\phi=\alpha\phi_B$ $(\alpha\in A)$ for some $B\in\mathfrak{A}$ with $B\supset A$. Provide each X_α $(\alpha\in I)$ with its discrete topology. Since X is compact and $[E_A \mid A\in\mathfrak{A}]$ is a class of nonvacuous closed subsets of X with the finite intersection property, there exists $\phi\in\bigcap_{A\in\mathfrak{A}}E_A$. The proof is completed. Corollary. A family of finite sets has a one-to-one choice function if and only if each of its finite subfamilies has a one-to-one choice function. [Cf. C. J. Everett and G. Whaples, *Representations of sequences of sets*, Amer. J. Math. vol. 71 (1949) pp. 287–293.] (Received January 16, 1950.)

295. S. T. Hu: *Chain transformations in Mayer chain complexes.*

A general theory of the extension and the classification of the equivariant chain transformations in Mayer chain complexes with a group W of operators is established in this paper by means of the obstruction methods analogous to those used in the theory of continuous maps. The presentable and the regular subgroups of the equivariant cohomology groups are defined in a natural way, and hence their quotient group $J_e^*(M, H_*(N))$, where M and N are Mayer chain complexes with operators W and $H_n(N)$ denotes the integral homology group of N. The main theorem of classification can be stated as follows: If there are two integers a and b such that $C_q(M)=0$ unless $a<q\leq b$ and if W operates freely on M, then the equivariant homotopy classes of the equivariant chain transformations of M into N are in a (1–1) correspondence with the elements of the direct sum $J_e^{a+1}(M, H_{a+1}(N)) \oplus \cdots \oplus J_e^b(M, H_b(N))$. A number of specializations are made in the later part of the paper. In particular, for any augmentable closure-finite abstract complex K, the condition $H^q(K, H_q(K))=0$ $(0\leq q<n)$ implies $H_q(K)=0$ $(0\leq q<n)$, where $H_0(K)$ denotes the reduced 0th integral homology group. (Received March 6, 1950.)

296. J. W. Keesee: *On a retract of a compact Hausdorff space.*

Let $C(X)$ denote the ring of real continuous functions on the compact Hausdorff space X. An ideal I contained in $C(X)$ is called a closed ideal if it is equal to the intersection of all maximal ideals in which it is contained. A compact Hausdorff space Y is homeomorphic to a retract of X if and only if $C(X)$ contains a closed ideal I and a subring R such that $C(X)=I+R$, $I\cap R=0$, and R is ring-isomorphic to $C(X)$. (Received March 6, 1950.)

297. V. L. Klee: *On certain intersection properties of convex sets.*

A collection of $n+1$ convex subsets of a Euclidean space E will be called an *n-set* in E provided each n of the sets have a common interior point although the intersection of all $n+1$ interiors is empty. It is well known that if $\{C_0, C_1\}$ is a 1-set, then C_0 and C_1 can be separated by a hyperplane. In the present note this result is generalized by showing that if $\{C_0, \cdots, C_n\}$ is an n-set in E, then there is a variety V of deficiency n in E such that V intersects no set Int C_i, although in each direction away from V, V has a translate which intersects some set Int C_i. This theorem is then used to prove the converse of Horn's recent generalization (Bull. Amer. Math. Soc. vol. 55 (1949) pp. 923–929, Theorem 4) of Helly's theorem on the intersection of convex sets. (Received March 6, 1950.)

298. A. D. Wallace: *Extension and reduction theorems.*

Let A, B be closed subsets of the fully normal ($=$paracompact) space X. Extension Theorem: If e is any element of $H^p(B, A\cap B)$ then there is an open set Q containing B and an element e_0 of $H^p(\bar{Q}, A\cap\bar{Q})$ whose image, under the natural homomorphism, is e. Reduction Theorem: If the element e of $H^p(X, A)$ is in the kernel of the natural homomorphism into $H^p(B, A\cap B)$, then, for some open set Q containing B, e is in the kernel of the natural homomorphism into $H^p(\bar{Q}, A\cap\bar{Q})$. These are proved for the Alexander-Kolmogoroff cohomology groups. Their validity for normal spaces is an open question. (Received January 30, 1950.)

299t. A. D. Wallace: *Extensional invariance.*

Theorem: Any completely regular Hausdorff space has a homeomorph X densely contained in a compact Hausdorff space X' so that (denoting closure in X' by $'$) if

A, B are closed in X then $(A\cap B)'=A'\cap B'$ provided that either (i) X is normal or (ii) X is locally compact and one of A, B is compact. This is merely the Tychonoff imbedding (as improved by Čech and Stone) together with the additional result about closure. Case (i) can be deduced from some results of Wallman but case (ii) cannot since Wallman's compaction is not Hausdorff unless X is normal. This result is applied to prove the extensional invariance of certain properties of X, for example unicoherence, if X has property S. (Received March 7, 1950.)

W. M. WHYBURN,
Associate Secretary

THE APRIL MEETING IN WASHINGTON

The four hundred fifty-eighth meeting of the American Mathematical Society was held at George Washington University, Washington, D.C., on Friday and Saturday, April 28–29, 1950. Registration headquarters were in the Hall of Government, and all sessions were held there. Hotel headquarters were at the Roger Smith Hotel. The attendance was about 200, including the following 168 members of the Society:

J. C. Abbott, R. D. Anderson, Miriam C. Ayer, R. P. Bailey, Joseph Barnett, P. T. Bateman, F. D. Bateman, T. J. Benac, Stefan Bergman, E. E. Betz, R. H. Bing, Garrett Birkhoff, D. H. Blackwell, W. A. Blankinship, J. H. Blau, Joseph Blum, R. P. Boas, T. A. Botts, J. R. Bowman, Jean M. Boyer, N. A. Brigham, F. E. Browder, Charlotte Brudno, Helen J. Buck, R. S. Burington, G. H. Butcher, B. F. Cheydleur, K. L. Chung, Randolph Church, A. H. Clifford, Natalie Coplan, Mary J. Cox, H. S. M. Coxeter, G. F. Cramer, A. R. Craw, J. M. Danskin, W. S. Dawkins, J. B. Diaz, C. H. Dowker, Yael N. Dowker, D. M. Dribin, W. D. Duthie, R. P. Eddy, Samuel Eilenberg, H. E. Ellingson, J. H. Engel, Bernard Epstein, G. C. Evans, William Feller, N. J. Fine, Wilhelm Fischer, E. E. Floyd, Bernard Friedman, H. M. Gehman, B. C. Getchell, Leonard Gillman, H. F. Gingerich, Michael Goldberg, Leon Goldstein, R. A. Good, E. C. Gras, T. N. E. Greville, Emil Grosswald, D. W. Hall, Philip Hartman, G. G. Harvey, R. M. Hayes, E. C. Higgison, H. A. Hill, L. Aileen Hostinsky, W. R. Hydeman, M. A. Hyman, S. B. Jackson, M. P. Jarnagin, Jr., Herbert Jehle, Walter Jennings, R. E. Johnson, F. E. Johnston, M. L. Juncosa, D. E. Kibbey, J. W. Kitchens, V. L. Klee, J. R. Kline, Paul Kopp, W. D. Lambert, R. E. Langer, Richard Latter, W. S. Lawton, R. A. Leibler, W. W. Leutert, J. H. Levin, D. C. Lewis, F. W. Light, Miriam A. Lipschutz, Lee Lorch, E. J. McShane, H. M. MacNeille, C. J. Maloney, Murray Mannos, M. H. Martin, W. T. Martin, Florence M. Mears, A. E. Meder, Jr., T. W. Moore, T. S. Motzkin, S. B. Myers, David Nelson, Paolo Nesbeda, Moris Newman, T. E. Oberbeck, R. E. O'Donnell, M. W. Oliphant, K. L. Palmquist, A. J. Penico, Harry Polachek, R. R. Reynolds, R. P. Rich, G. de B. Robinson, Murray Rosenblatt, J. H. Rosenbloom, P. C. Rosenbloom, H. E. Salzer, S. S. Saslaw, Max Sasuly, G. E. Schweigert, C. W. Seekins, Atle Selberg, Daniel Shanks, George Shapiro, Seymour Sherman, Ruth W. Shnider, D. T. Sigley, R. C. Simpson, A. C. Smith, D. C. Spencer, Vivian E. Spencer, W. J. Strange, G. R. Strohl, E. G. Swafford, Olga Taussky-Todd, William Clare Taylor, J. H. Taylor, Feodor Theilheimer, D. L. Thomsen, John Todd, Miriam M. Torrey, C. A. Truesdell, J. L. Vanderslice, A. H. Van Tuyl, Oswald Veblen, M. C. Waddell, J. L. Walsh, J. V. Wehausen, F. M. Weida, Alexander Weinstein, F. J. Weyl, C. R. White, P. M. Whitman, W. F. Whitmore, G. T. Whyburn, K. G. Wolfson, M. A. Woodbury, Arthur Wouk, J. G. Wozencraft, J. W. Wrench, J. W. Young, P. W. Zettler-Seidel, J. A. Zilber.

The Committee to Select Hour Speakers for Eastern Sectional Meetings invited two speakers. On Friday, April 28, at 4:15 P.M., Dr. R. P. Boas, Executive Editor of Mathematical Reviews, gave an address on *Trigonometric extremal problems*. Professor G. C. Evans presided. At 2:15 P.M. on Saturday, April 28, Dr. Atle Selberg of the

Institute for Advanced Study gave an address on *The prime number theorem for generalized primes and elementary methods*. President J. L. Walsh presided.

Sections for contributed papers were held at 2:00 P.M. and 3:00 P.M. on Friday and at 10:00 A.M. and 11:00 A.M. on Saturday. The presiding officers for these sessions were Professor William Clare Taylor, Professor G. T. Whyburn, Dr. Seymour Sherman, and Dr. J. V. Wehausen.

A resolution of thanks to George Washington University and to the Committee on Arrangements was moved by Professor G. de B. Robinson and adopted by those present at the session held at 2:15 P.M. on Saturday.

The Council met at 8:15 P.M. on April 28.

The Secretary announced the election of the following one hundred and thirty-eight persons to ordinary membership in the Society:

Professor Forrest Edwin Adams, University of Miami;
Mr. Mohammad Wassel Al-Dhahir, Columbia University;
Mr. Samuel I. Altwerger, Newtown High School and New School of Social Research, New York, N.Y.;
Professor Emil Amelotti, Villanova College;
Mr. William Edward Andrus, Jr., International Business Machines Corp., Endicott, N.Y.;
Professor Carolyn Ruth Bahous, Lynchburg College;
Mr. Dean Clifton Benson, Sioux Falls College;
Mr. Morris Carl Bergen, Citrus Junior College, Azusa, Calif.;
Mr. Herbert Ashley Bernhard, Columbia University;
Mr. John Emory Berterman, Cornell University;
Mr. Stanley Francis Block, University of Maryland;
Mr. Russell Newton Bradt, University of Kansas;
Dr. Abraham Brind, New York, N.Y.;
Mr. Walter Clarke Brown, University of Oklahoma;
Mr. Howard Walker Burnette, University of Georgia;
Mr. John Richard Byrne, University of Washington;
Professor Margaret C. Byrne, Saint Joseph's College, Brooklyn, N.Y.;
Professor Harry Nelson Carter, University of Tulsa;
Mr. Ernest Charles Casale, Temple University;
Mr. Harold Hsiao-Li Chang, Princeton, N.J.;
Mr. Robert Francis Clarke, United States Weather Bureau, Miami Springs, Fla.;
Mr. James William Cooley, Woodside, N.Y.;
Mr. Arthur Herbert Copeland, Jr., University of Michigan;
Mr. Hewitt David Crane, International Business Machines Corp., New York, N.Y.;
Professor Mary Honor Cummings, Rhode Island State College;
Mr. Peter M. Curran, Fordham University;
Mr. Max Alexander Dengler, Midwest Research Institute, Kansas City, Mo.;
Brother V. Dominic, Saint Mary's College, Calif.;
Mr. Edwin Richard Dunkle, Jr., Albright College;
Mr. Ralph Edwin Ekstrom, Westminster College, Fulton, Mo.;

Mr. Irving Jacob Epstein, Evans Signal Laboratory, Belmar, N.J.;
Mr. Walter Feder, City College, New York, N.Y.;
Professor Michael Fekete, Hebrew University, Jerusalem;
Commander Reid Puryear Fiala, United States Navy, Destroyer Division 602;
Professor Emery Louis Gerecz, Vincennes University;
Mr. Myron Leonard Greenberg, University of Michigan;
Mr. Urban S. Greene, Binghamton, N.Y.;
Mr. Joseph Arthur Greenwood, Manhattan Life Insurance Company, New York, N.Y.;
Professor Simon Gruenzweig, Lincoln University, Pa.;
Mr. Jack Cogswell Gysbers, University of Oregon;
Mr. William Bromley Hall, Jr., Long Island City, N.Y.;
Professor Wame Jacob Hallmark, San Antonio College;
Mr. Francis Robert Halpern, Bell Aircraft Company, Buffalo, N.Y.;
Mr. John Charles Harden, Jr., Clemson College;
Mrs. A. Z. Hays, Abilene Christian College;
Mr. Glenn Fairbanks Hewitt, Wright Branch, Chicago City Junior College;
Mr. Theodore Ware Hildebrandt, Massachusetts Institute of Technology;
Professor Philip Murray Iloff, Chico State College, Chico, Calif.;
Sister Mary Amata Kadlack, Saint Joseph's College for Women, Brooklyn, N.Y.;
Mr. Surindra Nath Kalra, University of Illinois;
Dr. Rosella Kanarik, University of Southern California;
Mr. Costas Kassimatis, University of Toronto;
Mr. J. Lawrence Katz, Brooklyn, N.Y.;
Professor Jack Thurston Kent, Agricultural & Mechanical College, College Station, Tex.;
Miss Muriel Clarissa King, Georgia State College;
Mr. Edward Harvey Kingsley, Roosevelt College, Chicago, Ill.;
Mr. Joseph Klein, Redbank, N.J.;
Mr. Robert J. Koch, New Orleans, La.;
Professor Motokiti Kondô, Kyûsyû University, Japan;
Professor Edward Michael Kovach, Cathedral College, New York, N.Y.;
Mr. Vernon Kuykendall, Clark Junior College, Vancouver, Wash.;
Mr. John Cornelius Laffan, Jr., Seton Hall College, South Orange, N.J.;
Mr. Orlie William Laing, Pasadena City College;
Mr. Seymour Landau, University of Connecticut;
Mr. Serge Lang, Princeton University;
Mr. Richard Kenneth Lashof, New York, N.Y.;
Mr. Richard F. Link, University of Oregon;
Professor Harry Maurice Linnette, Virginia State College;
Mr. Charles K. Loomis, Jr., Detroit Institute of Technology;
Mr. Robert Duncan Luce, Massachusetts Institute of Technology;
Dr. Robert Hastings McCobb, New Jersey State Teachers College, Glassboro, N.J.;
Professor Richard Joseph McCullough, Mount Saint Mary's College, Emmitsburg, Md.;
Miss Clarice Irene MacDonald, University of Washington;
Professor Duncan MacRae MacEwen, City College, New York, N.Y.;
Dr. Lawrence Earl Malvern, Carnegie Institute of Technology;
Mr. Robert Louis Marceau, University of Kansas;
Mr. Henry George Mazurkiewicz, University of Michigan;

Mr. José Carlos Mello E. Souza, Rua Redentor 92, Distrito Federal, Brazil;
Captain George Brown Miller, Jr., Georgia Military Academy, College Park, Ga.;
Professor James L. Miller, Villanova College;
Mr. Robert Elsworth Montgomery, University of Illinois;
Mr. John Coleman Moore, Brown University;
Miss Marian Alease Moore, Purdue University;
Dr. Charles Edward Mullan, Duquesne Light Company, Pittsburgh, Pa.;
Mr. George Max Muller, General Electric Company, Richland, Wash.;
Professor Sigurd Mundhjeld, Concordia College, Moorhead, Minn.;
Mr. James Henry Nash, Fordham University;
Mr. Abas Beaucan Neale, Naval Aircraft Factory, Philadelphia, Pa.;
Miss Elizabeth Jane Neiderberger, Purdue University;
Mr. Ralph Henry Niemann, Purdue University;
Dr. Fritz Oberhettinger, California Institute of Technology;
Mr. Edward Scott O'Keefe, University of Washington;
Miss Katharine Louisa Parks, Morgan Park Junior College, Chicago, Ill.;
Mr. Paul Bryan Patterson, University of Florida;
Mr. Paul Edwin Pfeiffer, Rice Institute;
Professor Harold David Reese, Drexel Institute of Technology;
Mr. Henry Gordon Rice, Syracuse University;
Professor Ralph A. Rodefer, Michigan College of Mining & Technology;
Mr. Jack Cree Rogers, Cornell University;
Miss Mary Lee Rogers, University of Washington;
Miss Florence Virginia Rohde, University of Kentucky;
Mr. Richard Irwin Rossbacher, New Haven, Conn.;
Mr. George Xavier Saltarelli, Buffalo, N.Y.;
Mr. James Sanders, Syracuse University;
Mr. Frank Salvatore Scalora, University of Illinois;
Professor Olive Crosbie Schooler, University of Tulsa;
Mr. William Howard Sellers, West Virginia University;
Miss Edna Sheinhart, University of Connecticut;
Mr. Alan B. Showalter, University of Kansas;
Professor Marvin Banks Sledd, Emory University;
Professor Richard Herbert Somers, University of New Hampshire;
Mr. William Kebren Spears, Keesler Air Force Base, Biloxi, Miss.;
Mr. Clifford Harvey Springer, Purdue University;
Mr. Jeremiah Milton Stark, Massachusetts Institute of Technology;
Mr. Gordian William Stremlau, St. Bede College, Peru, Ill.;
Professor Mary Virginia Sunseri, Stanford University;
Professor Victor G. Szebehely, Virginia Polytechnic Institute;
Professor Choy-tak Taam, University of Missouri;
Mr. Eugene Titus, Southern University and Agricultural and Mechanical College;
Mr. John William Toole, University of Illinois;
Mr. Grover Jackson Trammell, Jr., Tulane University of Louisiana;
Mr. Charles Hood Tyler, Lewis College, Lockport, Ill.;
Mr. Lewis Edes Ward, Jr., Tulane University of Louisiana;
Mr. Hans Felix Weinberger, Carnegie Institute of Technology;
Professor Annye Elizabeth Welch, Catawba College, Salisbury, N.C.;
Mr. Burton Garland West, Lincoln High School, Port Arthur, Tex.;
Professor John Elliott Westberry, Texas College;

Mr. Carl Edgar Weyland, College of South Jersey;
Professor Royal Samuel Weymouth, Idaho State College;
Mr. Robert Franklin Wheeling, Brown University;
Professor Clarence Reed White, Hampton Institute;
Mr. James Whittaker, University of California at Los Angeles;
Professor Roy Edwin Wild, University of Idaho;
Professor Ernest Williams, Alabama Polytechnic Institute;
Mr. Wilber Albert Wilson, Clemson College;
Mr. Henry Wolf, Brown University;
Professor Guy Aytch York, Georgia School of Technology;
Mr. Solomon Abraham Zadoff, Passaic, N.J.

It was reported that the following twenty-five persons had been elected to membership on nomination of institutional members as indicated:

Indiana University: Mr. Donald Henry Porter;
Institute for Advanced Study: Dr. Ernest Paul Specker;
University of Minnesota: Messrs. Martin Samuel Friberg and Arnold Jackson Tingley;
Northwestern University: Messrs. Donald Guy Austin, Robert Langley Sternberg, and Fred Marion Wright;
Princeton University: Messrs. Iain Thomas Arthur Carpenter Adamson, Jose Adem Chahin, Richard Charles Blanchfield, Eugenio Calabi, Francis Brock Fuller, John McCarthy, John Willard Milnor, John Forbes Nash, Jr., Dr. Christos Dimitriov Papakyriakopoulos, Messrs. Melvin Philip Peisakoff, Oscar Seymour Rothaus, Joseph Harold Sampson, Jr., Lloyd Stowell Shapley, William Tout Sharp, George Stephenson, John Torrence Tate, Jr., and Guillermo Torres Diaz;
University of Wichita: Professor Cecil Byron Read.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: London Mathematical Society: Mr. Cecil John Alvin Evelyn, London; Professor Werner Wolfgang Rogosinski, Durham University and Agricultural and Mechanical College, Stillwater, Okla.; Matematisk Forening, København: Mr. Bent Fuglede, Stanford University; Polish Mathematical Society: Professor Edward Arnold Otto, Warsaw Politechnike School; Miss Wanda Szmielew, University of California; Société Mathématique de France: Mr. Marcel Honore Bayard, Sous-Marino Telecommunications, Meudon; Dr. Jean Francois Braconnier, University of Lyon; Professor Jovan Karamata, University of Belgrade; Professor Georges de Rham, University of Geneve and Institute for Advanced Study; Mr. Dov Tamari, Boursier de l'A.R.S., Paris; Professor Miodrag Tomic, Belgrade Polytechnic Institute; Swiss Mathematical Society: Professor Hans Georg Haefeli, Boston College; Unione Matematica Italiana: Professor Francesco Elbano Sbrana, University of Genova.

Texas Christian University (Fort Worth, Texas) was elected to institutional membership in the Society.

The following actions taken by mail vote of the Council were reported: election of Professors Einar Hille and A. W. Tucker as members of the Executive Committee of the Council to serve for a period of two years beginning January 1, 1950; election of Professor W. T. Martin as a representative of the Society on the Policy Committee for Mathematics, to serve for a period of four years beginning January 1, 1950; authorization of the President to appoint a committee to study the question of reciprocity agreements with organizations on the American continents; authorization of the President to appoint a committee to cooperate with the American Institute of Graphic Arts in a study of printing problems.

The following appointments of representatives of the Society were reported: Professor R. G. Helsel at centenary celebration of University of Dayton on March 16, 1950; Professor Gillie A. Larew at inauguration of Orville Wentworth Wake as President of Lynchburg College on April 25, 1950; Professor S. P. Shugert at inauguration of H. Sherman Oberly as President of Roanoke College on April 14, 1950; Professor Helen Barton at inauguration of Dale Hartzler Gramley as President of Salem College and Academy on April 22, 1950; Professor V. H. Tingey at 100th anniversary of University of Utah on February 27–28, 1950; Professor Eric Reissner on Program Committee for 1951 London Conference in Heat Transmission (sponsored by American Society of Mechanical Engineers and Institute of Mechanical Engineers, London).

The following additional appointments by the President were reported: Professors T. F. Cope and B. P. Gill as auditors of the Society's accounts for 1950; Professors R. E. Langer (Chairman), J. L. Doob, M. H. Heins, T. H. Hildebrandt, and D. C. Spencer as a Committee to Study Individual Membership Dues; Professor R. L. Wilder as Chairman of the Committee on Places of Meetings for 1950; Professor J. M. Thomas as a member of the Committee on Places of Meetings for 1950–1952; Professors G. T. Whyburn (Chairman), Nelson Dunford, R. L. Jeffery, C. C. MacDuffee, and D. C. Spencer as a Committee to Study the Question of Reciprocity Agreements with Organizations on the American Continents; Professors Saunders MacLane (Chairman), C. C. MacDuffee, G. B. Price, A. C. Schaeffer, and J. M. Thomas as a Committee to Discuss Possible Reductions in Cost of Mathematical Composition; Professors S. S. Cairns (Chairman), E. F. Beckenbach, Einar Hille, Deane Montgomery, and R. L. Wilder as a Committee to Nominate Officers and

Members of the Council for 1951.

The following resolution on the death of Professor Virgil Snyder was adopted by the Council:

> The Council of the American Mathematical Society wishes to record its deep sense of loss in the death of Virgil Snyder, Emeritus Professor at Cornell University since 1938. He had been connected with Cornell University as a member of the faculty since 1895.
>
> Professor Snyder had an active and influential part in the affairs of the Society for fifty years, having served at various times as a member of the Council, vice president, president, as an editor of the Bulletin, and as a delegate to several international congresses.
>
> Professor Snyder's mathematical interest was in the field of algebraic geometry. His graduate study at Göttingen and his later study and association with the Italian geometers laid the foundation for his own outstanding work in this field and for his inspiring guidance of some forty graduate students in the preparation of their doctoral dissertations. His standing in this particular field was recognized when he was selected as chairman of the committee of the National Research Council to prepare a bibliography for algebraic geometry.
>
> During the first half of the twentieth century American interest in mathematics and American contributions to the development of the subject had a phenomenal growth. Now we are called upon once more to note the passing of one of the men who contributed largely to this advance. We do so with sincere appreciation of the man and of his work.

Dr. H. M. MacNeille, Executive Director, reported on the following problems: (1) that of obtaining suitable office space for the Society in New York and the related problem of the use of the Society's library as a bargaining asset in negotiations with Columbia University for office space; (2) that of establishing a new agreement with Johns Hopkins University in connection with the publication of the American Journal of Mathematics. The Council voted to recommend a change in the by-laws to reduce to two the number of representatives of the Society on the Editorial Board of the American Journal; (3) the Executive Director, at the request of the Board of Trustees, had gathered a list of candidates for the position of Executive Editor of Mathematical Reviews, for the consideration of the Editorial Committee.

The Council voted to approve the recommendation of the Committee on Applied Mathematics to hold the 1951 Symposium in Applied Mathematics at the University of Maryland and the United States Naval Ordnance Laboratory.

The Council voted to recommend to the Board of Trustees that Dr. J. V. Wehausen be appointed as Executive Editor of Mathematical Reviews, to succeed Dr. R. P. Boas.

The Policy Committe for Mathematics reported progress in its consideration of the following problems: the establishment of a

Mathematics Foundation to receive and administer funds for the general support of mathematics; National Science Foundation and the service such a foundation could render to mathematics; separate division for mathematics in the National Research Council; Union Conference to be held at Columbia University on August 27–29, 1950, for a discussion of a new International Mathematical Union.

The Council adopted a budget for the International Congress as recommended by the Organizing Committee; total estimated expenses for the Congress are $101,516.00.

It was reported that the dictionary containing a list of words useful in reading Russian mathematical articles and about thirty pages of grammar would shortly be ready for distribution. The Council voted to recommend to the Trustees that a thousand copies, in addition to the number required by the Navy under the contract, be printed for possible future sales. It was also reported that fifteen translations of Russian mathematical articles had been completed and that the Navy contract for the preparation of these translations had been extended to April 30, 1951.

The Council voted to recommend to the Trustees that for 1951 the list price of the Bulletin be set at $7.00 and that of the Proceedings at $11.00.

Abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 302 was read by Professor Fine, paper number 313 by Professor Epstein, paper number 335 by Dr. Rosenblatt, and paper number 338 by Professor Floyd. Mr. Hurley was introduced by Dr. H. W. E. Schwerdtfeger.

ALGEBRA AND THEORY OF NUMBERS

300*t*. Bailey Brown and N. H. McCoy: *Some theorems on groups with applications to ring theory.*

Let G be an additively written group, and F a mapping of G into the set of subgroups of G. Under certain restrictions on F relative to a set of endomorphisms of G, a normal subgroup N is defined, depending on F. When G is the additive group of a ring R, N becomes for special choices of F, respectively: (1) the greatest regular ideal in R (Bull. Amer. Math. Soc. Abstract 55-3-100), (2) the Jacobson radical of R, (3) the F-radical of R, (4) the radical of R (Amer. J. Math. vol. 69 (1947) pp. 46–58). Regularity is defined in, and (1) extended to, an arbitrary naring, and a proof is indicated that a full matrix naring over a regular naring is regular. By strengthening the assumptions on F, N becomes for any G the intersection of a certain class of normal subgroups of G, and a subdirect decomposition of G is obtained if $N=0$. When G is the additive group of R, the corresponding theorems for (2), (3), (4) now become special cases. Here R may be a naring or even a cluster. (Received March 27, 1950.)

301. A. H. Clifford: *A noncommutative simple linearly ordered group.*

In answer to Garrett Birkhoff's Problem 5 (Ann. of Math. vol. 43 (1942) p. 329) an example is given of a noncommutative linearly ordered group having no proper *l*-ideals. (Received March 14, 1950.)

302. N. J. Fine and G. E. Schweigert: *Concerning the group of homeomorphisms of a simple arc.*

Let G denote the group of all homeomorphisms of a simple arc onto itself. It is shown that every element of G is a product of at most four involutions (elements of order two). Certain subgroups of G are characterized, and G itself is determined among them by a maximality condition. A partial identification of the normal subgroups is made. An interesting class of commutative subgroups is studied in some detail, with special reference to their intersection properties. Problems similar to those above are considered for other (possibly nonseparable) spaces with order topologies. The results on G can be used to show that every element in the homeomorphism group of a simple closed curve is representable as a product of at most five involutions. (Received March 9, 1950.)

303. L. Aileen Hostinsky: *Direct decompositions in lattices.*

The refinement theorem proved by R. Baer for direct decompositions of operator loops (*Direct decompositions*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 62–98) is generalized by using results in the author's Bull. Amer. Math. Soc. Abstract 55-11-474 to translate into lattice theory Fitting's method of investigating refinement theorems. Relationships between this result and theorems of O. Ore, A. Kurosch, and M. Grayev are shown. (Received February 14, 1950.)

304*t*. A. C. Hurley: *The irreducible crystal classes in four dimensions.*

The problem of finding the crystal classes is tackled using the theory of group characters. Firstly all the types of elements which may occur are determined using the fact that the characteristic equation of the matrix must have integral coefficients (the invariants of the matrix). The second invariants of the matrices are shown themselves to constitute the character of a representation of the group: thus in using the result of character theory, that all power sums of the characters give totals which are multiples of the order of the group, the second invariants may be used in forming the power sums as well as the characters. Using this method the possible values for the order of a four-dimensional crystal group are found, as are also the number of elements in each group having a given value of the trace, second invariant, and determinant. Of these solutions of the character relations, some (38) are shown to represent irreducible crystal groups (derived from the regular four-dimensional polytopes and other methods), some are shown to be extraneous, whilst some (33) remain unaccounted for. (Received April 11, 1950.)

305*t*. A. C. Hurley: *The crystal classes in four dimensions.*

The investigation is based on a paper by Goursat (Ann. l'École Norm. vol. 16 (1889)). In this paper Goursat determines all finite four-dimensional rotation groups which contain $-I$, where I is the unit element. The trace and second invariant of all elements of Goursat's groups are determined, those groups with non-integral invariants being discarded. Of the remaining groups most are shown to be crystal groups. Some however are not; these may be eliminated using the results of a paper by Robinson (Proc. Cambridge Philos. Soc. (2) vol. 32 (1930) p. 12). The crystal groups not containing $-I$ are also found. The results of the character theory investigation given in the preceding abstract provide a useful check on the final list of groups, and in fact lead to the detection of an error of omission in the above-mentioned paper by Goursat. Using these methods it is shown that there are just 218 four-dimensional crystal groups, of which 45 are irreducible. (Received April 11, 1950.)

306*t*. D. J. Newman: *An evaluation of the constant in the asymptotic expression for $p(n)$.*

Hardy and Ramanujan proved in 1918 the asymptotic formula $p(n) \sim (1/4(3^{1/2})(e^{\pi(2n/3)^{1/2}}/n)$, where $p(n)$ represents the number of partitions of n (*Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. (1918) pp. 75–115). Erdős in 1942 proved, by elementary methods, that $p(n) \sim ae^{\pi(2n/3)^{1/2}}/n$ for some $a>0$, but did not prove that $a=1/4(3^{1/2})$ (*On an elementary proof of some asymptotic formulas in the theory of partitions*, Ann. of Math. (1942) pp. 417–430). In this paper the author proves that $a=1/4(3^{1/2})$ by elementary methods. The method used is to estimate the two series $\sum p(n)x^n \sim (e^{-\pi^2/12}/(2\pi)^{1/2})(1-x)^{1/2}e^{\pi^2/6(1-x)}$ and $\sum(e^{\pi(2n/3)^{1/2}}/n)x^n \sim 2e^{-\pi^2/12}(6/\pi)^{1/2}(1-x)^{1/2}e^{\pi^2/6(1+x)}$ from which an immediate application of the theorem $A_n \sim aB_n \rightarrow \sum A_n x^n \sim a\sum B_n x^n$ gives $(e^{-\pi^2/12}/(2\pi)^{1/2}(1-x)^{1/2} \cdot e^{\pi^2/6(1-x)} \sim a2e^{-\pi^2/12}(6/\pi)^{1/2}e^{\pi^2/6(1-x)}$ and this proves $a=1/4(3^{1/2})$. (Received March 6, 1950.)

307. R. P. Rich: *On partially ordered groups.*

A compositive convex normal subgroup of a partially ordered group G is called an l-ideal; this reduces to Birkhoff's definition (Ann. of Math. vol. 43 (1942) p. 310) if G is lattice ordered. The set $\mathfrak{L}$ of l-ideals of G is a complete lattice under inclusion. If G has the Riesz interpolation property (Birkhoff, loc. cit. p. 328), then $\mathfrak{L}$ is a distributive sublattice of the lattice $\mathfrak{N}$ of normal subgroups of G, and the complemented l-ideals in $\mathfrak{L}$ form a Boolean algebra. Answering Birkhoff's Problem 5 (loc. cit. p. 329): there exists a non-archimedean lattice ordered group H which is simple, that is, without proper l-ideals; H is a subgroup of the group of homeomorphisms of an interval considered by Everett and Ulam (Trans. Amer. Math. Soc. vol. 57 (1945) p. 212). If C is a linearly ordered group with an l-ideal A such that A and C/A are both archimedean, then (i) if A is in the center of C, the factor set of C/A in A is arbitrary; (ii) if A is not in the center of C, then the factor set is associate to the identity. (Received March 14, 1950.)

308*t*. C. E. Rickart: *Isomorphic groups of linear transformations: the unitary and simpletic cases.*

In another paper (to appear in Amer. J. Math., cf. also Bull. Amer. Math. Soc. abstracts 54-11-440, 55-11-487) the structure of isomorphisms of certain groups of linear transformations which reduce in the finite-dimensional case to full linear groups was discussed. In the present paper, a similar investigation is carried out for groups of linear transformations which reduce in the finite-dimensional case to unitary (or orthogonal) groups on the one hand and to simplectic groups on the other. The main result is that the group isomorphisms are essentially generated by isomorphisms of the underlying vector spaces on which the transformations act. The result in the finite-dimensional simplectic case reduces essentially to a result announced by Dieudonné (C. R. Acad. Sci. Paris vol. 225 (1947) pp. 914–915). (Received March 20, 1950.)

309. G. de B. Robinson: *Induced representations and invariants.*

Consider the subroup $H=S_m\times S_m\times\cdots\times S_m$ (n factors) and its normalizer $\mathfrak{N}(H)$ in the symmetric group S_{mn}. If $[\alpha]$ is an irreducible representation of S_m, then the representation $[\alpha]^n$ of H can be extended by means of a representation $[\beta]$ of S_n to yield an irreducible representation $[\alpha;\beta]$ of $\mathfrak{N}(H)$, and this representation $[\alpha;\beta]$ induces a representation $[\alpha]\odot[\beta]$ of S_{mn}. In a previous paper the author gave an explicit formula for the reduction of $[\alpha]\odot[\beta]$ in terms of group characters. This formula has since been obtained by J. A. Todd (to appear shortly in the Canadian Journal of Mathematics) via the corresponding theory of the full linear group. However, so far as the applications to invariants are concerned, it would be desirable to avoid the complications arising from such a dependence upon character tables. The present method accomplishes this end through a new "integration" process which makes it possible to pass from n to $n+1$ for a fixed $[\alpha]$. It is necessary to add a "boundary condition" which depends on the $\overline{n+1}$-hook structure of the Young diagrams so obtained. (Received March 20, 1950.)

310. Olga Taussky: *Symmetrization of systems of linear equations.*

Let $A=(a_{ik})$ be a real nonsingular $n\times n$ matrix, $\lambda_{\max}$, $\lambda_{\min}$ its characteristic roots of largest, resp., smallest modulus, A' the transpose of A and (b_{ik}) its inverse. The following two "condition numbers" of A have been introduced recently: (i) $|\lambda_{\max}|/|\lambda_{\min}|$; (ii) $(1/n)(\sum a_{ik}^2)^{1/2}/(\sum b_{ik}^2)^{1/2}$ (see J. von Neumann and H. H. Goldstine, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1021–1099, and A. M. Turing, Quarterly Journal of Mechanics and Applied Mathematics vol. 1 (1948) pp. 287–308). It is shown that both numbers increase if A is replaced by AA'. Thus symmetrization of a system of linear equations, in general, leads to a worsening of the condition. Further, the class of the matrices AA' is studied for which (ii) is constant. (Received March 14, 1950.)

311. M. C. Waddell: *On properties of regular rings.*

A ring R is regular if, for each $a\in R$, there exists $x\in R$ with $axa=a$. A regular ring is called t-regular if every principal ideal has a unit element, and is called s-regular if it contains no nonzero nilpotent element. Several properties, analogous to those of classical semi-simple ring theory, are obtained, including the following. In a regular ring every ideal is the meet of all generalized prime ideals (in the sense of McCoy, Amer. J. Math. vol. 71 (1949) p. 823) containing it, and the ideals form a distributive lattice. In a t-regular ring every ideal is the meet of all maximal ideals containing it, and if every maximal ideal has nonzero annihilator, then every ideal contains a minimal ideal, the ring is isomorphic to a subdirect sum of its minimal ideals, and the ideals form a Boolean algebra. Nine necessary and sufficient conditions for a regular ring to be s-regular are given, including (i) every idempotent element is in the center of the ring, (ii) only two-sided ideals exist, (iii) every generalized prime ideal is prime, (iv) every ideal is the meet of all prime ideals containing it, (v) every ideal is a radical ideal (that is, $a^n\in A$ implies $a\in A$). Assuming s-regularity, the sets of prime, maximal, and indecomposable ideals coincide. (Received March 14, 1950.)

ANALYSIS

312t. R. P. Boas: *Differential equations of infinite order.*

The equations are of the form $A(D)f(z)=g(z)$, where $D=d/dz$, $A(t)=\sum_0^\infty a_n t^n$ is an entire function, $g(z)$ is regular in a (sufficiently large) neighborhood of 0, and the solution $f(z)$ is to be regular in some neighborhood of 0. By means of exponential transforms [Boas, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 481–483] some results of Sheffer (Duke Math. J. vol. 3 (1937) pp. 593–609) and Muggli (Comment. Math. Helv. vol. 11 (1938) pp. 151–179) are obtained when $A(t)$ is of exponential type. In addition, it is shown that there is always a solution when $A(t)$ belongs to a wider class of entire functions, whose essential property is that they increase at their maximum rate in at least three directions which are the normals to the sides of a closed convex polygon; $A(t)=\exp(t^3)$ is a typical example. (Received March 15, 1950.)

313. Joseph Lehner and Bernard Epstein: *Behavior of modular functions and forms near the boundary.*

The authors prove the following results, x representing a real number: (1) If $J(\tau)$ is the absolute invariant of the full modular group, and if x is irrational, then along no curve terminating at x does $J(\tau)$ approach a limit, finite or infinite, while it is known that $J(\tau)\to\infty$ as τ approaches any rational x in any Stolz angle. (2) If $f(\tau)$ is a modular form belonging to the full group, then $f(\tau)$ possesses no limit at irrational x except on a certain set E where $f(\tau)\to 0$ or ∞ in any Stolz angle according as $f(\tau)$ is of positive or negative dimension. The set E consists of those irrationals whose expansions in regular continued fractions have bounded partial quotients. (3) These results generalize to certain subgroups of the full group. (4) There exists an angle α independent of x such that, with the exception of a countable set of x, the image in the w-plane under the mapping $w=\lambda(\tau)$ of the Stolz angle of aperture α with vertex at x covers infinitely often a certain domain of the w-plane. (Received March 27, 1950.)

314*t*. Dorothy Maharam: *Decompositions of measure of algebras and spaces.*

If A is a given σ-subalgebra of a σ-finite measure algebra (E, μ), it is shown that (E, μ) is isometric to a subalgebra of a principal ideal in the direct product of two σ-finite measure algebras, one of which is algebraically isomorphic to A. (Cf. Gleason, Bull. Amer. Math. Soc. Abstract 55-3-144; Nikodým, C. R. Acad. Sci. Paris vol. 228 (1949), for related results when μ is finite.) This is proved by introducing a suitable abstract-valued measure in E and applying the author's representation theorem (Trans. Amer. Math. Soc. vol. 65 (1949)). Applications are made to normal measure spaces (Ω, μ); for example, if (Ω, μ) is acted on by a measure-preserving transformation, the direct sum decomposition of Ω into ergodic parts can be imbedded, in a simple way, in a direct product decomposition. Other results due to von Neumann, Halmos, and Dieudonné are extended similarly. (Received March 10, 1950.)

315*t*. E. F. Moore: *Density ratios and* $(\phi, 1)$ *rectifiability in n-space.*

If ϕ is a linear measure over Euclidean n-space E_n, a set $A\subseteq E_n$ is said to be $(\phi, 1)$ rectifiable if and only if for every $\epsilon>0$ there is a rectifiable set B such that $\phi(A-B)<\epsilon$. If $\phi(A)<\infty$, the $(\phi, 1)$ rectifiability of A is shown to be equivalent to inequalities holding for ϕ almost all x in A between constant multiples of certain upper and lower density functions defined locally in terms of ϕ and A. This extends (with the same constants used in the inequalities) results proved in the case $n=2$ by Morse and

Randolph (Trans. Amer. Math. Soc. vol. 55 (1944) pp. 236–305). (Received April 17, 1950.)

316. Ruth E. O'Donnell: *k-polar polynomials.*

The purpose of this paper is to prove a generalization of the theorem of Grace on apolar polynomials. The polynomials $f(z)=\sum_{j=0}^{n} C_{n,j}a_jz^j$ and $g(z)=\sum_{j=0}^{n} C_{n,j}b_jz^j$ are k-polar if their coefficients satisfy the k^2 conditions $\sum_{j=0}^{n-k+1} (-1)^j C_{n-k+1,j}a_{s-j}b_{j+h}$ $(h=0, \cdots, k-1; s=n, \cdots, n-k+1)$. It is proved that the k-polarity of two polynomials is sufficient to insure that every circular domain containing all the zeros of one of the polynomials shall contain at least k zeros of the other. (Received March 16, 1950.)

317t. J. E. L. Peck: *An ergodic theorem for a certain non-commutative semigroup of linear operators.*

If $\mathcal{L}$ is a semigroup of bounded linear operators on a Banach space, which is totally bounded in the strong topology of operators, and which satisfies the pseudo-commutative restrictions (1) given S_1, S_2 in $\mathcal{L}$ there exists S_3, S_4 in $\mathcal{L}$ with $S_1S_3=S_2S_4$ and (2) given S_5, S_6 in $\mathcal{L}$ there exist S_7, S_8 in $\mathcal{L}$ with $S_5S_6=S_7S_5=S_6S_8$; then $\mathcal{L}$ is an ergodic semigroup in the sense of Eberlein (Trans. Amer. Math. Soc. vol. 67 (1949) pp. 217–240) and every element of the Banach space is ergodic in the same sense. This theorem is proved by making use of a fixed point theorem of Kakutani (Proc. Imp. Acad. Tokyo vol. 14 (1938) pp. 242–245). (Received April 17, 1950.)

318. P. C. Rosenbloom: *A problem in Hilbert space.*

Let A be a non-negative bounded Hermitian operator, x_0 a given element in Hilbert space, and t a given positive constant. We seek the maximum of $|(x, x_0)|$ under the conditions $\|x\|=1$ and $(Ax, x)=t$. This maximum and the extremal element x can be determined in terms of the function $F(\lambda)=((\lambda I-A)^{-1}x_0, x_0)$, which is a Stieltjes transform. The behavior of the maximum as $t\to 0$ depends on that of $F(\lambda)$ as $\lambda\to 0-$. We obtain a known inequality of Carlson as a special case, and also the best possible inequalities for the maximum modulus of an analytic function on a given circle when the mean square on two given circles has prescribed values. Other applications to analytic and harmonic functions are obtained. (Received March 23, 1950.)

319. George Shapiro: *On the Dirichlet series associated with Ramanujan's function τ(n).* Preliminary report.

Let $F(s)=\sum_{n=1}^{\infty} \tau(n)n^{-s-11/2}$ for $\sigma>2/5$. This series, as well as the infinite product representation $F(s)=\prod_{k=1}^{\infty}(1-\tau(p_k)p_k^{-s-11/2}+p_k^{-2s})^{-1}$, is known to be absolutely convergent when $\sigma>1$; hence, $F(s)\neq 0$ there. Let G be the (simply-connected) region obtained from the half plane $\sigma>1/2$ by deleting all horizontal segments $\sigma+it_0$, $1/2<\sigma\leqq\sigma_0$, where σ_0+it_0 is a zero of $F(s)$ in $\sigma>1/2$ (if such exists). By log $F(s)$ in $\sigma>1/2$ is meant the analytic continuation into G of log $F(s)$ in $\sigma>1$. Let $F_n(s)=\prod_{k=1}^{n}(1-\tau(p_k)p_k^{-s-11/2}+p_k^{-2s})^{-1}$. By iteration of the alternate use of a result of Carlson (Arkiv för Matematik, Astronomi och Fysik vol. 19A (1926) no. 25) and the functional equation of $F(s)$, it is shown that $|F(\sigma+it)|^2$ has a finite mean value for every fixed $\sigma>13/20$. From this it follows, by a method similar to that of Bohr (Acta Math. vol. 40 (1916) p. 67), that for every fixed $\sigma>13/20$, the sequence $\{\log F_n(s)\}$

converges in relative measure to log $F(s)$. On the basis of Ramanujan's conjecture that $\tau(n)=O(n^{11/2+\epsilon})$, the same results are obtained in $\sigma>1/2$. (Received March 14, 1950.)

320. K. G. Wolfson: *On the spectrum of a boundary value problem with two singular end points.*

Let $p=p(t)>0$ and $q=q(t)$ be continuous functions for $-\infty<t<+\infty$ such that (*) $(px')'+(q+\lambda)x=0$ is of "Grenzpunkt" type at $t=-\infty$ and $t=+\infty$. Let the spectrum of the boundary value problem thus determined be denoted by S, and the set of its cluster points by S'. For $\epsilon>0$ and $a<b$, let $N(a, b, \lambda, \epsilon)$ be the number of eigenvalues in the interval $[\lambda-\epsilon, \lambda+\epsilon]$ belonging to the Sturm-Liouville problem determined by (*) and the boundary conditions $x(a)=x(b)=0$. Put $n(\lambda, \epsilon)=\limsup N(a, b, \lambda, \epsilon)$, as $a\rightarrow-\infty$ and $b\rightarrow+\infty$. Finally, let $\delta(\lambda_0)$ denote the number (0, 1, or 2) of the half-lines $(-\infty, 0]$, $[0, \infty)$ on which (*) is oscillatory for λ near, but exceeding, λ_0. Then λ_0 is in S if and only if $n(\lambda_0, \epsilon)\geqq\delta(\lambda_0)+1$ for all $\epsilon>0$; while λ_0 is in S' if and only if $n(\lambda_0, \epsilon)=\infty$ for all $\epsilon>0$. (Received March 14, 1950.)

321. Arthur Wouk: *Difference equations and J-matrices.* I. Preliminary report.

Let $J=(j_{ik})$ be a real symmetric infinite matrix which is of the Jacobi type, that is, $j_{ik}=0$ if $|i-k|>1$ and $j_{i,i+1}>0$. A connection between J and a Sturm difference equation $\Delta(p_n\Delta x_n)+q_nx_{n+1}=0$, where $p_n=j_{n,n+1}$, $q_n=j_{nn}+j_{n-1,n}+j_{n,n+1}$, is implicit in Hellinger's paper on *Continued fraction theory*, Math. Ann. vol. 86 (1922) pp. 18–29. Methods common in the theory of the Sturm differential equation are applied to the difference equation to obtain criteria for J to be self-adjoint and to obtain a characterization of the spectrum of J. (Received March 14, 1950.)

322*t*. D. M. Young: *The rate of convergence of an improved iterative method for solving the finite difference analogue of the Dirichlet problem.*

Using overrelaxation combined with the Liebmann method, the number of iterations required to solve the finite difference analogue of the Dirichlet Problem for an n-dimensional network with mesh size h is asymptotically proportional to h^{-1} if the overrelaxation factor ω_1 is suitably chosen. This compares with h^{-2} for the Liebmann method. If μ_1 is the largest eigenvalue of the Richardson method (Philos. Trans. Roy. Soc. London Ser. A vol. 210 (1910) pp. 307–357) $\mu_1^2\omega_1^2=4(\omega_1-1)$. For a rectangular network with sides I_ih, $\mu_1=(1/n)\sum_{i=1}^{n}\cos(\pi/I_i)$. For other regions $(1-\mu_1)$ can be estimated by comparison theorems. If $(1-\mu_1)$ is replaced by $\theta(1-\mu_1)$ $(0<\theta<1)$, the relative increase in the required number of iterations is about $(1/\theta^{1/2}-1)$. The eigenvalues for the method defined by any overrelaxation factor $\omega\geqq\omega_1$ each have absolute value $(\omega-1)$, and the normal form of the matrix is diagonal except if $\omega=\omega_1$. If $\omega=\omega_1$, the normal form has precisely one non-diagonal element. The formula for ω_1 is valid for any self-adjoint second order partial difference equation in two variables. The above statements about the normal matrix form are also valid. (Received March 6, 1950.)

APPLIED MATHEMATICS

323. J. B. Diaz: *Concerning scalar products and certain minimum and maximum principles of mathematical physics.*

Let A and B be points of three-dimensional Euclidean space; P the plane through B perpendicular to the straight line joining A and B; and S the sphere through A and tangent to P at B. Then (1) (non-negative) minimum principle: for a point C of P, $d(B, A) \leqq d(C, A)$ (d denotes distance); and (2) (non-negative) maximum principle: for a point C of S, $d(C, A) \leqq d(B, A)$. Simply derived analogues of (1) and (2) for a linear vector space with a positive semi-definite scalar product $(,)$ yield—for various choices of the scalar product—many minimum and maximum principles (some hitherto unknown) in hydrodynamics, elasticity, electromagnetic theory, and so on. All of these are thus seen to have a common origin. As an example, suppose A is the solution of a boundary value problem consisting of a single linear differential equation plus some linear boundary conditions. Four distinct principles for A are obtained. Two of them, one a minimum and one a maximum principle, characterize A among functions satisfying the boundary conditions. The other two, one a minimum and one a maximum principle, characterize A from among the solutions of the differential equation. (Received March 14, 1950.)

324. M. A. Hyman: *Solution of boundary value problems as initial value problems.*

A technique is presented for obtaining numerical solutions of boundary value problems over elementary regions, such as rectangles or rings. The method is applicable whenever a formula (F) giving an exact solution is available. If the differential (or difference) equation is linear and the region R elementary, one can usually obtain (F) by introducing normal or uncoupled coordinates. One evaluates the derivatives (or differences) of (F) on a part P of the complete boundary, then seeks the solution of the differential (or difference) equation in R with initial values given at P. This initial-value problem can be solved step-wise with simple operations at each step. Several "initial lines" may be used, on the boundary and inside the region (one motive being to control error growth). Using this method, Laplace's equation (in difference form) has been solved over a rectangle. A solution over a general region may be effected by transforming to an elementary region, provided the corresponding transform of the differential (or difference) equation is tractable. Since this technique, when applied to a linear difference equation, in effect solved a system of linear algebraic equations, it constitutes a new approach to high-order matrix inversion. (Received March 16, 1950.)

325. Herbert Jehle: *Complex and real representations of the Lorentz group without reflections.*

The relationship between Dirac's and Majorana's wave equations as well as that between the wave equations of Majorana and of the present author are investigated. (Cf. Serpe, Physical Review (1949), Jehle, Bull. Amer. Math. Soc. vol. 56 (1950)). (Received March 15, 1950.)

326t. C. N. Mooers: *Generation of unique ciphers for a finite network.*

It is desired to assign a unique cipher of letters and numerals to each chemical compound of finite size representable by a network. Such assignment is shown to be possible with a fixed number of rules making a closed set: Cuts are made at arbitrary positions in rings and at branching points to give linear segments. Segments are placed end to end in arbitrary order, and are translated into letters and signs by use of conventional chemical abbreviations, hyphens for bonds, and semicolons to indicate breaks. Each character and mark is given a number, starting with one at the left. Connection into molecule of the end of each segment is shown by the corresponding number, zero for a terminal, and these numbers are inserted into expression at ends of each segment. The result is a non-unique cipher. Process is repeated for all possible cuts and orderings, giving set of all possible ciphers for molecule. By an alphabetization of characters, the set of ciphers is ordered, and the first cipher is designated as unique. The large but finite number of steps required is a consequence of the requirement for a fixed number of rules. (Received March 15, 1950.)

327*t*. Seymour Sherman: *Games and sub-games.*

Suppose a two-person zero-sum game matrix B is decomposed into MN sub-matrices $\{B^i_j \mid 1 \leqq i \leqq M, 1 \leqq j \leqq N\}$, where the rows of B involved in the sub-matrix B^i_j are independent of j and the columns independent of i, and that the value of the game associated with B^i_j is v^i_j. If, for each i, there is a common optimal strategy for the first player in the games, $\{B^i_j \mid 1 \leqq j \leqq N\}$, then v, the value of game associated with B, is greater than or equal to $\bar{v}$, the value of game with matrix, $\overline{B} = \{v^i_j\}$. Under analogous conditions "down columns" an upper bound for the value of the game and a strategy for the second player preventing the payoff from exceeding the upper bound are given. In the event that both conditions, that is, "across rows and down columns," are satisfied, the value of the game and a set of optimal strategies are determined from the values and optimal strategies of the games, B^i_j, and the value and optimal strategies of the game, $\overline{B}$. (Received February 29, 1950.)

328. C. A. Truesdell: *On Poincaré's analogy between vorticity and mass-density.*

Let r be the radius vector, let its polyadic nth power be defined by $r^{(0)} = 1$, $r^{(n)} = r^{(n-1)}r$, let w be the vorticity vector of a continuous motion, let $\{r^{(n)}w\}$ stand for $r^{(n)}w + r^{(n-1)}wr + \cdots + wr^{(n)}$. Then $W_n = \int \{r^{(n)}w\} dV$ taken over a finite volume is expressible as an integral over the boundary surface. Hence the vector moments W_n are all zero in a motion bounded by finite walls, to which the material adheres without slipping, or in a motion vanishing at ∞ to a sufficiently high order; under somewhat weaker conditions, they are constant. These purely kinematical results, generalizing some vortex theorems of Poincaré, exhibit a strong regularity in the average distribution of vorticity in many common types of three-dimensional motions. The conditions for their validity are never satisfied by a plane motion. (Received February 28, 1950.)

329*t*. L. A. Zadeh: *The correlation function of the response of a variable system.*

Consider a linear varying-parameter system in which one or more parameters are stationary functions of time. In a recent paper (Proceedings of the I.R.E. vol. 38 (1950) pp. 291–299) the *system function* of a linear varying-parameter system was defined as a function $H(j\omega; t)$ such that $H(j\omega; t)e^{j\omega t}$ is the response of the system to an input of the form $e^{j\omega t}$. Define the correlation function of the system as $\psi(\tau; \omega) = \overline{H(j\omega; t)H(-j\omega; t+\tau)}$. Let the input to the system be stationary and uncorrelated with $H(j\omega; t)$, and let its spectral density be $S(\omega)$. It is shown that the correlation function of the response of the system to this input is given by the relation $\psi(\tau)$

$=(2\pi)^{-1}\int_{-\infty}^{\infty}\psi(\tau;\ \omega)S(\omega)e^{j\omega\tau}d\omega$. This result may be stated in the form of a theorem as follows: *The correlation functions of the input and the output of a variable system N may be regarded as the input and output of variable system N* whose system function is the correlation function of the system function of N.* This theorem has numerous practical applications. (Received March 15, 1950.)

330*t*. Othar Zaldastani: *Operator methods in the theory of one-dimensional unsteady compressible fluid flow.*

If the ratio, κ, of specific heats equals $(2N+1)/(2N-1)$, N being an integer, the nonlinear partial differential equations of the unsteady rectilinear motion of a compressible fluid can be solved by using the Legendre transform U of the stream function. Using this procedure, the author extends the results of Beckert (Annals of Physics vol. 5 (1940) p. 38). In particular, he gives general equations for the particle lines in the u, v-plane, u being the velocity and v a function of the velocity. For $\kappa=1.4$, the problem of the study of the flows is reduced to the integration of $U_{uu}-U_{vv}=-4U_v/v$, for which the general solution $U=\int v\int v(f+g)dvdv$ is used, where f and g are arbitrary functions respectively of $u+v$ and $u-v$. Bergman's integral operator of the second kind (Bergman, Proceedings of the First Symposium in Applied Mathematics, American Mathematical Society, 1949, pp. 19–40) supplies another form of a general solution, which is particularly useful in applications, since the generating function of the later operator (for N integer) becomes a Bernoulli polynomial. Several initial and boundary value problems are considered, and a complete study of the continuous motion of a fluid between the fixed pistons is given for $\kappa=1.4$. (Received March 30, 1950.)

GEOMETRY

331. H. S. M. Coxeter: *Integral Lorentz transformations.*

An integral Lorentz transformation is a linear transformation with integral coefficients leaving the form $t^2-x^2-y^2-z^2$ invariant. It is known (see A. Schild, Canadian Journal of Mathematics vol. 1 (1949) p. 47) that the group of all such transformations is generated by the changes of sign of the coordinates, the permutations of x, y, z, and the reflection that changes (r, x, y, z) into $(2t-x-y-z, t-y-z, t-z-x, t-x-y)$. The transforms of (1, 0, 0, 0) are the integral solutions of the Diophantine equation $t^2-x^2-y^2-z^2=1$. When regarded as points in hyperbolic space with the absolute quadric $t^2-x^2-y^2-z^2=0$, these are the vertices of the hyperbolic honeycomb whose Schläfli symbol is $\{4, 4, 3\}$. (In the same notation the ordinary cubic lattice in Euclidean space is $\{4, 3, 4\}$.) (Received March 6, 1950.)

332*t*. Jack Levine: *Collineations in generalized spaces.* II.

It is shown that the equations $\Delta\,\Pi^i_{jk}=0$ (Buchin Su, *Descriptive collineations in spaces of K-spreads*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 495–507) defining the projective collineations of a generalized space H_n can be expressed in the equivalent simpler form $\Delta P^i=0$ (where Δ is the Lie derivative operator). The components Π^i_{jk} of projective connection are obtained in the form $2\,\Pi^i_{jk}=\partial^2 P^i/\partial p^j\partial p^k$, and $P^i=H^i-p^iF/(n+1)$, $F=\partial H^i/\partial p^i$. The paths of an H_n are given by $d^2x^i/dt^2+\Pi^i_{jk}(x,\ p)p^jp^k+p^iF(x,\ p)/(n+1)=0$, where $p^i=dx^i/dt$. The P^i satisfy the identity $\sum\partial P^i/\partial x^i=0$. All generalized spaces of two dimensions admitting real groups G_r of projective collineations are determined by solving the equations $\Delta P^i=0$. There are nine such types of H_2 and $r\leqq 3$. (Received March 15, 1950.)

333. T. S. Motzkin: *Duality for general varieties.* Preliminary report.

Let $p \neq 0$ be the characteristic of the (algebraically closed) coordinate field. For a hypersurface whose general point P is not parabolic, duality is birational and involutory. For other hypersurfaces the tangent hyperplane T at P touches with multiplicity 2 or p^e; in both cases the dual may have, or not have, smaller dimension; and in the latter case the variety of points whose tangent hyperplane is T need not be linear, nor split into linear varieties. For a surface in 6-space duality may be non-involutory even if it is birational and even if $p=0$. For curves, cf. Bull. Amer. Math. Soc. Abstracts 56-1-25, 56-3-193, 56-3-194. Illustrations are given by monomial varieties $x_i = c_i \prod t_k^{ik}$, other abelian varieties, and Fermat hypersurfaces $\sum x_i^n = 0$. (Received May 1, 1950.)

STATISTICS AND PROBABILITY

334t. K. L. Chung and Mark Kac: *Remarks on fluctuations of sums of independent random variables.*

Let $X_1, X_2, \cdots$ be independent random variables each having the same symmetric stable distribution with exponent $\alpha(0<\alpha\leqq 2)$. Let $s_k = X_1 + \cdots + X_k$ and denote by N_n the number of changes of sign in the finite sequence $s_1, s_2, \cdots, s_n$. For $1<\alpha\leqq 2$ the limiting distribution of $N_n/n^{1-1/\alpha}$ is determined explicitly and the answer is related to the Mittag-Leffler function $E_\alpha(-x)$. For $0<\alpha<1$ the limiting distribution of $N_n/\log n$ is proved to be exponential with the exponent depending on α. For $\alpha=1$ (Cauchy distribution) the limiting distribution of $N_n/(2\pi^2(\log n)^2$ is exponential $(1-e^{-x})$. (Received February 16, 1950.)

335. Mark Kac and Murray Rosenblatt: *On diffusion with absorbing barriers in more than* 1 *dimension.*

Let $\bar{r}(t)$ be an element of the product space of n independent Wiener spaces. Let C be the bounding surface of the region R in the space of n-vectors $\bar{r}$. It is assumed that C satisfies the regularity conditions of Poincaré and that R contains the origin. The probability $u(\bar{x}, t) = \Pr(\bar{r}(t) = \bar{x}, \bar{r}(\tau) \in R, 0 \leqq \tau < t)$ is shown to be the fundamental solution of $\partial u/\partial t = \nabla^2 u/2$ with boundary value zero at C and singularity at the origin. Let $V(\bar{r})$ be the characteristic function of the complement of R. The expectation $E(\exp(-u\int_0^t V(\bar{r}(\tau))d\tau) \mid \bar{r}(t) = \bar{x}) \exp(-\bar{x}^2/2t)(2\pi t)^{-n/2}$ is considered. The derivation uses the fact that $u(x, t) = \lim_{u\to\infty} \exp(-\bar{x}^2/2t)(2\pi t)^{-n/2} E(\exp(-u\int_0^t V(r(\tau))d\tau) \mid \bar{r}(t) = \bar{x})$ and the method of barriers. (Received April 28, 1950.)

336t. C. N. Mooers: *Logic of selective systems.*

In the digital description and selection of non-numerical information, one technique (Zatocoding) employs a collection of information carriers, such as cards or frames of film strip, each bearing a multiposition digital symbol field with each position taking a mark or blank, assigns a random pattern of marks in the field to each descriptor of information, and combines the several patterns in each field by Boolean addition. Descriptor patterns defining a selection of cards are combined by Boolean addition in the selector, and selection occurs when the total pattern in the field includes the total selector pattern. Aside from the statistically-expected "noise" in the selection, selection is found to be according to the logical product of the selector descriptors. One-step selection according to logical sum is found to be incompatible with adequate control of "noise" for all variations by complementation or duality when pattern inclusion rules the selections. (Received March 15, 1950.)

TOPOLOGY

337t. R. D. Anderson: *On collections of pseudo-arcs.*

The author studies collections of pseudo-arcs. Bing has recently shown that if H and K are chained hereditarily indecomposable continua, H is topologically equivalent to K. The pseudo-arc of Moise is, by construction, a chained hereditarily indecomposable continuum. The author shows, using Bing's result, that there exists a continuous collection of pseudo-arcs filling up the plane. It is known that such a collection must be topologically equivalent to the plane. In demonstrating the above theorem the author proves a lemma which implies, among other things, that there exists a continuous collection of pseudo-arcs filling up a closed annulus such that this collection is topologically equivalent to a simple closed curve. (Received March 14, 1950.)

338. R. H. Bing and E. E. Floyd: *Coverings with connected intersections.*

Suppose S is a compact locally connected metric continuum. There is a countable basis for S such that the common part of the elements of any subcollection of this basis is connected and uniformly locally connected. Furthermore, the collection of closures of the elements of this basis is a family G of continuous curves such that the common part of the elements of any subcollection of G is either null or a continuous curve. (This is an extension of a result of R. D. Anderson [*Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 451–460].) These results are proved by making use of brick partitionings of S. (Received March 7, 1950.)

339. V. L. Klee: *A proof that Hilbert space is homeomorphic with the surface of its unit sphere.*

Suppose that E is a nonreflexive normed linear space. By use of the fact that E must contain a decreasing sequence of bounded closed convex sets whose intersection is empty, it is proved that (a) ***every hyperplane in*** E ***is homeomorphic with the surface of the unit sphere of*** E; (b) ***there is a homeomorphism of period*** 2 ***without fixed points of*** E ***onto itself which takes the*** (*solid*) ***unit sphere onto itself.*** These statements also apply to many reflexive spaces, and in particular to Hilbert space (since (l^1) is homeomorphic with (l^2)). Thus certain questions of S. Kakutani and P. A. Smith are answered. (Received March 14, 1950.)

340t. J. E. L. Peck: *The embedding of topological groupoids in topological quasigroups.*

An alternation topological groupoid is a topological set, closed under a binary, continuous operation, in which any four elements satisfy the relation $(ab)(cd) = (ac)(bd)$. An element is a *canceller if all its powers may be cancelled on both sides. A topological quasigroup is a topological groupoid in which any one of the three elements, related by $a=bc$, is a unique, continuous function of the other two. It is proved that an alternation topological groupoid, in which the set of *cancellers is open and in which the operation satisfies a further continuity restriction, may be embedded in an alternation topological groupoid, whose set of *cancellers form a topological quasigroup, containing the original *cancellers. This topological theorem is based on the algebraic results of Scholander (Bull. Amer. Math. Soc. vol. 55 (1949) pp. 746–757). A similar theorem is proved for noncommutative topological semigroups, using the algebraic results of Dubreil (*Algèbre*, Paris, 1946). It is also shown that a compact topological semigroup, with a cancellation law, is a topological group. (Received April 17, 1950.)

H. M. MacNeille,
Executive Director

THE APRIL MEETING IN CHICAGO

The four hundred fifty-seventh meeting of the American Mathematical Society was held at the University of Chicago, Chicago, Illinois, on Friday and Saturday, April 28–29, in conjunction with meetings of the Institute of Mathematical Statistics. Approximately 500 persons attended, including the following 208 members of the Society:

A. A. Albert, W. R. Allen, D. N. Arden, K. J. Arnold, Max Astrachan, W. L. Ayres, Reinhold Baer, R. G. Bartle, A. F. Bausch, J. H. Bell, L. D. Berkovitz, S. F. Bibb, K. E. Bisshopp, L. M. Blumenthal, W. M. Boothby, D. G. Bourgin, H. R. Brahana, Richard Brauer, C. F. Briggs, Ellen F. Buck, R. C. Buck, E. L. Buell, I. W. Burr, B. G. Carlson, R. E. Carr, E. D. Cashwell, T. E. Caywood, Lamberto Cesari, Herman Chernoff, E. W. Chittenden, Helen M. Clark, Mary Dean Clement, Harvey Cohn, J. B. Coleman, J. A. Colon, B. H. Colvin, E. G. H. Comfort, J. J. Corliss, M. L. Curtis, M. M. Day, John DeCicco, R. F. Deniston, Flora Dinkines, N. J Divinsky, D. G. Duncan, P. S. Dwyer, W. F. Eberlein, H. P. Evans, H. S. Everett, Chester Feldman, W. L. Fields, D. A. Flanders, L. R. Ford, J. S. Frame, Evelyn Frank, Cleota G. Fry, R. E. Fullerton, M. P. Gaffney, B. A. Galler, Murray Gerstenhaber, H. A. Giddings, J. K. Goldhaber, C. C. Goldman, Michael Golomb, R. D. Gordon, R. N. Goss, M. J. Gottlieb, S. H. Gould, L. M. Graves, V. G. Grove, John Gurland, M. M. Gutterman, Marshall Hall, P. R. Halmos, P. C. Hammer, W. L. Hart, H. D. Hartstein, M. Agnes Hatke, Ruth Heinsheimer, R. G. Helsel, Fritz Herzog, G. F. Hewitt, E. H. C. Hildebrandt, T. H. Hildebrandt, G. P. Hochschild, R. E. von Holdt, D. L. Holl, W. A. Howard, C. C. Hsiung, S. P. Hughart, H. K. Hughes, Ralph Hull, P. E. Irick, W. E. Jenner, Meyer Jerison, S. M. Johnson, R. V. Kadison, G. K. Kalisch, L. H. Kanter, Leo Katz, J. B. Kelley, L. M. Kelly, W. M. Kincaid, E. C. Kleinhans, L. A. Knowler, Jacob Korevar, H. G. Landau, E. P. Lane, Leo Lapidus, Sim Lasher, C. G. Latimer, J. S. Leech, K. B. Leisenring, F. C. Leone, D. J. Lewis, O. I. Litoff, A. J. Lohwater, F. W. Lott, W. S. Loud, A. W. McGaughey, Saunders MacLane, W. G. Madow, Morris Marden, A. M. Mark, Kenneth May, D. M. Merriell, H. L. Meyer, Josephine M. Mitchell, J. T. Moore, C. W. Moran, M. L. Mousinho, Leopoldo Nachbin, Zeev Nahari, W. J. Nemerever, K. L. Nielsen, R. Z. Norman, E. P. Northrop, F. S. Nowlan, Rufus Oldenburger, H. W. Oliver, Daniel Orloff, E. H. Ostrow, G. K. Overholtzer, Gordon Pall, Katharine Parks, Mary H. Payne, Marilia C. Peixoto, M. M. Peixoto, George Piranian, G. B. Price, L. E. Pursell, A. L. Putnam, Gustave Rabson, Tibo Rado, G. Y. Rainich, O. W. Rechard, W. T. Reid, Haim Reingold, P. R. Rider, R. A. Roberts, W. G. Rosen, I. A. Rosenberg, Arthur Rosenthal, J. M. Sacks, Hans Samelson, L. J. Savage, R. H. Scherer, O. F. G. Schilling, W. T. Scott, I. E. Segal, B. R. Seth, M. Anice Seybold, M. E. Shanks, A. S. Shapiro, Jack Sherman, J. H. Siedband, G. F. Simmons, I. M. Singer, M. L. Slater, M. F. Smiley, G. W. Smith, Elizabeth S. Sokolnikoff, E. H. Spanier, G. L. Spencer, M. D. Springer, W. L. Stamey, B. M. Stewart, A. F. Svoboda, R. L. Swain, P. M. Swingle, J. V. Talacko, R. M. Thrall, E. A. Trabant, E. F. Trombley, C. H. Tyler, Eugene Usdin, F. A. Valentine, Bernard Vinograde, D. R. Waterman, H. F. Weinberger, L. M. Weiner, M. E. Wescott, J. B. Wright, L. C. Young, J. W. T. Youngs, R. K. Zeigler, Daniel Zelinsky, Antoni Zygmund.

Sessions for contributed papers were held on Friday at 10:45 A.M. and 3:30 P.M., and on Saturday at 10:30 A.M. and 11 A.M. The last session was held jointly with the Institute of Mathematical Statistics. Presiding officers for the various sessions were Professors Marshall Hall, Saunders MacLane, G. B. Price, and G. Y. Rainich. Dr. John Gurland presided over the joint sessions with the Institute of Mathematical Statistics.

At 2:00 P.M. Friday, by invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor Hans Samelson, of The University of Michigan, delivered an address entitled *Topology of Lie groups*. Professor A. A. Albert, of the University of Chicago, was the presiding officer.

There was a tea for visiting members and their guests held in the lounge of Eckhart Hall on Friday afternoon.

Abstracts of papers presented follow below. Abstracts whose numbers are followed by the letter "*t*" were read by title. Paper number 357 was read by Professor Buck, paper number 359 by Mr. Klein, paper number 364 by Professor Piranian, and paper number 370 by Mr. Stamey. Mr. Luke was introduced by Professor Eric Reissner, Dr. Zahorski, Mr. Calderon and Mr. Klein by Professor Antoni Zygmund.

ALGEBRA AND THEORY OF NUMBERS

341. A. A. Albert: *A theory of power-associative commutative algebras.*

Let A be a commutative power-associative ring whose characteristic is prime to 30 and suppose that the equation $2x=a$ has a unique solution x in A for every a of A. We show that if A is simple and contains a pair of orthogonal idempotents whose sum is not the unity quantity of A then A is a Jordan ring. The result may be applied to commutative power-associative algebras A over a field F of characteristic not 2, 3, or 5. Every simple algebra not a nilalgebra has a unity quantity e and we may extend the center F of A to a field K such that $e=e_1+\cdots+e_t$ for pairwise orthogonal idempotents e_i of A_K. The maximal value of t is called the degree of A and we show that if $t>2$ then A is a Jordan algebra. All simple Jordan algebras of degree $t>2$ are shown to be classical Jordan algebras, that is, the same type of algebras that arise when F has characteristic zero. Define the radical N of any power-associative commutative algebra A of characteristic prime to 30 to be the maximal nilideal of A. Then if e is a principal idempotent of A we show that $A_e(1/2)+A_e(0)$ is contained in N. Thus every semisimple A has a unity quantity and we show that A is uniquely expressible as the direct sum of simple algebras with unity quantities. Our theory thus includes the theory of Jordan algebras of characteristic $p\neq 2, 3, 5$. (Received March 20, 1950.)

342*t*. Rheinhold Baer: *Free mobility and orthogonality.*

Suppose that V is an n-dimensional vector space over the [not necessarily commutative] field F, and that P is a domain of positivity of F. If $b_1, \cdots, b_k$ are independent elements in V, then the subsets $\sum_{j=1}^{i-1} Fb_j + Pb_i$, for $i=1, \cdots, k$, of V form an i-dimensional chain of incident sub-half-spaces of V. The problem is to characterize the groups of linear transformations of V which are simply transitive on the totality of these i-dimensional chains of incident sub-half-spaces of V. This problem is solved under the additional hypotheses that $1<i$ and $2<n$. (Received February 2, 1950.)

343*t*. J. R. Brown and A. B. Showalter: *On a problem connected with the theory of the Riemann zeta-function.*

In his paper (*On some approximate Dirichlet-polynomials in the theory of the zeta-function of Riemann*, Det Kgl. Danske Videnskabernes Selskab. vol. 24 (1948)) Turán proved that if there is an n_0 such that for $n>n_0$ the partial-sums $U_n(s)=1/1^s+1/2^s + \cdots +1/n^s$ do not vanish in the half-plane $R(s)>1$, then Reimann's hypothesis concerning the roots of the zeta-function is true. Turán also discussed the cases for $n=1, 2, 3, 4, 5$ and showed that $U_n(s)\neq 0$ for these values of n. In this note the authors prove $U_n(s) \neq 0$ for $n=6$ by showing the still stronger result: $R[U_6(s)]>0$ for $R(s)>1$. (Received March 13, 1950.)

344*t*. F. Marion Clarke: *On the factorisation of polynomials in n variables.* IV.

The theorem of the author's Bull. Amer. Math. Soc. Abstract 55-7-399 concerning the uniqueness of the splitting field of polynomial in n variables over a coefficient field of characteristic zero is generalized for polynomials over fields of any characteristic. (Received March 17, 1950.)

345. Nathan Divinsky: *Power-associative crossed extension algebras.*

This paper is a study of power-associative crossed extension algebras, that is, algebras built from an algebra A, a group of nonsingular linear transformations G of A, a subset H of G, and a set of elements g of A. These algebras include crossed product algebras and we first show that a power-associative crossed product is associative. When A is a simple associative algebra over a field of characteristic not 2, we show that the power-associative crossed extension C is either associative or is an extended Cayley algebra. Then we show that if A is a G-simple semi-simple associative algebra over a field of characteristic not 2 and if the identity is the only semi-identity transformation in G, then either C is associative or A is simple when the above result holds. In proving these results we show that every automorphism T of a simple noncommutative ring with minimum condition such that $x \cdot x^T = x^T \cdot x$ for every x must be the identity, and we extend this to semi-simple rings. Finally we generalize the definition of crossed products to include Jordan algebras and find sufficient conditions for these generalized crossed products to be simple Jordan algebras. (Received April 24, 1950.)

346. D. G. Duncan: *Note on a formula by Todd.* Preliminary report.

In a recent paper (Cambridge Philosophical Society, vol. 45, part 3) Todd has given the following formula: $\{m\} \otimes S_n = \sum_\sigma \theta_\sigma \{\sigma\}$. Here $\otimes$ denotes Littlewood's "new" or induced multiplication of S functions. In this note a new characterization is given to the coefficients θ_σ and a short method provided for determining them in terms of the removal of hooks from the Young diagram associated with the partition $[\sigma]$. θ_σ is shown to be zero if the removal of hooks of length n leaves a core and otherwise is plus or minus one according as m plus the sum of the heights of the removed hooks is even or odd. (Received April 24, 1950.)

347. J. K. Goldhaber: *Conditions ordering the characteristic roots of matrices.*

Let $\mathfrak{A}$ be a subalgebra of a total matric algebra of order n^2 over an algebraically closed field $\mathfrak{F}$. Suppose that the identity matrix is in $\mathfrak{A}$. Let λ_{ij}, $j=1, \cdots, n$, denote the characteristic roots of $A_i \in \mathfrak{A}$. The following three properties are shown to be equivalent. I: The characteristic roots of every pair of matrices $A_1, A_2 \in \mathfrak{A}$ may be so ordered that the characteristic roots of A_1+A_2 are $\lambda_{1j}+_{2j}$; II: The characteristic roots of every finite set of matrices $\{A_i\}_{i=1}^{k} \in \mathfrak{A}$ may be so ordered that the characteristic roots of $\sum_{i=1}^{k} a_i A_i$ are $\sum_{i=1}^{k} a_i \lambda_{ij}$ for all $a_i \in \mathfrak{F}$; III: Let $f(x_1, \cdots, x_k)$ be any polynomial. The characteristic roots of every finite set of matrices $\{A_i\}_{i=1}^{k} \in \mathfrak{A}$ may be so ordered that the characteristic roots of $f(A_1, \cdots, A_k)$ are $f(\lambda_{1j}, \cdots, \lambda_{kj})$. This ordering is the same for every f. In the proof of this equivalence the following mapping theorem is used: Let $\mathfrak{A}$ and $\mathfrak{B}$ be linear algebras over F. Let Φ be a mapping of $\mathfrak{A}$ onto $\mathfrak{B}$ satisfying (a) $\Phi(e)=e'$ where e and e' are the units of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, (b) $\Phi(\sum_{i=1}^{k} a_i A_i) = \sum_{i=1}^{k} a_i \Phi(A_i)$, and (c) if $\prod_{i=1}^{k} A_i = 0$ then $\prod_{i=1}^{k} \Phi(A_i) = 0$. Then Φ is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}-\mathfrak{N}$, where $\mathfrak{N}$ is the radical of $\mathfrak{B}$. (Received March 13, 1950.)

348. J. B. Kelly: *A closed set of algebraic integers.*

An algebraic integer, θ, will be said to belong to class S_1 if all conjugates of θ, save θ itself, lie inside the unit circle and to class S_2 if all conjugates of θ, save θ and one additional conjugate, θ', lie inside the unit circle. The subclass of real elements of S_2 will be denoted by S_2', the subclass of non-real elements by S_2''; if $\theta \in S_2''$, then $\theta' = \bar{\theta}$. Salem (Duke Math. J. vol. 11 (1944) pp. 103–108; Duke Math. J. vol. 12 (1945) pp. 153–172) proved that S_1 is a closed set. It is shown here that although the sets S_2, S_2', and S_2'' are not closed, the set $S_1 \cup S_2''$ is closed. Further, $S_1 \cup S_2''$ has no limit points on the unit circle. (Received April 24, 1950.)

349. D. J. Lewis: *Cubic homogeneous equations in p-adic number fields.*

It is shown that every cubic homogeneous polynomial equation $f(x_1, x_2, \cdots, x_n) = 0$ with coefficients in a p-adic number field has a nontrivial integral solution in the field of the coefficients provided $n \geqq 10$. Simple examples of equations having $n \leqq 9$ which possess only the trivial solution are given. Thus ten is the minimum number of variables necessary to insure the existence of nontrivial solutions. (Received March 20, 1950.)

350*t*. A. R. Schweitzer: *Grassmann's extensive algebra and modern number theory.*

Grassmann's algebra is associated with modern number theory through the "algebraic keys" (clefs algébriques) of A. Cauchy (C. R. Acad. Sci. Paris vol. 36) and through researches of E. Lasker. In this paper the author discusses Grassmann's theory in the light of the following articles by Lasker: (I) *An essay on the geometrical calculus*, Proc. London Math. Soc. vol. 28 (1896–1897) p. 217; (II) Ibid. p. 500; (III) *A new method in geometry*, Amer. J. Math. vol. 30 (1908) p. 65; (IV) *Zur Theorie der Moduln und Ideale*, Math. Ann. vol. 60 (1905) p. 20. (I) constitutes, in effect, a survey of Grassmann's theories of 1844 and 1862 as interpreted by Lasker in terms of algebraic identities, modern algebra, and projective geometry. (II) is construed by the author as a logical continuation of (I) and as a means of transition from (I) to (III); (II) and (III) together are regarded as a connecting bond between Grassmann's Ausdehnungslehre and modern number theory represented by (IV). The latter article constitutes, as is well known, a development of Kronecker's modular systems, that is, ideals of polynomial rings. Reference is made to W. Krull, *Idealtheorie*, Berlin, 1935. (Received March 13, 1950.)

351*t*. A. B. Showalter: *On a problem connected with the extended Riemann hypothesis.*

Let $\chi(n)$ denote a complex character (mod k) and $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$ for $R(s) > 0$. It has been conjectured ("extended Riemann hypothesis") that if $L(s, \chi) = 0$ for complex s [with $R(s) > 0$], then $R(s) = 1/2$. However, it has not even been proved that $L(s, \chi) \neq 0$ for $s > 0$. The latter would be true if (1) $R[L(s, \chi)] > 0$ for $s > 0$. Chowla has conjectured that (1) is true. In this paper the author finds by numerical calculations that (1) is true in a large number of special cases. The author also finds that (2) $R[L(1, \chi)] > 0$ for all complex χ(mod k) where $k \leqq 37$. To prove (1) in special cases the author uses the method described in Acta Arithmetica (vol. 1 (1936) pp. 115–116). To prove (2) he also uses two new methods devised by the author. (Received March 13, 1950.)

352*t*. A. B. Showalter: *Remark on partial sums of L-series.*

Let $f(m \cdot n) = f(m)f(n)$, $f(1) = +1$, and let each $f(n) = +1, -1$, or 0. Let $\lambda(n)$ denote Liouville's number theoretic function. The author proves that if $\sum_{n=1}^{k} \lambda(n)/n > 0$ for all $k \leqq i$, then $\sum_{n=1}^{i} f(n)/n \geqq \sum_{n=1}^{i} \lambda(n)/n$. (Received March 13, 1950.)

353. Bernard Vinograde: *Hermitean classification of matrices.* Preliminary report.

The space M of unitary vectors X such that $AX = A^*X$ for a complex matrix A will be called the hermitean domain of A. A subspace $M_0 \subset M$ is invariant if and only if it completely reduces A unitarily. The usual theory of a Hermitean A is readily modified and extended. Any ring contained in the set of matrices whose hermitean domains contain M is completely reduced by M and easily characterized. (Received March 15, 1950.)

ANALYSIS

354*t*. R. P. Agnew: *Mean values and Frullani integrals.*

Using results of Iyengar (Proc. Cambridge Philos. Soc. vol. 37 (1941) pp. 9–13) and Agnew (Duke Math. vol. 9 (1942) pp. 10–19) or, alternatively, of Ostrowski (Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 612–616) and a theorem on mean values of functions, we obtain the following incisive theorem involving Frullani integrals and the familiar mean values of $f(x)$ at 0 and at ∞. Let $f(t)$ be Lebesgue integrable over each finite interval $0 < m \leqq t \leqq M < \infty$. If the Frullani integral in the left member of (*) $\int_0^\infty t^{-1}[f(at)-f(bt)]dt=\lambda[\lim_{x\to\infty} x^{-1}\int_0^x f(t)dt-\lim_{x\to 0} x^{-1}\int_0^x f(t)dt]$ exists for each $\lambda=\log (a/b)$ in a set of positive measure, then the mean values in the right member exist and (*) holds for each pair of positive numbers a and b. On the other hand if the mean values in the right member of (*) exist, then the left member exists and (*) holds for each pair of positive numbers a and b. (Received April 11, 1950.)

355*t*. E. F. Beckenbach: *Generalized convex curves.*

In a domain D of the plane, let there be given a family $\{C\}$ of Jordan arcs C each separating D into two components, and such that for each pair of distinct points A and B of D there is exactly one C_{AB} of $\{C\}$ passing through both A and B. Then a Jordan arc K in D shall be said to be $\{C\}$-convex to one of its sides s provided that, for each pair of points A and B of K, the arc between A and B, of the above C_{AB}, lies on the side of K opposite to s. Properties of the family $\{C\}$ and of these generalized convex curves, analogous to known properties of generalized convex functions, are investigated; in particular, generalized convex regions and their boundaries are defined and discussed. (Received March 10, 1950)

356. L. D. Berkovitz: *Spherical summation and localization of double trigonometric series.* Preliminary report.

Let $U=\sum a_{mn}e^{i(mx+ny)}$ be a double trigonometric series with coefficients $a_{mn}=o(((m^2+n^2)^\gamma)^{1/2})$, $\gamma\geqq -1$. An operator ∇^{-2} is defined, which on each term of U acts as a formal inverse to the Laplacian. To U, ∇^{-2} is applied a sufficient number of times, and a series is obtained which converges ***uniformly*** to a function $F(x, y)$. If the function F vanishes on a rectangle R, contained in the square bounded by $x=\pm\pi$, $y=\pm\pi$, then for (x, y) in any rectangle R', contained in R, the series U is uniformly circularly summable (in the sense of Bochner) $(C, \gamma+1)$ to zero. This fact is a consequence of localization formulae which for the case $\gamma=-1$, say, read: $(\sum_{m^2+n^2\leqq R^2} a_{mn}e^{i(mx+ny)}-(1/4\pi^2)\int_0^{2\pi}\int_0^{2\pi} F(u, v)\lambda(u, v)\nabla^2 D_R(u-x, v-y)dudv)\to 0$ where $\lambda(u, v)$ is a sufficiently regular localizing function, and $D_R(x, y)$ is the analogue of the Dirichlet Kernel for circular summation of double Fourier Series. The method of proof uses formal multiplication of series in a manner somewhat analogous to that developed for single series by Rajchman and Zygmund, but applied to circular sums and means. (Received March 15, 1950.)

357. R. P. Boas and R. C. Buck: *Expansions of entire functions in polynomial series.*

J. M. T. Whittaker has obtained conditions under which an entire function has a convergent expansion in a basic series $\sum C_n p_n(z)$ (*Interpolatory function theory*, Cambridge, 1938). One of the authors has observed that in the case of Appell polynomials, convergence may occur for a much larger class of functions, if the coefficients $\{C_n\}$ are chosen differently. In the present paper, a general theory for the expansion of entire functions into convergent or summable series of Appell-type polynomials is obtained. (Received March 16, 1950.)

358. A. P. Calderon and George Klein: *On an extremal problem concerning trigonometrical polynomials.*

This paper is concerned with the following generalization of a theorem of P.

Erdős: suppose $\phi(x)$ is a function defined for non-negative x and satisfies the condition that $(\phi(x)-\phi(0))/x$ be a non-negative non-decreasing function of x, $x \geqq 0$. Then the maximum of the integral $\int_0^{2\pi}\phi(|S'(x)|)dx$ for all real trigonometrical polynomials $S(x)$ of order n, bounded in absolute value by 1, is achieved by the Tchebycheff polynomial $\cos(nx+\alpha)$. If ϕ is not a constant function, then among all such polynomials the Tchebycheff polynomial only achieves this maximum. This theorem applies, for example, to nondecreasing functions and the theorem of Erdős referred to above for arc length is obtained by setting $\phi(x)=(1+x^2)^{1/2}$. (Received April 28, 1950.)

359t. C. C. Camp: *An expansion associated with an irreducible partial differential equation.* Preliminary report.

The equation $u_x+u_y+\lambda a(x, y)u=0$ for which $a(x, y)$ is not separable into $g(x)+h(y)$ cannot be reduced by Bernoulli's method to a system of ordinary differential equations. The boundary conditions $W_1=\int_0^1[1-u(x, 1)\ v(x, 0)]dx=0$, $W_2=\int_0^1[u(1, y)v(0, y)-1]dy=0$, where $v=1/u$ is a solution of the adjoint equation, likewise cannot be carried over to equivalents for an ordinary boundary value problem unless $a(x, y)$ is properly restricted. Although a formal expansion can be set up, even with a second parameter μ, such that $u=\exp[-\lambda H(x, y)+\mu(x-y)]$ and $H_x+H_y=a(x, y)$, no valid expansion exists since it can be shown that the only characteristic value for λ is $\lambda=0$. If $H(x, 1)=H(x, 0)$ and $H(1, y)=H(0, y)$, then λ will have a countably infinite number of characteristic values and the usual convergence proof involving a Green's function and a double contour integral in λ, μ can be carried through. Without the second parameter μ the formal expansion can be shown not valid. This theory is being extended to more than 2 independent variables. (Received March 15, 1950.)

360. W. F. Eberlein: *Decomposition of weak almost periodic functions.* Preliminary report.

Let $x(t)$ $(-\infty<t<\infty)$ be weakly almost periodic (for this concept cf. W. F. Eberlein, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 217–240). Then $x(t)$ admits the unique decomposition $x(t)=x_1(t)+x_2(t)$, where $x_1(t)$ is uniformly almost periodic, and $x_2(t)$ is a weakly almost periodic function all of whose Fourier coefficients vanish. The point spectrum of a weakly almost periodic function is thus determined. Preliminary results on extension of the theorem to arbitrary locally compact Abelian groups are obtained. (Received February 13, 1950.)

361. R. E. Fullerton: *Characterizations of L and C spaces.*

Let X be a Banach space whose unit sphere S is the convex closure of its set of extreme points. Then X is a space of integrable functions or of continuous functions if any one of the following three equivalent conditions hold: (1) If $\|x\|=1$, $x\in F$ where F is a maximal convex subset of the surface of S and S is the convex set determined by F and $-F$. (2) Every extreme point of S is the vertex of a cone C having the property that $S=C\cap(-C)$. (3) If $u\in S$ is an extreme point of S, then for any λ, $0\leqq\lambda<2$, $S\cap((\lambda/2)u+S)=(\lambda/2)u+(1-\lambda/2)S$. (Received March 15, 1950.)

362. L. M. Graves: *Induced linear transformations on spaces of continuous functions.*

Let $\mathfrak{U}$ and $\mathfrak{V}$ be Banach spaces, and $\mathfrak{S}$ a compact metric space. Let $\mathfrak{X}$ and $\mathfrak{Y}$ denote the Banach spaces consisting of all continuous functions on $\mathfrak{S}$ to $\mathfrak{U}$ and to $\mathfrak{V}$, respectively. Let Ω_0 be the space of all linear continuous transformations K mapping $\mathfrak{U}$ *onto* $\mathfrak{V}$. Let $L(K, v) = \text{g.l.b.}\|u\|$ for $Ku = v$, $I(K) = \text{l.u.b.}\ L(K, v)$ for $\|v\| = 1$. By a theorem of Banach, $I(K) < \infty$ on Ω_0. Now let $K(s)$ be continuous on $\mathfrak{S}$ to Ω_0, and let $y = \kappa x$ denote the induced transformation defined by $y(s) = K(s)x(s)$. Then κ maps $\mathfrak{X}$ *onto* $\mathfrak{Y}$, and $I(\kappa) = \max I(K(s))$ on $\mathfrak{S}$. Moreover, if $\mathfrak{S}$ is a linear interval, and $N > I(\kappa)$, then for each y there exists x with $\kappa x = y$ and $\|x(s)\| < N\|y(s)\|$ on $\mathfrak{S}$. In this special case, the proof is made by constructing a sequence of polygonal approximations to a solution $x(s)$. By induction the theorem may be extended to the case when $\mathfrak{S}$ is an interval in n-space, and then to the Hilbert parallelotope. From this result the general case is obtained with the help of a device used by Hestenes. The theorem has numerous applications in securing existence theorems for the solution of functional equations. (Received March 13, 1950.)

363. Josephine M. Mitchell: *A convergence theorem for multiple Fourier series summed by spherical means.*

Let $S_R = \sum_{\nu \leqq R} a_{m_1 \cdots m_k} \exp\,(i(x_1 m_1 + \cdots + x_k m_k))$ $(\nu = m_1^2 + \cdots + m_k^2)$ be the spherical partial sum of the Fourier expansion of a Lebesgue square integrable function $f(x_1, \cdots, x_k)$ defined on the k-dimensional hypercube $D(-\pi \leqq x_j \leqq \pi, j = 1, \cdots, k)$. The Lebesgue function $(1/(2\pi)^k)\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left| \sum_{\nu \leqq R} \exp i \sum m_j(x_j - y_j) \right| dy_1 \cdots dy_k = O(R^{k/4})$ from which it follows that $S_R = O(R^{k/8})$. Consequently we may prove the theorem that if $\sum (m_1^2 + \cdots + m_k^2)^{k/4} a_{m_1 \cdots m_k}^2 < \infty$, then $\lim_{R \to \infty} S_R$ exists almost everywhere in D. For the cases $k = 2$ and 3 the exponent $k/4$ may be made considerably smaller. The proofs of these statements use the classical methods of Fourier and orthogonal series and represent an improvement over results obtained by different methods (cf, for example, Chandrasekharan and Minakshisundaharan, Duke Math. J. vol. 14 (1947) and Chandrasekharan, Bull. Amer. Math. Soc. vol. 52 (1946)). (Received March 15, 1950.)

364. George Piranian and A. J. Lohwater: *Linear accessibility in Jordan regions.*

A point P on the boundary C of a Jordan region R is said to be ***linearly accessible*** if there exists a rectilinear segment QP whose points lie interior to R, except for P. There exists a Jordan curve C in the ζ-plane such that, under the mapping $z = f(\zeta)$ of the simply-connected region determined by C onto the unit circle, the set of linearly accessible points of C is mapped into a set of Lebesgue measure zero. The proof is based on the construction of an appropriate Taylor series with large gaps. (Received March 15, 1950.)

365*t*. R. K. Ritt: *Algebraic functions in an Abelian normed ring.* Preliminary report.

Let $K = R(e, a)$ be an Abelian normed ring, $\sigma(a)$ the spectrum of a. This paper examines the relationship between the topology of $\sigma(a)$ and the possibility of solving the equation $u^n = b$, $b \in K$. Applications are made to transformations in a Banach space. (Received March 20, 1950.)

366*t*. I. E. Segal: *Decomposition of operator algebras.* II. *Multiplicity theory.*

Every commutative W^*-algebra (=weakly closed and self-adjoint algebra of bounded linear operators on a Hilbert space) is a direct sum of W^*-algebras of uniform multiplicity, an algebra of the latter type of multiplicity $n>0$ being unitarily equivalent to an n-fold copy of a maximal abelian W^*-algebra. This last algebra is unitarily equivalent to the algebra of all multiplications by bounded measurable functions on L_2 over a suitable measure space. An arbitrary W^*-algebra is determined within unitary equivalence by the Boolean ring (of all measurable sets modulo null sets) associated thereby with each multiplicity. It follows that any commutative W^*-algebra is algebraically isomorphic to a maximal abelian W^*-algebra in a fashion which is weakly bicontinuous and which preserves the operational calculus. The decomposition of W^*-algebras into parts of uniform multiplicity is valid also for non-commutative algebras, and any W^*-algebra of finite uniform multiplicity $n>0$ is unitarily equivalent to an n-fold copy of a W^*-algebra A for which A' is abelian. (Received April 7, 1950.)

367. B. R. Seth: *Some solutions of the wave equation.*

A number of solutions of the two-dimensional wave equation $\Delta^2\phi=\partial^2\phi/\partial t^2$ have been obtained by B. R. Seth, D. G. Christopherson, and P. N. Sharma. In all these solutions the boundary is rectilinear and either ϕ or the normal derivative vanishes over the boundary. In the present paper the boundary condition is taken as $\phi=$a constant, and the corresponding forms of ϕ are obtained for a number of boundaries. For an equilateral and some rhombus and pentagonal boundaries ϕ is obtained in a finite number of trigonometrical terms. (Received February 17, 1950.)

368t. Zygmunt Zahorski: *On the first derivative.* (*Sur la première dérivée.*)

This paper is concerned primarily with characterizing five classes of functions in terms of the properties of the sets $[f'(x)>a]$, where $f'(x)$ exists everywhere. In the case where $f'(x)$ is bounded, necessary and sufficient conditions are developed. For the cases where $f'(x)$ is either finite or assumes the values $\pm\infty$, only necessary conditions are developed. The five classes of functions considered comprise a descending sequence intermediate between the first class of Baire with Darboux property and the class of approximately continuous functions. In particular, the following theorem is established: Let $f(x)$ be a function which passes through all mean values and such that, allowing infinite values, $f'(x)$ exists everywhere, with the possible exception of the points of an enumerable set, and $f'(x)\geqq 0$ almost everywhere; then $f(x)$ is continuous and non-decreasing. This theorem is established without making any assumption as to the measurability of $f(x)$. (Received April 24, 1950.)

APPLIED MATHEMATICS

369. Y. L. Luke: *Some notes on integrals involving Bessel functions.*

In a number of applied mathematics problems, the results depend on the values of $\int_0^x e^{ku}u^{\nu}Z_n(\lambda u)du$ where $Z_n(u)$ represents any of the Bessel functions of the first three kinds or the modified Bessel functions. The purpose of this note is to develop some recursion formulae and power series expansions, which can be used to compute the above integral. The results are applied to show that $\int_0^x e^u K_0(u)du=xe^x[K_0(x)+K_1(x)]-1$. Thus one obtains an alternative proof that the constant term in the asymptotic expansion of the latter integral is -1. (H. Bateman and R. C. Archibald, *Guide to* *tables of Bessel functions*, MTAC, vol. 1, no. 7, July 1944, p. 226.) Other applications also are presented. (Received March 2, 1950.)

GEOMETRY

370. L. M. Blumenthal and W. L. Stamey: *Characterization of pseudo-$S_{n,r}$ sets.*

A semimetric space S is pseudo-$S_{n,r}$ provided each of its $(n+2)$-tuples is congruently imbeddable in the convexly metrised n-sphere $S_{n,r}$ of diameter πr, but S itself is not. If no two points of a pseudo-$S_{n,r}$ set S have distance πr, it was shown by Blumenthal and Thurman (Amer. J. Math. vol. 62 (1940) pp. 835–854) that for every integer k, the matrix $\|\cos (p_i p_j / r)\|$ $(i, j=1, 2, \cdots, k)$ of each k-tuple $p_1, p_2, \cdots, p_k$ of points of S has (upon multiplication by -1 of appropriate rows and the same numbered columns) every element equal to 1 or $-1/(n+1)$, provided S contains more than $n+3$ points. In this note, the characterization is completed by showing that the restriction that S be without diametral point-pairs may be dropped. The Blumenthal-Thurman theorem remains valid provided S contains more than $2(n+3)$ points, or if S contains $n+k+3$ points $(k>0)$, not more than $k-1$ of its pairs be diametral. (Received March 15, 1950.)

371. H. R. Brahana: *Finite metabelian groups and the lines of a projective four-space.*

The groups dealt with are metabelian (that is, of class 2); their elements are all of order p; their centrals and commutator subgroups coincide; and each is generated by five elements. There is one largest group G, of order p^{15}, with these properties; every other group is a quotient group of G. The cyclic subgroups of the central quotient group of G can be represented on the points of a finite projective four-space X. The lines of X correspond to the commutators of G; the lines of X can be represented on points of a projective nine-space S, these points constitute a locus V of order five and dimension six. The problem of the determination of the groups is the problem of the possible relations of points, lines, planes, and so on, in S to V. This paper determines the 85 groups of order p^{15} to p^{11} (there are 54 of order p^{11}). Canonical forms are given which are immediately translatable into defining relations of the groups. The geometric relations which determine the canonical forms are invariant under collineations in X and translate into properties of the groups which are independent of the generating elements. (Received March 16, 1950.)

372. G. Y. Rainich: *The neglected part of the Riemann tensor.*

A tensor of rank four is considered (in flat Euclidean four-space) which has the algebraic properties of the Riemann tensor and also satisfies the Bianchi equations. On this tensor is imposed the condition that its contracted tensor vanish. The remaining components are arranged into a matrix M of three rows and six columns such that the two halves of it are three-by-three symmetric matrices of zero trace. The transformation properties of these matrices under rotations of the four-space are described by presenting such rotation as product of two elementary rotations (for each of which every vector belongs to an invariant plane) of opposite type. Each of these does not affect one half of M and transforms the other half as a rotation in three-space transforms a symmetric tensor of rank two. The Bianchi relations reduce to a system of differential equations which may be considered as generalizations of the Cauchy-Riemann equations for analytic functions. The equations for each row of M are also the exact Euclidean analogues of Maxwell's equations but the three sets of these Maxwell equations are coupled by having some components in common. (Received March 14, 1950.)

373. J. B. Wright: *Meta-projective geometry.*

A geometry without points and without hyerplanes is formulated as an extension of projective geometry. An equivalence relation produces a dichotomy of elements in which those of one equivalence class are related to those of the other, as points are related to planes. The reflexivity, symmetry, and transitivity of this relation follow from the first four axioms which make use of chains (sequences of elements each joined to the next by incidence). The axiom that every five-chain is open implies that every five-chain can be closed (transitivity of incidence). An infinite-dimensional counter example is eliminated by an axiom guaranteeing the existence of a finite set of equivalent elements without common incident. One says that n elements are independent if the incidence of $n-1$ of them to an element does not imply the incidence of the remaining. The axiom that n independent elements have a unique common incident makes it possible to prove the usual properties of bases and dimensionality of the space. Axioms for n-dimensional projective geometry over a division ring become theorems in our system. In three dimensions the Moebius theorem assures the commutativity of the number system. (Received February 10, 1950.)

STATISTICS AND PROBABILITY

374. P. E. Irick: *A geometric method for finding the distribution of standard deviations when the sampled population is arbitrary.* Preliminary report.

For an ordered random sample, $x_1 \leqq x_2 \leqq \cdots \leqq x_n$, chosen from a population, $f(x)$, $a \leqq x \leqq b$, let $r_i = x_{i+1} - x_i \geqq 0$, $i = 1, 2, \cdots, n-1$. Then make the transformation $r_i = -((i-1)/2i)^{1/2} r'_{i-1} + ((i+1)/2i)^{1/2} r'_i$, and call U' the $1/n!$ portion of the r' space bounded by the $n-1$ sphere and hyperplanes, $\sum_{i=1}^{n-1} r_i'^2 = 2ns^2$, $r'_i = ((i-1)/(i+1))^{1/2} r'_{i-1}$, $i = 1, 2, \cdots, n-1$, where s is the sample standard deviation. The point density in U', $\delta(r')$, is the transform of $\delta(r) = \int_{x_1=a}^{b-\Sigma r} f(x_1) f(x_1+r_1) \cdots f(x_1+r_1+\cdots+r_{n-1}) dx_1$. Change to generalized polar coordinates and call U the outer hyperspherical boundary of U' whereon the density is designated by $\delta((2n)^{1/2}s, \phi)$. Then $p(s)$, the probability law for s, is given by $p(s)ds = n! n^{n/2} s^{n-2} ds \int_{\phi_1} \cdots \int_{\phi_{n-2}} \delta((2n)^{1/2}s, \phi) \sin^{n-3}\phi_1 \cdots \sin \phi_{n-3} d\phi_{n-2} \cdots d\phi_1$, where arc cos $((n/(n-i)(i+1))^{1/2} \leqq \phi_i \leqq$ arc cos $[((i-1)/(i+1))^{1/2}/\tan \phi_{i-1}]$, $i = 1, 2, \cdots, n-2$, whenever b is infinite. The distribution of sample range is readily found in U' and is expressible in the same form as $p(s)$ with the same limits of integration. When b is finite, the complete integral holds only for $0 \leqq s \leqq (b-a)/(2n)^{1/2}$, there being $n^2/4$ connected arcs in $p(s)$ if n is even, and $(n^2-1)/4$ arcs if n is odd. The axes are rotated to give relatively simple formulas for $p(s)$ when $n \leqq 4$, the case of $n = 5$ also being discussed. The method readily produces previously reported results for $p(s)$. In the application of the method, particular attention has been paid to the Type III and polynomial Type I populations. The density function provides much information concerning the form of $p(s)$ for various populations, and contours of constant δ in U' are of theoretical interest. (Received March 15, 1950.)

375. W. M. Kincaid: *Analysis of a one-person game.* Preliminary report.

The problem of allocation of supplies is one which arises in many military and economic connections. The present report discusses a game constructed as a model of a simple situation of this type. The player is given a supply of cards, and receives payments for giving these up when certain random events occur during the period of play. The optimal strategy, which maximizes the expected value of these payments, is governed by certain critical times such that the player's response to a particular event depends on whether it occurs before or after one of these times. (Received March 3, 1950.)

376. W. S. Loud: *Probability of a correct result with a certain rounding-off procedure.*

Consider the problem of the addition of n numbers expressed in the base B of numeration. Supposing each number known to arbitrary accuracy, to obtain the sum accurate to k places, one may round off each number to $(k+1)$ places, add, and round the sum to k places. If the numbers are assumed uniformly distributed, the probability that the above procedure gives the correct result may be found explicitly by use of characteristic functions. If the base B is odd, the result is $2(\pi B)^{-1}\int_0^\infty \sin^{n-1} u \cdot \sin^2 Bu\, u^{-n-1}du$, and if the base B is even, $2(\pi B)^{-1}\int_0 \sin^{n-1} u \sin^2 Bu \cos u\, u^{-n-1}du$. Both formulas have the asymptotic formula $6^{1/2}B(\pi n)^{-1/2}$ as n becomes infinite. (Received March 13, 1950.)

377. P. R. Rider: *The distribution of the quotient of ranges in samples from a rectangular population.*

The distribution of the quotient of two independent, random samples from a continuous rectangular population is derived. The distribution is independent of the population range and can be used to test the hypothesis that two samples came from the same rectangular population, just as the distribution of the variance ratio is used to test whether two samples came from the same normal population. (Received March 13, 1950.)

TOPOLOGY

378*t*. E. A. Michael: *Topologies on subsets.*

Two topologies on the collection 2^X of closed subsets of a topological space X are studied. The finite topology (L. Vietoris, Monatshefte für Mathematik und Physik, 1922) is the coarsest topology on 2^X in which $\{E \in 2^X | E \subset A\}$ is closed if A is closed and open if A is open. The uniform topology (N. Bourbaki, *Topologie générale*, chap. 2, p. 97), defined on 2^X if X is uniform, generalizes Hausdorff's topology for a metric X. It is shown that these topologies agree on $C(X)$, the collection of compact subsets of X, for any uniform X; they agree on all of 2^X if X is normal, and if the uniform structure is induced by the Stone-Čech compactification. Various properties of 2^X and $C(X)$, known for the Hausdorff metric topology, are extended. Among new results, the following is typical: Let X be a topological space with topology T. Then a necessary and sufficient condition for the existence of a continuous function from 2^X to X which maps each element of 2^X into a point of itself is that there exist a linear ordering on X such that the order topology is coarser than T and such that every closed (T) subset of X has a first element. (Received March 16, 1950.)

379. P. M. Swingle: *The closure of types of connected sets.*

Using where necessary the hypothesis of the continuum and Zermelo's axiom, it is shown in any connected domain, D, in the plane there exist subsets W, B, F, and T such that W is widely connected, B is a biconnected set without dispersion point, F is a finitely-containing connected set, and both T and $D-T$ are indecomposable connexes—and D, W, B, F, T, and $D-T$ all have the same closure. (Received March 8, 1950.)

J. W. T. YOUNGS,
Associate Secretary

THE APRIL MEETING IN BERKELEY

The four hundred fifty-ninth meeting of the American Mathematical Society was held at the University of California, Berkeley, California, on Friday and Saturday, April 28–29, 1950. Over 100 persons attended, including the following 97 members of the society:

F. C. Andrews, Richard Arens, W. G. Bade, E. M. Beesley, Richard Bellman, H. F. Bohnenblust, F. A. Butter, E. H. Clarke, Paolo Comba, K. L. Cooke, F. G. Cresse, P. H. Daus, D. B. Dekker, J. E. Denby-Wilks, C. R. De Prima, Douglas Derry, R. C. H. Dieckmann, S. P. Diliberto, R. P. Dilworth, Worthie Doyle, Roy Dubisch, L. K. Durst, Arthur Erdélyi, E. A. Fay, J. M. G. Fell, Werner Fenchel, Harley Flanders, G. E. Forsythe, A. L. Foster, Marianne R. Freundlich, R. A. Fuchs, Bent Fuglede, P. R. Garabedian, M. A. Girshick, G. E. Gourrich, J. W. Green, K. D. C. Haley, G. J. Haltiner, J. G. Herriot, M. R. Hestenes, J. L. Hodges, D. W. Hullinghorst, R. C. James, S. L. Jamison, Irving Kaplansky, Samuel Karlin, William Karush, J. L. Kelley, S. H. Lachenbruch, R. M. Lakness, Cornelius Lanczos, D. H. Lehmer, Hans Lewy, Charles Loewner, A. T. Lonseth, J. E. McLaughlin, Rhoda Manning, W. A. Mersman, A. B. Mewborn, E. D. Miller, C. B. Morrey, F. R. Morris, A. P. Morse, Ivan Niven, Andrewa R. Noble, Fritz Oberhettinger, T. K. Pan, R. S. Phillips, George Pólya, F. P. Pu, Edris P. Rahn, W. C. Randels, Hugo Ribeiro, Julia B. Robinson, R. M. Robinson, J. B. Rosser, Herman Rubin, H. M. Schaerf, Abraham Seidenberg, Max Shiffman, W. H. Simons, W. M. Stone, Irving Sussman, Otto Szász, Gabor Szegö, Alfred Tarski, F. B. Thompson, F. G. Tricomi, A. W. Tucker, R. L. Vaught, R. K. Wakerling, S. S. Walters, W. R. Wasow, P. A. White, A. L. Whiteman, A. R. Williams, František Wolf.

The meeting opened Friday afternoon with an invited address by Professor Richard Arens of the University of California, Los Angeles, on *Representation of rings and functionals in spectral theory*. Professor J. L. Kelley presided. Following the invited address there was a session for contributed papers in analysis and applied mathematics, Professor C. B. Morrey presiding. A second invited address, *On the differential geometry of closed space curves* by Professor Werner Fenchel of the Technical University of Denmark and the University of Southern California was delivered on Saturday morning, Professor Hans Lewy presiding. On Saturday afternoon there were three sections: algebra and number theory, geometry and topology, and late papers. These were presided over by Professor Ivan Niven, Abraham Seidenberg, and George Pólya, respectively.

Abstracts of the papers presented at the meeting follow. Abstracts whose titles are followed by the letter "*t*" were presented by title. Paper 386 was presented by Mr. Fell, paper 388 by Mr. Jackson and paper 390 by Professor Karlin.

Algebra and Theory of Numbers

380. R. P. Dilworth: *Infinite distributivity in function lattices.*

A lattice L is said to be ***infinitely distributive*** if whenever $\bigcup S$ exists for a subset S of L, $\bigcup(a\cap S)$ also exists and $a\cap\bigcup(S)=\bigcup(a\cap S)$. A distributive lattice need not be infinitely distributive. Nevertheless, as first noticed by Tarski and von Neumann, every complemented distributive lattice (Boolean algebra) is infinitely distributive. In this note it is proved that the lattice of all bounded continuous real functions on a topological space is infinitely distributive. (Received March 15, 1950.)

381. Worthie Doyle: *An arithmetical theorem for partially ordered sets.*

Let S be a partially ordered set satisfying the ascending chain condition. An element $q\in S$ is ***irreducible*** *if* $q=$g.l.b. (X) implies $q\in X$ for all finite subsets X of S for which g.l.b. (X) exists. The ascending chain condition implies that every element of S is representable as a meet of irreducibles. A subset A of S is an ***ideal*** if (1) $x\in A$ and $y\geqq x$ imply $y\in A$, (2) $x=$g.l.b. (X) where X is a finite subset of A implies $x\in A$. The ideals form a complete lattice L. If $a\in S$, the $(a)=(x|x\geqq a)$ is a ***principal*** ideal. S is ***upper semimodular*** if B covers (a), $C\supseteq(a)$, and $C\not\supseteq B$ imply $B\cup C$ covers C for all $a\in S$ and B, $C\in L$. Let U_a denote the union of ideals covering (a) and let L_a denote the quotient lattice $U_a/(a)$. The following theorem is proved: If S is a partially ordered set satisfying the ascending chain condition, then every element of S is uniquely representable as a reduced meet of irreducibles if and only if S is upper semimodular and each L_a is a Boolean algebra. This result extends to partially ordered sets a known theorem for lattices. (Received March 15, 1950.)

382. Harley Flanders: *Algebraic field extensions.*

The purpose of this paper is to prove systematically certain basic theorems of the theory of algebraic extensions of algebraic fields. These theorems include, for example, the transitivity of norm, discriminant, and separability, the existence of a maximal separable subfield, and the equivalence of p-independence to separability. The proofs depend strongly on the use of the regular representation, trace, and discriminant and almost completely avoid the use of conjugate elements and decomposition into towers of simply generated extensions. (Received March 15, 1950.)

383. J. E. McLaughlin: *Simple quotient lattices in relatively complemented lattices.*

In a simple complemented modular lattice every quotient lattice is simple. While this is not true for arbitrary relatively complemented lattices, the following theorem gives a large class of simple quotient lattices: Let L be a simple relatively complemented lattice of dimension $n>1$. If p is any point and k is any integer such that $k<[(n+1)/2]$, let C_p^k be the class of points projective to p is not more than $2k$ transposes, and let $a_p^k=\bigcup(C_p^k)$. Then a_p^k/z is a simple lattice of dimension at least $2k+1$. (Received March 16, 1950.)

384. R. M. Robinson: *Arithmetical definitions in the ring of integers.*

It is shown that the set of natural numbers cannot be defined arithmetically in the ring of integers by a formula containing just one quantifier. There is however a suitable formula involving two quantifiers. (Received March 15, 1950.)

385. J. B. Rosser: *Real zeros of real Dirichlet L-series.*

A long standing conjecture is that if χ is real then $L(s, \chi)$ has no positive real zeros. By a certain computational procedure the conjecture has been verified for each individual $k \leqq 67$. In the present paper, this computational procedure was tried for each $k \leqq 227$ and failed for $k=163$. An improved computational procedure is given in the present paper, but even by the new procedure the case $k=163$ presents difficulty. Finally, a new formula for $L(s, \chi)$ is presented which makes it possible to treat many values of k simultaneously. By means of the new formula, the case $k=163$ is readily handled. (Received March 8, 1950.)

386. Alfred Tarski and J. M. G. Fell: *On algebras whose factor algebras are Boolean.*

Consider algebras $\mathfrak{A}=\langle A, +\rangle$ having one binary operation with zero element 0. Two subalgebras B and C are called complementary factors if $B \times C=A$, where $\times$ is the operation of direct multiplication. A subalgebra B is a factor of $\mathfrak{A}$ if for some subalgebra C, $B \times C=A$. The factor algebra of $\mathfrak{A}$ is called Boolean if the algebra of all factors of $\mathfrak{A}$, with the operation $\times$, is a disjunctive Boolean algebra. (For the notions involved here see Tarski, *Cardinal algebras*.) The following three conditions are shown to be equivalent: (i) the factor algebra of $\mathfrak{A}$ is Boolean, (ii) every factor for $\mathfrak{A}$ has precisely one complementary factor, (iii) if B and C are any complementary factors, B is not homomorphic to any subalgebra of the center of C containing elements different from 0. The results extend to algebras with many operations. Further, if the factor algebra of $\mathfrak{A}$ is Boolean, the same applies to any algebra obtained from $\mathfrak{A}$ by adding new operations (result of Jónsson and Tarski). The results obtained show directly that the factor algebras of various special classes of algebras—namely, cyclic groups, centerless algebras, rings with unit, and the so-called zero-equivalent algebras—are Boolean. The class of zero-equivalent algebras includes groups identical with their commutator subgroups and algebras with an infinity element ($x+\infty=\infty+x=\infty$). (Received March 22, 1950.)

387. A. L. Whiteman: *Cyclotomy and Jacobstahl sums.*

Let g be a fixed primitive root of a prime p. Let e be a divisor of $p-1$ and write $p-1=ef$. The Jacobstahl sum $\phi_e(n)$ is defined by $\sum_{h=1}^{p-1} (h/p)((h^e+n)/p)$, where (h/p) denotes the quadratic character of h with respect to p. Of the theorems established in this paper, the following is typical. If e is odd, then $\sum_{k=1}^{e-1} \phi_e(g^k)\phi_e(g^{j+k})=e^2p-e(p-1)$ or $-e(p-1)$ according as $j=0$ or $j\neq 0$, $0 \leqq j \leqq e-1$. Among the applications is the result: if $e=4$ and f is even, then $p=(\phi_4(1)/4)^2+2(\phi_4(g)/4)^2$. The paper also contains a detailed study of the connection between Jacobstahl sums and the theory of cyclotomy. (Received March 7, 1950.)

ANALYSIS

388. E. F. Beckenbach and L. K. Jackson: *A generalization of subharmonic functions.*

The idea of a subharmonic function is generalized as follows, in analogy with E. F. Beckenbach, *Generalized convex functions*, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 363–371. Let there be given a family of functions $\{F(x, y)\}$ which are (a) continuous in a domain D, (b) uniquely determined by continuous boundary values on simple closed contours Γ lying in D, (c) such that, if $F_1(x, y) \leqq F_2(x, y)$ on a simple closed contour Γ bounding a subdomain G, then $F_1(x, y) \leqq F_2(x, y)$ in G, the strict inequality holding throughout G if it holds at any point of Γ, and (d) such that if $F_1(x, y) = F_2(x, y) + K$ on Γ then $F_1(x, y) \leqq F_2(x, y) + K$ throughout G for $K \geqq 0$. Then a function $f(x, y)$ may be said to be a sub-$\{F\}$ function provided (1) $f(x, y)$ is continuous in D, and (2) if $f(x, y) \leqq F(x, y)$ on Γ, then $f(x, y) \leqq F(x, y)$ in G. Results of Littlewood and others concerning subharmonic functions are generalized to sub-$\{F\}$ functions. Also, results of the following type are obtained, if $\{F(x, y)\}$ consists of the solutions of $L(F) \equiv \Delta F + a(x, y)F_x + b(x, y)F_y + c(x, y)F + e(x, y) = 0$, $c(x, y) \leqq 0$ in D, then a function $f(x, y)$ is sub-$\{F\}$ if and only if $L(f) \geqq 0$. (Received February 15, 1950.)

389*t*. Richard Bellman: *On the asymptotic behavior of solutions of linear differential equations.*

A method is given for obtaining the asymptotic behavior of the solutions of $u'' - (1 + a(t))u = 0$ under the assumptions that $a(t) \to 0$ as $t \to \infty$ and that there exists an n for which $\int_0^\infty |a(t)|^n dt < \infty$. The case $1 \leqq n \leqq 2$ has recently been treated by Hartman, Trans. Amer. Math. Soc. (1948). The method is a combination of a device of Poincaré and the standard technique of Poincaré-Liapounoff stability theory. The general nth order equation $u^{(n)} + a_1(t)u^{(n-1)} + \cdots + a_n(t)u = 0$ is amenable to the same treatment and results similar to those of Levinson, Duke Math. J. (1948), may be derived under weaker assumptions. (Received March 10, 1950.)

390. H. F. Bohnenblust and Samuel Karlin: *On positive operators.*

Let E be a Banach space in which a closed cone K (x, y in K and λ, $\mu \geqq 0$ imply $\lambda x + \mu y \in K$; x and $-x$ in K imply $x = 0$) is given. It is assumed that K spans E. A bounded linear operator T is said to be positive ($T > 0$) if $T(K) \subset K$. Let $\sigma(T)$ denote the spectrum of T and let $\lambda_0 = \sup |\lambda|$ for λ in $\sigma(T)$. By extending results on power series with non-negative coefficients to abstract-valued functions, the following result are established: (1) λ_0 is in the spectrum, (2) If λ_0 is a pole of $R(\lambda, T)$ of order k then $n^{-k}\Gamma(k+1)\sum_1^n \lambda_0^{-i} T^i$ tends to $(\lambda_0 I - T)^{k-1}E$ uniformly, where E denotes a projection on the manifold of characteristic vectors of λ_0, (3) If $n^{-1}\sum_1^n \lambda_0^{-i} T^i$ tends to E uniformly, then λ_0 is a pole of order 1 of $R(\lambda, T)$, (4) the value λ_0 can be characterized as the sup of all λ ($\lambda \geqq 0$) for which there exists an x in K with $\lambda x \leqq Tx$. This holds for completely continuous positive operators and also for any strictly positive operators, that is, those which map K into the interior of K. (Received March 17, 1950.)

391. J. E. Denby-Wilkes: *Structure of commutative normed rings.*

Let A be a complex normed ring and S_u its unit sphere about the unit element u. It is shown that the origin θ is a vertex of S_u in the sense that there exists a total family of functionals which are all supporting planes of S_u at θ. In case A is finite-dimensional it is also shown that the cone formed by the radical R of A has only the point θ in common with S_u. If A is finite-dimensional and has a star operation, under suitable restrictions a representation of A as a direct sum of three rings $A = A_1 \oplus A_2 \oplus A_3$ is obtained in which the product in A splits up into three parts: in A_1 scalar, in A_2 convolution (or Cauchy), in A_3 void ($x_3, y_3 \in A_3 \rightarrow x_3 y_3 = \theta$). This decomposition yields in particular for A a "natural" norm with a polyhedral unit sphere ($\|x\|=1 \rightarrow \exists\epsilon$, $|\epsilon|=1$ such that $\|(x+\epsilon u)/2\|=1$); it also gives a product in A^* making A^* a normed ring and such that $A^{**}=A$. (Received March 22, 1950.)

392. Bent Fuglede: *A commutativity theorem for normal operators in Hilbert space.*

The following theorem on linear operators in Hilbert space is established: "If a bounded operator B commutes with a bounded, normal operator N: $BN=NB$, then B commutes also with the adjoint operator N^* of N: BN^*NB." Therefore, it follows that B commutes with all projectors in the canonical spectral representation of N. The theorem holds also if N is nonbounded. It is not known whether it holds for a bounded, normal operator N and a closed, but nonbounded, operator B. If B and N are both nonbounded the concept of commutativity is not generally defined. As a conversion of the above theorem we find that a *bounded* operator N must necessarily be normal if N and N^* commute with exactly the same bounded operators B. This follows at once by the choice $B=N$. For nonbounded, even closed, operators N this conversion does not hold in general. In the author's note in Proc. Nat. Acad. Sci. U.S.A. vol. 36 where the above theorem is proved is also given an example illustrating the last remark by defining a closed operator which does not commute with any bounded operator at all. This operator is, of course, not normal. (Received March 20, 1950.)

393*t*. Edwin Hewitt and H. S. Zuckerman: *A group-theoretic method in approximation theory.*

The following general theorem is proved. Let G be an Abelian group, and let Σ be any group of characters of G. Let x be any character of Σ, let $\sigma_1, \cdots, \sigma_m$ be any finite subset of Σ, and let ϵ be any positive real number. Then there exists an element $g \in G$ such that $|x(\sigma_i)-\sigma_i(g)|<\epsilon$ $(i=1, 2, \cdots, m)$. This result is used to give new proofs of Kronecker's approximation theorems, and to prove a number of new approximation theorems, of which the following is typical. Let $\omega_1, \cdots, \omega_m$ be real-valued functions of bounded variation on $(-\infty, +\infty)$ which are rationally independent. Let $\alpha_1, \cdots, \alpha_m$ be any real numbers and let ϵ be any positive real number. Then there exists a real-valued continuous function f on $(-\infty, +\infty)$ such that $\lim_{x\to\pm\infty} f(x)=0$ and such that $|\alpha_i-\int_{-\infty}^{\infty} f(x)d\omega_i(x)|<\epsilon$ $(i=1, 2, \cdots, m)$. (Received January 31, 1950.)

394. R. S. Phillips: *On Fourier-Stieltjes integrals.*

A function $\phi(\tau)$ on $(-\infty, \infty)$ satisfies condition (A) if for some constant C, $|\sum a_n\phi(\tau_n)| \leqq C\|\sum a_n \exp(i\tau_n s)\|$ for all finite sets (τ_n) and (a_n) where $\|f(s)\| = \text{Lub } |f(s)|$. If $\phi(\tau)$ is measurable and satisfies (A), then there exists a unique function of bounded variation $y(s)$ continuous on the right with $y(-\infty)=0$ such that $\phi(\tau)=\int_{-\infty}^{\infty} \exp(i\tau s)dy(s)$ a.e. One obtains the Bochner, Riesz result for positive definite functions from this theorem by means of the lemma: Any non-negative trigonometric polynomial can be expressed as the sum of absolute value squares of periodic trigonometric polynomials. As a corollary one shows that a function satisfying the positive definiteness condition only for finite commensurate sets of τ's will be positive definite. An analogous result is obtained for condition (A) by means of a decomposition of the space of almost periodic functions into the direct product of a nondenumerable set of limited periodic spaces. The paper also contains generalized

Fourier-Stieltjes integral representation theorems for arbitrary complex valued functions satisfying (A) and for measurable Banach-space valued functions satisfying (A) and a weak compactness condition. (Received March 3, 1950.)

395*t*. R. S. Phillips: *On one-parameter semi-groups of linear transformations.*

Let X be a Banach space and let $E(X)$ be the Banach algebra of bounded linear transformations on X to itself. A one-parameter semi-group of operators in $E(X)$ is defined to be a function $T(\xi)$ on $(0, \infty)$ to $E(X)$ such that $T(\xi_1+\xi_2)x=T(\xi_1)T(\xi_2)x$ for $0<\xi_1$, $\xi_2<\infty$, and $x\in X$. $T(\xi)$ is said to be weakly measurable if the numerically valued function $f[T(\xi)x]$ is measurable for each $x\in X$ and $f\in \overline{X}$. $T(\xi)$ is said to be strongly measurable if, for each x, $T(\xi)x$ is the limit almost everywhere of a sequence of step functions. It is shown that if $T(\xi)$ is strongly measurable, then $\|T(\xi)\|$ is bounded in each interval $[\delta, 1/\delta]$ where $\delta>0$. An example is given of a one-parameter semi-group of transformations $T(\xi)$ which is weakly measurable but such that $\|T(\xi)\|$ is unbounded in every finite subinterval of $(0, \infty)$. (Received January 30, 1950.)

396. Max Shiffman: *Minimal surfaces, analytic functions, and symmetrization.*

In connection with boundary curves bounding several different minimal surfaces, the following problem was posed by T. Rado in the session on *Analysis in the large* of the Princeton Bicentennial Conferences on the Problems of Mathematics: show that a minimal surface bounded by two parallel circles, with the line of centers perpendicular to the planes of the circles, is a surface of revolution. This, and a more general result, is established by showing that a certain angle defined at each point of the minimal surface is a harmonic function on the surface, and that the curvature at each point of a family of plane curves on the minimal surface satisfies a certain partial differential equation of elliptic type. A discussion of the second eigenvalue of an associated elliptic equation yields the result. These theorems are generalizations and analogues of theorems in the theory of analytic functions. In fact, minimal surface theory can be put in the form analogous to the Cauchy-Riemann equations. The particular case of Rado can also be done by an extension of the notion of symmetrization, but the more general case treated leads to an open geometric question in this domain. (Received March 16, 1950.)

397. Otto Szász: *Gibbs phenomenon for Hausdorff means.*

We discuss here the Gibbs phenomenon for the Hausdorff means of Fourier series. These means for a sequence s_n are defined by $h_n=\sum_0^n C_{n,\nu}s_\nu\int_0^1 r^\nu(1-r)^{n-\nu}d\psi(r)$, where $\psi(r)$ is of bounded variation. The main result is: The series $\sum \sin nt/n=(\pi-t)/2$ presents a Gibbs phenomenon at $t=0$ if and only if $\max_{\tau>0}\int_0^1\{1-\psi(r)\}(\sin r\tau/r)dr>\pi/2$. It includes known results for Cesàro and for Euler means. Applied to Hölder means of order $p>0$, it is found that there is a constant γ so that Hölder means present a Gibbs phenomenon for $p<\gamma$, but not for $p\geqq\gamma$. Numerical calculation yields $\gamma=0.58\cdots$. For Cesàro means the corresponding constant is, according to Gronwall, $0.439\cdots$. (Received March 8, 1950.)

398. F. G. Tricomi: *On the finite Hilbert transformation.*

The finite Hilbert transformation $\mathfrak{T}[\phi] \equiv \pi^{-1}\int_{-1}^{1}(y-x)^{-1}\phi(y)dy$ (Cauchy's principal value of the integral) furnishes the necessary basis for the rigorous study of the singular integral equations of *Carleman's type*: $a(x)\phi(x) - \lambda\mathfrak{T}[\phi] = f(x)$. Some, but not all, of the properties of the $\mathfrak{T}$-transformation can be deduced from those of the corresponding *infinite* transformation treated carefully in Titchmarsh's *Theory of Fourier integrals*. In particular to these properties is added a kind of *convolution theorem* which asserts that $\mathfrak{T}\{\phi_1\cdot\mathfrak{T}[\phi_2]+\phi_2\cdot\mathfrak{T}[\phi_1]\} = \mathfrak{T}[\phi_1]\cdot\mathfrak{T}[\phi_2]-\phi_1\cdot\phi_2$ provided that the functions ϕ_1, ϕ_2 belong to suitable L^p classes. Precise conditions are also given for the *inversion* of the transformation $\mathfrak{T}$, that is, for the solution of the *airfoil equation* $\mathfrak{T}[\phi] = f$. (Received March 1, 1950.)

399. S. S. Walters: *Locally bounded linear topological spaces of analytic functions.*

A method for norming the conjugate space of a locally bounded linear topological space (LBLTS) is presented. Let $\Delta = E_z[|z| < 1$, z a complex number$]$, and $\mathfrak{A}$ be all complex valued f analytic on Δ. Then a subspace $\mathfrak{B}$ of $\mathfrak{A}$ is said to be of type I provided $\mathfrak{B}$ is a LBLTS$\ni \gamma_{0,z} \in \mathfrak{B}^*$ (conjugate space of $\mathfrak{B}$) for all $z \in \Delta$ and provided $\exists$ a real-valued function $N(r) \ni \|\gamma_{0,z}\| \leqq N(r)$ for $|z| \leqq r < 1$. Here one defines $\gamma_{n,z}(f) = f^{(n)}(z)/n!$, $n = 0, 1, \cdots, f \in \mathfrak{A}$, $z \in \Delta$. Then, in a space $\mathfrak{B}$ of type I, weak convergence is meaningful since $\mathfrak{B}^*$ distinguishes elements in $\mathfrak{B}$. It is shown that $f_n \rightarrow^{w} f$ (f_n converges weakly to f) implies $f_n(z) \rightarrow f(z)$ uniformly on compact subsets of Δ. For arbitrary $f \in \mathfrak{A}$ let $T_r f: T_r f(z) = f(rz)$, $0 \leqq r < 1$, $z \in \Delta$, and let $U_n: U_n(z) = z^n$, $n = 0, 1, \cdots$. Then $\mathfrak{B}$, a space of type I, is of type II provided: (a) $\{U_n\}$ is a bounded subset of $\mathfrak{B}$, (b) $\sum_{n=0}^{\infty}\gamma_{n,0}(f)W^n U_n$ converges in $\mathfrak{B}$ for all $f \in \mathfrak{B}$, $w \in \Delta$, (c) $T_r f \rightarrow^{w} f$ as $r \rightarrow 1-$ for all $f \in \mathfrak{B}$, and (d) $\mathfrak{B}$ is complete. Let $\mathfrak{D}$ denote all $G \in \mathfrak{A} \ni \lim_{r\rightarrow 1}\int_0^{2\pi} f(\rho e^{i\theta})G((r/\rho)e^{-i\theta})d\theta$, $r < \rho < 1$, exists for all $f \in \mathfrak{B}$. It is then shown that $\mathfrak{B}^*$ and $\mathfrak{D}$ are algebraically isomorphic, and if $\gamma \rightarrow G_\gamma$ is the isomorphism, then $\gamma(f) = \lim_{r\rightarrow 1}\int_0^{2\pi} f(\rho e^{i\theta})G_\gamma((r/\rho)e^{-i\theta})d\theta$, $r < p < 1$, for every $f \in \mathfrak{B}$. An example of a space of type II is H^p, $0 < p < 1$, where the topology of H^p is that determined by the metric $d(f, g) = \sup_{0 \leqq r < 1}\int_0^{2\pi}|f(re^{i\theta}) - g(re^{i\theta})|^p d\theta$. (Received May 22, 1950.)

400*t*. W. R. Wasow: *A study of the solutions of the differential equation $y^{(4)}+\lambda^2(xy''+y)=0$ for large values of λ.*

The differential equation $y^{(4)}+\lambda^2(xy''+y)=0$ belongs to a type studied in a previous paper by the author (Ann. of Math. vol. 49 (1948) p. 852). By means of Laplace contour integrals and the method of steepest descent the character of the solutions for large λ is studied more completely than is possible with the methods of the former paper. The results obtained include, in particular: (1) An asymptotic description of the solutions in a full neighborhood of the origin. This requires three sets of formulas, one valid when $|x| \geqq$ const. >0, the second when $|x| \leqq$ const., but $\lambda^{2/3}x$ very large, the third when $|\lambda^{2/3}x| \leqq$ const. (2) Asymptotic expressions for the solutions called "balanced" in the paper mentioned above that are valid in the sector where these solutions diverge. They show that these solutions are there of the order of $\lambda^{-3/2}\exp[\lambda(2/3)(-x)^{3/2}]$. (3) Asymptotic formulas valid on the "Stokes lines." (Received March 10, 1950.)

APPLIED MATHEMATICS

401. William Karush: *An iterative method for finding characteristic vectors of a symmetric matrix.*

The following iteration procedure for finding a characteristic vector of a real symmetric matrix A is treated. Let $s>1$ be a fixed positive integer. Given the vector x^i, construct the space $\mathcal{A}^i$ spanned by $x^i, Ax^i, \cdots, A^{s-1}x^i$. Then there is a unique vector x^{i+1} of the form $x^i+\eta$, $(x^i, \eta)=0$, which minimizes (maximizes) the Rayleigh quotient $(x, Ax)\cdot|x|^{-2}$ for x in $\mathcal{A}^i$. The initial vector x^0 is chosen arbitrarily. Let $\lambda_{\min}$ $(\lambda_{\max})$ be the least (greatest) characteristic value of characteristic vectors y for which $(x^0, y)\neq 0$. Then $x^i \rightarrow y_{\min}$ $(y_{\max})$, where $y_{\min}$ $(y_{\max})$ has characteristic value $\lambda_{\min}$ $(\lambda_{\max})$. The proof is based upon an investigation of the polynomials associated with an orthogonalization of the spaces $\mathcal{A}^i$. The iteration scheme is essentially a gradient method. For $s=2$ the vector η is a multiple of the gradient of the Rayleigh quotient; for $s>2$ the vector η contains higher correction terms. (Received March 7, 1950.)

402. Cornelius Lanczos: *Inversion of the Laplace transform.*

The customary method of inverting the Laplace transform makes use of integration along the imaginary axis from $-i\infty$ to $+i\infty$, closing the path of integration through the negative infinite semi-circle. The integration is then reduced to loops around the singular points or lines. This method becomes cumbersome if the singularities of the transform are not of a simple type. The present procedure maps the infinite imaginary axis into the unit circle, with the help of reciprocal radii. Inside of this circle the transform is regular and allows expansion into a Taylor series. In the inversion the series can be integrated term by term. The resultant $f(x)$ appears as an infinite expansion into modified Laguerre functions, orthogonal in the range between 0 and ∞ with respect to the weight factor x. These expansions combine the simplicity of the Taylor coefficients, obtained by successive recursions, with the advantages of an approximation in the large, usually obtained by cumbersome integrations. The method yields also a simple algorithm for designing an electric network of definite order which approximates a prescribed pulse response as closely as possible. (Received March 17, 1950.)

403. W. R. Wasow: *On random walks and eigenvalues of elliptic difference equations.*

Let B be a finite domain bounded by a polygon C whose vertices are points of a square lattice of mesh length h. A particle starting with mass one from a point P in B performs a random walk in the lattice with equal transition probabilities for the four possible directions, until it is absorbed by the boundary. At each step the mass is multiplied by the value of a given positive function $k(x, y)$ of the position. Let $E(P, R)$ be the expected value of the mass absorbed by a preassigned boundary point R. If $E(P, R)$ is finite, it is a solution of the difference equation $\Delta u+(4/h^2)(1-1/k)u=0$, where Δu is the usual finite difference analogue of the Laplace operator. It is shown that $E(P, R)$ is finite if and only if the difference expression above is positive definite in B. With appropriate modifications a similar result holds for infinite domains. This random walk procedure can be used for the approximate numerical solution by means of sampling methods of boundary and eigenvalue problems for the corresponding partial differential equations. (Received March 10, 1950.)

GEOMETRY

404. D. B. Dekker: *Some generalizations of hypergeodesics.*

In a previous paper (*Hypergeodesic curvature and torsion*, Bull. Amer. Math. Soc.

vol. 55 (1949) pp. 1151–1168) the author studied an arbitrary family of hypergeodesics on a general surface in ordinary euclidean space. In the present paper a more general class of curves is obtained by replacing in the usual tensor differential equation of a family of hypergeodesics the homogeneous cubic form in the first derivatives of the surface coordinates with respect to arc length by a rational function homogeneous of degree three in the derivatives. This more general class of curves contains the hypergeodesics as a subclass and possesses most of their properties. It is found that a curve of a family is a plane curve if and only if it is a curve of a special related intersector net of a certain complex of lines determined by the family. If n and $n-3$ are the degrees of the numerator and denominator of the rational homogeneous function then in general the special intersector net consists of $3n-1$ one-parameter families of curves on the surface. (Received March 8, 1950.)

405. Douglas Derry: *The duality theorem for curves of order n in n-space.*

Let C_n be a curve in real projective n-space for which (1) all osculating r-subspaces, $1 \leqq r < n$, are defined for each curve point and (2) no hyperplane cuts the curve in more than n points. P. Scherk showed (*Über differensierbar Kurven und Bogen*, Časopis pro pěstování, matematiky a fysiky vol. 66 (1937) pp. 172–191) by a fixed point theorem applied to a closed C_n that the dual curve of C_n possesses properties (1) and (2). The present note gives another proof of this result which is based on a variation of Rolle's theorem. (Received March 15, 1950.)

406. T. K. Pan: *Hypergeodesics and dual hypergeodesics.*

A system of curves on a general proper analytic surface in ordinary metric space called dual hypergeodesics is defined, which includes a system of dual union curves as a special case. First, an extended relation R with respect to a general net is defined and an equation of a line L in extended relation R to L' is derived. Then the envelope of L as L' generates the osc-cone at a point of a family of hypergeodesics is found, which leads from the definition to the intrinsic differential equations of dual hypergeodesics. The unicity and the reciprocity properties of hypergeodesics and dual hypergeodesics lead to the definition of reciprocal families of hypergeodesics, which is followed by the pairing of hypergeodesics of one or two reciprocal families into a conjugate net and the relationships between the associated cones and cubics. Torsal curves associated with a family of hypergeodesics are defined and the non-asymptotic torsal curves are shown to be the nonlinear plane hypergeodesics contained in the family. (Received March 14, 1950.)

TOPOLOGY

407. P. A. White: *Regular convergence in terms of Čech cycles.*

In a compact Hausdorff space closed subsets $\{A_i\}$ converging to A are said to converge n-regularly if for each finite covering $\mathcal{V}$ of S by open sets, there is a finite refinement $\mathcal{V}$ of U and an integer N such that if z^r is a Čech cycle (with field coefficients) on A_i for $i > N$ and $r \leqq n$ with diameter less than $\mathcal{V}$, then z^r bounds a chain on A_i with diameter less than U. This differs from the original definition of G. T. Whyburn in that Čech cycles without assuming a metric are used instead of Vietoris cycles with a metric. It is proved that under n-regular convergence the limit set A is an lc^n, the Betti numbers $p^r(A_i) = p^r(A)$ for all $r \leqq n$ and almost all i, and if each A_i is an

BOOK REVIEWS

Electromagnetic theory. (Proceedings of Symposia In Applied Mathematics, vol. 2.) New York, American Mathematical Society, 1950. 2+91 pp. $3.00.

The present book gives an account of the Second Symposium in Applied Mathematics of the American Mathematical Society, held at the Massachusetts Institute of Technology, July 24–31, 1948. It covers all fields of modern development in the electromagnetic theory, ranging from quantum electrodynamics to practical applications of the electromagnetic theory in communication engineering.

H. Feshbach presents in the first paper an excellent and very clear summary of the problems of quantum electromagnetic theory. He explains the advances made in recent years, especially the work of Schwinger on the elimination of infinite terms and the correct discussion of the Lamb-Rutherford correction.

J. L. Synge discusses a problem of importance for general relativity, and investigates how far it may be possible to dissociate electromagnetism from metric definitions. This is a central problem in relativity theory, and it had retained the attention of the late P. Langevin in many of his (unpublished!) lectures at the College de France. Synge's paper goes back to the fundamental definitions and represents a very important contribution to the problem.

W. H. Watson investigates the possibility of introducing discontinuities in the classical electromagnetic theory, and especially a sort of discontinuous motion. The idea is to obtain a translation of Maxwell's theory that could be more easily connected with the quantum statistical methods. This interesting paper shows how such an adaptation of the classical theory can be logically attempted and practically performed.

L. Infeld discusses the factorization method and its application to differential equations in theoretical physics, a very interesting problem on account of the power of the method for practical resolution of problems.

R. J. Duffin treats of nonlinear networks and indicates the method that can be used for such problems.

C. L. Pekeris compares the ray theory to the normal mode method, and shows that they are related by a Poisson transform.

A. E. Heins discusses systems of Wiener-Hopf integral equations and their applications to some boundary value problems in electromagnetic theory.

Other papers, by A. H. Taub, S. O. Rice, A. F. Stevenson, R. Truell E. G. Ramberg, M. Kac, N. Wiener, Y. W. Lee, H. Wallman, E. Feenberg, are short abstracts of articles published in different scientific reviews.

L. BRILLOUIN

The real projective plane. By H. S. M. Coxeter, New York, McGraw-Hill, 1949. 10+196 pp. $3.00.

This book is an admirable introduction to the subject for students who know a fair amount of ordinary plane geometry, including at least something about conics, but have no idea at all of projective geometry. It begins by the introduction of points and the line at infinity by means of the "vanishing line" or horizon of a central projection from one plane onto another, and goes on to illustrate the distinction between affine and projective geometry by proving Desargues' theorem by projection, from the properties of similar and similarly situated triangles. In the second chapter a set of five axioms of incidence (one of which is Desargues' theorem) is given, the principle of duality explained, and the quadrilateral and quadrangle and the harmonic relation studied in an elementary manner. The perspective relation between two lines is defined, likewise its dual (also called perspective). Chapter III introduces the idea of order, defined in terms of the separation relations of two pairs of points, and its properties deduced from six simple axioms, of which one states that order is invariant under perspective correspondence. The familiar separation properties of the harmonic relation are very simply deduced. Then as a temporary expedient Enriques' theorem to the effect that an ordered correspondence which relates an interval to an interior interval has a first invariant point in the interval is introduced as an axiom of continuity.

Chapter IV is on one-dimensional projectivities, defined as correspondences that preserve the harmonic relation. The fundamental theorem is proved from the axiom of continuity; Pappus' theorem and the axis of projectivity follow simply. Projectivities are classified into direct and opposite according to their effect on sense and into elliptic, parabolic, and hyperbolic, according to the number of their invariant points. Involutions are studied, and the involutory property of the quadrangular set proved.

Chapter V is a similar treatment of two-dimensional projectivities, both collineations and correlations, a collineation being defined as any point-to-point correspondence that preserves collinearity. It is shown from the fundamental theorem that there cannot be more than one collineation in which one given quadrangle corresponds to another, but curiously it is never shown, though constantly taken for granted in the sequel, that there is always one such collineation. The proof of this, though a simple consequence of Desargues' theorem, is surely not sufficiently self-evident to be passed without even a mention. Oddly enough, the corresponding result for correlations is proved with care, using the fundamental theorem.

Polarities are introduced, as symmetrical correlations, and are classified in the next chapter as hyperbolic or elliptic according as there are points lying in their own polars or not. This leads to the definition of a conic as the locus of such points. Early in the chapter this is shown to be equivalent to Steiner's definition of the conic as locus of intersections of corresponding lines of projectively related pencils. Chapters VI and VII contain a quite full account of conics, pencils of conics, and projectivities on conics, and a hint at the transformation by conics through three points (in the form of "polarity" with respect to a triangle).

Chapters VIII and IX give an illuminating account of the derivations of affine and metrical from projective geometry, by the special treatment of a particular line and a particular elliptic involution on it. This is carried as far as the circles connected with a triangle and the focal properties of conics.

Chapter X is probably the best in the book. It is a reexamination of the problem of continuity, based on the axiom that every monotonic sequence of points has a limit; from this it is proved that the line is perfect. The chapter concludes with proofs of the fundamental theorem, Dedekind's axiom, and Enriques' theorem, which was temporarily taken as an axiom in Chapter III.

The last two chapters deal with the introduction and use of coordinates. The addition and multiplication of points on a line or on a conic are defined by the conditions that (P_∞, P_∞), (A, B), $(P_0, A+B)$ and (P_0, P_∞), (A, B), (P_1, AB) are three pairs of an involution, and the ring obtained is shown by continuity methods to be isomorphic with the field of real numbers. In the plane, homogeneous coordinates are defined by the ingenious but rather baffling method of drawing a conic through the vertices of the triangle of reference, the unit point, and the point whose coordinates are required; the coordinates of the point are then the parameters, on the conic, of the vertices of the triangle with regard to a triad consisting of the remaining two points and an arbitrary sixth point of the conic, whose choice provides the arbitrary coefficient of homogeneity. This is remarkably neat, but is it really the best way to show a student new to the business what homogeneous coordinates really mean geometrically? In the last chapter the start is made at the other end. A point is defined by a set of coordinates and a line by a linear equation; it is verified that this system obeys all the axioms, and the general collineation and correlation are expressed as linear transformations.

It will be seen from the foregoing that the work is severely and carefully argued from beginning to end, and that within its limitations to the real field and to two dimensions it covers just about everything that one could think of including. The shelving of the serious discussion of continuity to a late stage, by assuming one of its chief results as a temporary axiom, probably makes greatly for the intelligibility of the book to beginners. A great number of admirably clear diagrams (probably more than one to every page on an average) illustrate the ideas. The proofs are lucid, and in nearly every case lay bare the fundamental ideas that are being used rather than obscuring these in a mass of detail. The whole book, indeed, is most readable; there are interesting historical notes on the genesis of the ideas presented and a very good bibliography. An appendix of only a couple of pages briefly indicates the nature of the step from real to complex geometry.

PATRICK DUVAL

Extrapolation, interpolation, and smoothing of stationary time series with engineering applications. By Nobert Wiener. Cambridge, Technology Press of Massachusetts Institute of Technology, and New York, Wiley, 1949. 10+163 pp. $4.00.

This is the second book by Professor Wiener on time series and communication engineering published since 1948. While the first book, *Cybernetics*, treated the subject from a general standpoint and was more philosophical than mathematical, the present book is more technical than theoretical, and is intended to give a useful tool for engineers working in the field of electrical communication and related subjects. This book is essentially a reproduction of a pamphlet which had limited circulation during the war.

The main problem discussed in this book is the following: Let $\{x_t\}$ be a stationary time series of class L^2, where the parameter t runs through all integers (discrete case) or all real numbers (continuous case); given a random variable y of class L^2 and a set T of the values of the parameter t, how can we approximate y by finite linear combinations of x_t with t from T?

In the terminology of Hilbert space, this problem can be formulated in a different manner. Let $\{x_t\}$ be a "sequence" (discrete case) or a "curve" (continuous case) in a Hilbert space $\mathfrak{H}$ such that the inner product (x_s, x_t) depends only on $s-t$. This is equivalent to saying that there exists a unitary transformation U defined on $\mathfrak{H}$ and an element x_0 of $\mathfrak{H}$ such that $U^t x_0 = x_t$ for all integers t (discrete case), or there exists a one-parameter group $\{U^t\}$ of unitary transformations U^t defined on $\mathfrak{H}$ and an element x_0 of $\mathfrak{H}$ such that $U^t x_0 = x_t$ for all real numbers t (continuous case). For any set T of the values of the parameter t, let $\mathfrak{M}_T$ be a closed linear subspace of $\mathfrak{H}$ spanned by $\{x_t \mid t \in T\}$.

Given an element y of $\mathfrak{H}$, the main problems are: (I) how can we compute the distance $d(y, \mathfrak{M}_T)$ of y from $\mathfrak{M}_T$ and (II) how can we find an element y_T of $\mathfrak{M}_T$ which attains or approximates this minimum distance to y? If $t_1 < t_0$, $T = \{s \mid -\infty < s \leqq t_1\}$ and $y = x_{t_0}$, then this is nothing but a problem of prediction (that is, stochastic extrapolation). If $t_1 < t_0 < t_2$, $T = \{s \mid -\infty < s \leqq t_1 \text{ or } t_2 \leqq s < \infty\}$ and $y = x_{t_0}$, then this is reduced to a problem of stochastic interpolation. It is easy to see that $d(y, \mathfrak{M}_T) = d(h)$ depends only on $h = t_0 - t_1$ in the first case, and that $d(y, \mathfrak{M}_T) = d(h_1, h_2)$ depends only on $h_1 = t_0 - t_1$ and $h_2 = t_2 - t_0$ in the second case. Further, if $x_t = x_t' + x_t''$, where $\{x_t'\}$ and $\{x_t''\}$ are stationary time series, and if $T = \{s \mid -\infty < s \leqq t_1\}$, $y = x_{t_0}'$, then this becomes a problem of filtering, that is, a problem of estimating the future, present, or past values of x_{t_0}', according as $t_0 - t_1$ is >0, $=0$, or <0, when the past values of x_t up to present $(t = t_0)$ are known. If we consider x_t' as a real message, x_t'' as random noise, and $x_t = x_t' + x_t''$ as actual information, then this is a problem of eliminating unnecessary noise x_t'' from the information x_t.

Let us consider the continuous case. In this case $\phi(t) = (x_t, x_0) = (U^t x_0, x_0)$ is a positive definite function, and hence, by Bochner's theorem, there exists a distribution function $F(\lambda)$ defined on the infinite interval such that

$$\phi(t) = \int_{-\infty}^{\infty} e^{it\lambda} dF(\lambda). \tag{1}$$

It is then easy to see that the problem of extrapolation and interpolation is equivalent to the problem of finding a trigonometrical polynomial $P(\lambda)$ which attains or approximates the infimum of

$$\int_{-\infty}^{\infty} \left| e^{it_0\lambda} - P(\lambda) \right|^2 dF(\lambda), \tag{2}$$

where $P(\lambda) = \sum_{k=1}^{n} a_k e^{it_k\lambda}$ is any finite linear combination of $e^{it_k\lambda}$ with $t_k \in T$, $k = 1, \cdots, n$.

In this book the discussion is limited to the case when $F(\lambda)$ is absolutely continuous. It turns out that either $d(h)>0$ for all $h>0$ or $d(h)\equiv 0$ for all $h>0$, and the first case happens if and only if

$$\int_{-\infty}^{\infty}\frac{\log F'(\lambda)}{1+\lambda^2}d\lambda > -\infty. \tag{3}$$

Here the problem is further reduced to the discussion of the possibility, and the concrete method, of representing $F'(\lambda)$ as a product of two functions $\Phi_+(\lambda)$ and $\Phi_-(\lambda)$ which are the boundary values on the real line of functions $\Phi_+(\lambda+i\mu)$ and $\Phi_-(\lambda+i\mu)$ defined and analytic in the upper and the lower half plane, respectively.

These problems are not only mathematically interesting, but also extremely important in application. This book gives a detailed discussion of the theory of approximation of this type which has been developed by the author himself. As is pointed out by the author, the theory of extrapolation and interpolation for the discrete case has been investigated independently by A. Kolmogoroff (Bull. Acad. Sci. USSR, Ser. Math. vol. 5 (1941) pp. 3–14) by using the method of operator theory in Hilbert space. It is easy to see that in the discrete case $F(\lambda)$ becomes a distribution defined on the set of real numbers mod. 1, or equivalently on the unit circle. Kolmogoroff does not assume that $F(\lambda)$ is absolutely continuous. In case $F(\lambda)$ is absolutely continuous, the problem is again reduced to that of finding a representation of $F'(\lambda)$ as a product of two functions which are the boundary values on the unit circle of functions defined and analytic inside and outside the unit circle, respectively.

It is to be remarked that Kolmogoroff is mainly interested in problem (I), that is, in the problem of finding the exact values of $d(h)$ and $d(h_1, h_2)$ in terms of h, h_1, h_2, and $F(\lambda)$, while the author is more interested in problem (II), that is, in the problem of finding a concrete method of obtaining a linear combination which attains or approximates the minimum distance. The method of the author is straight forward and is based on Fourier analysis. This fact makes it possible to solve practical prediction problems experimentally in many important cases by constructing a suitable electrical network. It is remarkable to note that the theory of generalized harmonic analysis developed by the author some twenty years ago is exactly the right tool for this purpose.

The whole treatment is based on the author's theory of generalized harmonic analysis (Acta Math. vol. 55 (1930) pp. 117–258) and as in many other writings of the author, technical terms from electrical engineering are often used to explain the intuitive meaning of various mathematical concepts. These technical terms are very useful and illustrative in view of its application to practical problems. On the other hand, it seems that there are many places in this book in which geometrical language from Hilbert space could have been successfully introduced to clarify the situation. Also, the interpretation of each $x_t = x_t(\omega)$ as a random variable, and the observation of the fact that, because the ergodicity of the system in case $F(\lambda)$ is absolutely continuous, the "time average"

$$\lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} x_{s+u}(\omega)x_{t+u}(\omega)du \tag{4}$$

is equal to the "space average"

$$(x_s, x_t) = \int x_s(\omega)x_t(\omega)d\omega \tag{5}$$

would have simplified the argument substantially.

The book consists of an introduction, five chapters, and three appendices. After explaining the general outline of the problem in the introduction, the author gives in Chapter I a review of generalized harmonic analysis which is necessary for the understanding of the following chapters. Chapters II and III are devoted to the problems of prediction and filtering respectively. In Chapter IV there is given a brief account of the theory of multiple prediction, that is, the theory of prediction when we deal with more than one time series at the same time. This theory is still incomplete, and the problem analogous to that of factorization of $F'(\lambda)$ is not yet solved. Finally, in Chapter V there is given a short discussion on the application of similar methods to a problem of approximate differentiation.

The book concludes with three appendices: a numerical table of Laguerre functions which are useful in practical application, and two papers by N. Levinson reprinted from Journal of Mathematics and Physics vol. 25 (1946) pp. 261–278, vol. 26 (1947) pp. 110–119. The first of these notes by N. Levinson gives a practical method of eliminating random noise (filtering) in communication engineering by using electrical networks, and the second note is an excellent, heuristic but very clear, exposition which explains the structure of Wiener's theory of prediction and filtering.

Shizuo Kakutani

NOTES

Brown University announces its Third Symposium on Plasticity to be held September 8–9, 1950 under the auspices of the Graduate Division of Applied Mathematics.

The University of Maryland announces a Conference on Differential Equations to be held on September 8–9, 1950.

The John Simon Guggenheim Memorial Foundation has announced awards in mathematics to the following: Dr. J. H. Bigelow, Institute for Advanced Study; Professor Samuel Eilenberg, Columbia University; Associate Professor N. E. Steenrod, Princeton University; Dr. R. P. Boas, Jr., American Mathematical Society; and Associate Professor Philip Hartman, Johns Hopkins University.

Professor H. S. M. Coxeter of the University of Toronto has been elected a fellow in the Royal Society.

Professor J. G. van der Corput of the University of Amsterdam is on leave of absence and has been appointed to a visiting professorship at Stanford University.

Professor Emeritus W. B. Carver of Cornell University has been appointed to a visiting professorship at Southern Illinois University.

Assistant Professor W. B. Caton of the State College of Washington has been appointed to an assistant professorship at DePaul University.

Associate Professor John DeCicco of the Illinois Institute of Technology has been appointed to a professorship at DePaul University.

Mr. R. J. Learson of the Western and Southern Insurance Company has accepted a position as Associate Manager of Selection with the Mutual Life Insurance Company of New York.

Professor H. B. Mann of Ohio State University is on leave of absence and has been appointed to a visiting professorship at the University of California.

Professor B. C. Patterson of Hamilton College has been appointed president of Washington and Jefferson College.

Professor A. C. Schaeffer of Purdue University has been appointed to a professorship at the University of Wisconsin.

Dr. Dorothy M. Stone has been appointed to a visiting professorship at Wellesley College.

Dr. J. V. Wehausen of David Taylor Model Basin, Navy Department, Washington, has accepted appointment to the position of Executive Editor of Mathematical Reviews to succeed Dr. R. P. Boas.

Professor F. B. Wiley of Denison University has retired with the title Emeritus.

The following promotions have been announced:

F. S. Beckman, Pratt Institute, to an assistant professorship.

R. G. Blake, University of Florida, to an assistant professorship.

H. K. Brown, Northeastern University, to an associate professorship.

R. G. Helsel, Ohio State University, to a professorship.

Chosaburo Kato, Denison University, to a professorship.

Howard Levi, Columbia University, to an associate professorship.

G. B. Lang, University of Florida, to an associate professorship.

C. J. Rees, University of Delaware, to dean of the School of Graduate Studies.

E. P. Vance, Oberlin College, to an associate professorship.

M. R. Spiegel, Rensselaer Polytechnic Institute, to an assistant professorship.

The following appointment to an instructorship is announced: Oregon State College: Mr. F. E. Ehlers.

Professor G. H. Livens of University College, Cardiff, Wales, died March 26, 1950. He had been a member of the Society for twenty-eight years.

The note about Dr. Meyer Jerison which appeared under instructorships on p. 275 of volume 56 of the Bulletin, May 1950, is in error and should read "University of Illinois: Dr. Meyer Jerison."

The note about Professor Fritz Rothberger on p. 274 of volume 56 of the Bulletin, May 1950, is in error and should read "professorship at the University of New Brunswick."

NEW PUBLICATIONS

Blanc, E. See Picard, E.

Bol, G. *Projektive differential geometrie.* Vol. 4, part 1. Gottingen, Vandenhoeck and Ruprecht, 1950. 8+365 pp. 20 DM.

Boulanger, G. R. *Contribution à la théorie générale des abaques à plans superposés.* Paris, Gauthier-Villars, 1949. 117 pp. 1200 Fr.

Bouligand, G., and Desgranges, J. *Le declin des absolus mathematico-logiques.* Paris, Sedes, 1949. 270 pp. 450 Fr.

Bourbaki, N. *Espaces fonctionnels.* Paris, Hermann, 1949. 104 pp.

Callandreau, E. *Célèbres problèmes mathématiques.* Paris, Michel, 1949. 477 pp.

Curry, H. *A theory of formal deducibility.* University of Notre Dame, 1950. 10+126 pp.

Delachet, A. *Calcul vectoriel et calcul tensoriel.* (Oeuvres de Henri Poincaré, vol. 5.) Paris, Gauthier-Villars, 1950. 8+552 pp. 7200 Fr.

Denis-Papin, M., and Kaufmann, A. *Cours de calcul opérationnel. Transformation de Laplace.* Paris, Michel, 1950. 237 pp. 1200 Fr.

Denjoy, A. *Leçons sur le calcul des coefficients d'une serie trigonometrique.* Part 4. Paris, Gauthier-Villars, 1949. 390 pp. 3700 Fr.

Desgranges, J. See Bouligand, G.

Goldstein, H. *Classical mechanics.* Cambridge, Addison-Wesley, 1950. 12+399 pp.

Good, I. J. *Probability and the weighing of evidence.* New York, Hafner, 1950. 8+119 pp. $3.00.

Halmos, P. *Measure theory.* New York, Van Nostrand, 1950. 12+304 pp. $5.90.

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Madelung, E. *Die mathematischen Hilfsmittel des Physikers.* (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, vol. 4.) Berlin, Springer, 1950. 18+531 pp. 49.70 DM.

von Neumann, J. *Functional operators.* Vol. 2: *The geometry of orthogonal spaces.* (Annals of Mathematics Studies, no. 22.) Princeton University Press, 1950, 107 pp. $2.25.

Orange, W. B. See Urner, S. E.

Picard, É. *Leçons sur quelques équations fonctionnelles avec des applications à divers problèmes d'analyse et de physique mathématique.* Ed. by E. Blanc. Paris, Gauthier-Villars, 1950. 3+187 pp. 400 Fr.

Picard, É. *Leçons sur quelques types simples d'équations aux dérivées partielles avec des applications à la physique mathématique.* Paris, Gauthier-Villars, 1950. 3+214 pp. 700 Fr.

Schwartz, L. *Theorie des distributions,* Part 1. Paris, Hermann, 1950. 148 pp.

Urner, S. E., and Orange, W. B. Elements of mathematical analysis. New York, Ginn, 1950. 7+562 pp. $4.00.

FUNCTIONS ON LOCALLY COMPACT GROUPS

GEORGE W. MACKEY[1]

I. INTRODUCTION

1. **Background.** The subject of this address is a branch of mathematics which may be regarded as a combination of the classical theory of representations of finite groups by matrices and that part of analysis centering around the theory of Fourier series and integrals. The connection between these two apparently diverse subjects arises simply enough from the fact that the real numbers and the real numbers modulo 2π form groups under addition.

In one form of the theory of representations of a finite group G a central role is played by the so-called group ring or group algebra. This is usually defined as the set of all formal linear combinations of group elements $c_1s_1+c_2s_2+ \cdots +c_ns_n$, where each $s_i \in G$ and each c_i is a complex number. Two such expressions are added in the obvious manner and are multiplied by writing down the formal product and simplifying by means of the distributive law and the given multiplication of group elements. It may also be defined (and this is the definition we shall use) as the vector space of all complex-valued functions on G with multiplication defined by the formula

$$(f*g)(s) = \sum_{t\in G} f(st^{-1})g(t).$$

If we regard $c_1s_1+c_2s_2+ \cdots +c_ns_n$ as the function whose value at s_i is c_i, it is not difficult to see that these two definitions amount to the same thing. A basic result in the theory of group representations asserts that the group algebra is a direct sum of minimal two-sided ideals. Now it is easy to see that a linear subspace M of the group algebra is a left (right) ideal if and only if it is closed with respect to left (right) translation; that is, if and only if $f\in M$ implies that ${}_af(f_a)$ is in M for all a in G where ${}_af(s)=f(as)$ and $f_a(s)=f(sa)$. Thus every function on G is uniquely expressible as a sum of functions each of which has the property that its right and left translates generate a linear subspace which is minimal with respect to translation invariance.

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[1] The author is a fellow of the John Simon Guggenheim Memorial Foundation.

On the other hand, Fourier's theorem asserts that every member of a large class of functions on the line with period 2π (that is, functions on the group of the reals modulo 2π) may be written in the form

$$\sum_{n=-\infty}^{\infty} c_n \exp (inx)$$

where the c_n are complex numbers, n is an integer, and $i^2 = -1$. Here the functions $\exp (inx)$ generate one-dimensional and hence minimal subspaces which are certainly translation invariant.

The analogy is clear. In each case there is a theorem asserting that a fairly general function on a group is a sum of functions having especially simple translation properties. It is natural to seek general theorems about the translation properties of functions on groups which include these as special cases. In this address I shall describe some of the work that has been done in this direction in the last twenty years or so and indicate its connections with other branches of mathematics.

There will be no attempt at completeness of coverage except from the special point of view here adopted. Many topics more or less closely related will not be discussed at all. In particular the emphasis will be on theorems valid for quite general groups and on the papers in the literature where these theorems are established in general. Thus in spite of his pioneering work in the harmonic analysis of functions on the line, there will be little mention of the name of Norbert Wiener.

In gathering this material together I have benefited greatly from conversations with I. Kaplansky, L. H. Loomis, F. I. Mautner, and I. E. Segal, and I wish to take this opportunity to acknowledge my indebtedness to them.

2. **Preliminaries.** In the theory of Fourier series and integrals the topology of the underlying group plays a role. Accordingly it is natural to consider topological groups; that is, groups which are at the same time topological spaces in such a manner that the group operations are continuous. Since every group is a topological group with respect to the topology under which every set is closed, discrete and in particular finite groups are not excluded. Among topological groups, those admitting a compact neighborhood of the identity, the so-called locally compact groups, present themselves as especially suitable for study. According to a refinement of an important

theorem of A. Haar [34],[2] every such group[3] admits an essentially unique Borel measure[4] which is invariant under right (left) translation. When G is the real line, this measure reduces to ordinary Lebesgue measure. In the general case its existence makes it possible to extend most of the methods, notions, and results of the theory of Lebesgue integration to the study of functions on arbitrary locally compact groups. In particular one may speak of the class $\mathcal{L}^1(G)$ of functions summable on the group of the class $\mathcal{L}^2(G)$ of functions of summable square, of the integral $\int f(x)dx$ of a function in $\mathcal{L}^1(G)$, and so on.

It is only when G is finite that a satisfactory theory of the set of all functions on G can be given. In general it seems to be necessary to restrict attention to a large but not all inclusive subfamily. This family is usually taken to be the set of all functions on the group having some simple topological or measure theoretic property. It is ordinarily linear and translation invariant and provided with a natural topology. The problem[5] is then to give as complete an analysis as possible of the structure of the set of closed translation invariant linear subspaces. Of the families of functions one might consider, two have properties which make them especially suitable for such a study and have received a great deal of attention. These are the two measure theoretically defined families mentioned in the preceding paragraph, $\mathcal{L}^1(G)$ and $\mathcal{L}^2(G)$. The first may be made into an algebra which is a generalization of the group algebra of a finite group, multiplication being defined by the formula

$$f*g(s) = \int f(st^{-1})g(t)dt$$

where integration is with respect to right invariant Haar measure. Whenever f and g are in $\mathcal{L}^1(G)$, the right-hand side exists for almost all s and defines a function in $\mathcal{L}^1(G)$. It is not difficult to show that a linear subspace of $\mathcal{L}^1(G)$ closed in the usual norm is right (left) invariant under translation if and only if it is a right (left) ideal in the group algebra. In this case then the general problem becomes that of studying the ideal theory of a certain Banach algebra. $\mathcal{L}^2(G)$ on the other hand, while not in general an algebra, is a Hilbert space and possesses the especially simple structure of such spaces. In particular the possibility of taking orthogonal complements makes its particularly suitable for direct sum-like decompositions.

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

[3] Haar considered only separable groups and did not prove uniqueness. Uniqueness was first proved by von Neumann [55] and the extension to the nonseparable case was first carried out by Weil [77]. Cf. also von Neumann [56] and Kakutani [38].

[4] In the terminology of Halmos [36].

[5] Cf. [66, p. 865].

II. COMPACT GROUPS

3. Two-sided invariant subspaces and the Peter-Weyl theorem. When the group in question is not only locally compact but compact, a rather complete extension of the theory for finite groups may be given. This theory was first developed by F. Peter and H. Weyl [60] in 1927. These authors restricted themselves to Lie groups, but only because the general measure of Haár, which incidentally for compact groups is always both left and right invariant, was not then available. Their theory goes over to the general case without change.[6] The fundamental result of this theory—the celebrated Peter-Weyl theorem—may be stated in the following form[7]: $\mathcal{L}^2(G)$ is the direct sum of finite-dimensional minimal two-sided invariant subspaces $\mathfrak{I}_\alpha$. Each consists entirely of continuous functions. Every continuous function on G is a uniform limit of finite sums of functions from the $\mathfrak{I}_\alpha$. The theory of the closed two-sided invariant subspaces of $\mathcal{L}^2(G)$ now follows at once. Every such subspace is the direct sum of the $\mathfrak{I}_\alpha$'s which it contains and every family of $\mathfrak{I}_\alpha$'s defines a unique such subspace. In particular there is a natural inclusion preserving correspondence between the subfamilies of the family of all $\mathfrak{I}_\alpha$'s and the closed two-sided invariant subspaces of $\mathcal{L}^2(G)$. The theory of $\mathcal{L}^1(G)$ is hardly less simple. Each $\mathfrak{I}_\alpha$ is a minimal two-sided ideal and every closed two-sided ideal is the closed linear span of the $\mathfrak{I}_\alpha$'s which it contains. For compact groups there is a natural one-to-one correspondence between the closed invariant subspaces of $\mathcal{L}^2(G)$ on the one hand and those of $\mathcal{L}^1(G)$ on the other.

4. One-sided invariant subspaces and representations. Every closed right invariant subspace R of $\mathcal{L}^2(G)$ or $\mathcal{L}^1(G)$ is the closed linear span of the set of intersections $R \cap \mathfrak{I}_\alpha$. Thus the analysis of the right invariant subspaces of $\mathcal{L}^2(G)$ or $\mathcal{L}^1(G)$ reduces to that of the $\mathfrak{I}_\alpha$. Now it is a well known theorem of Wedderburn that each $\mathfrak{I}_\alpha$, being simple as an algebra, is isomorphic to the algebra of all linear transformations of an n-dimensional vector space into itself. Here $n = n_\alpha$ is a positive integer depending on α. It follows that each $\mathfrak{I}_\alpha$ is a (non-unique) direct sum of n_α minimal right invariant subspaces and that the family of all right invariant subspaces of $\mathfrak{I}_\alpha$ is order isomorphic to the family of all subspaces of an n_α-dimensional vector space.

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[6] As far as separable groups are concerned this was pointed out by Haar himself as a corollary to the existence of his measure.

[7] Cf. Köthe [42].

The finite-dimensional right invariant subspaces of $\mathcal{L}^2(G)$ furnish examples of what are called representations of G, that is, continuous homomorphisms of G into the group of unitary transformations of a finite-dimensional Hilbert space into itself. Indeed if M is such a subspace, $f \in M$, $a \in G$, and $U_a(f) = f_a$ where $f_a(x) = f(xa)$, then U_a is unitary and the map $a \rightarrow U_a$ is a representation of G. The close and important connection between the theory of the representations of G and the foregoing may be described briefly as follows. If $M^{(1)}$ and $M^{(2)}$ are minimal right invariant subspaces of the same $\mathfrak{I}_\alpha$, then the associated unitary representations $U^{(1)}$ and $U^{(2)}$ are unitary equivalent in the sense that there exists a unitary map V of $M^{(1)}$ on $M^{(2)}$ such that $VU_a^{(1)}V^{-1} = U_a^{(2)}$ for all $a \in G$. Moreover any unitary representation $a \rightarrow U_a$ of G which is irreducible in the sense that there is no subspace invariant under all U_a is unitary equivalent to the representations associated with one and only of the $\mathfrak{I}_\alpha$. If $\phi_1, \phi_2, \cdots$ is a basis in the space in which the U_a act, then the "matrix elements" $(U_a\phi_i, \phi_j)$ are continuous complex-valued functions defined on G. Their linear span is independent of the basis and in fact coincides with $\mathfrak{I}_\alpha$. It follows that the matrix elements of the irreducible representations of G are dense in $\mathcal{L}^2(G)$, and it is in this form that the Peter-Weyl theorem is often quoted.

Since $\mathcal{L}^2(G)$ is the direct sum of the $\mathfrak{I}_\alpha$, each $f \in \mathcal{L}^2(G)$ is of the form $f = \sum_\alpha f_\alpha$ where $f_\alpha \in \mathfrak{I}_\alpha$. If we use representations, a compact and suggestive formula for f_α may be obtained. If U_f^α is defined for each α and each $f \in \mathcal{L}^2(G)$ by the formula

$$U_f^\alpha = \int U_{x^{-1}}^\alpha f(x)dx$$

where U^α is associated with $\mathfrak{I}_\alpha$, then

$$f_\alpha = n_\alpha \text{ Trace } [U_x^\alpha \; U_f^\alpha].$$

This last equality implies and is implied by the equality

$$\int |f(x)|^2 dx = \sum_\alpha n_\alpha \text{ Trace } [U_f^\alpha (U_f^\alpha)^*],$$

which may be regarded as a generalization of Parseval's equation in the theory of Fourier series.

5. **Almost periodic functions.** Bochner's reformulation [4] of Bohr's definition of almost periodic functions on the line makes sense for arbitrary groups. For compact groups all the continuous functions turn out to be almost periodic. J. von Neumann in a celebrated paper [54] showed that the Peter-Weyl theory for continuous functions on a compact group could be obtained as a special case of an analogous theory of continuous almost periodic functions on an arbitrary topological group.[8] When the group is the real line this theory reduces to that of Bohr. A central tool is a mean value defined for each almost periodic function and taking the place of the Haar integral. The main result asserts that the almost periodic functions on a group are related to the minimal finite-dimensional invariant subspaces of continuous functions and hence to the representations of the group just as are the continuous functions on a compact group. Later A. Weil[9] [76; 78] showed that the theory of von Neumann could be deduced from the Peter-Weyl theory. More precisely he showed that for an arbitrary topological group G there exists a continuous homomorphism ϕ of G onto a dense subgroup of a certain compact group K with the property that a function f on G is almost periodic if and only if there exists a continuous g on K such that $f(x)=g(\phi(x))$.

III. ABELIAN GROUPS

6. **Duality and the Plancherel theorem.** The theory of compact groups assumes an especially simple form when the group G is Abelian as well as compact. The irreducible representations are all one-dimensional and so are simply continuous homomorphisms of G into the group of complex numbers of modulus one, that is, characters of G. The minimal invariant subspace $\mathfrak{I}_\alpha$ of $\mathcal{L}^2(G)$ associated with the representation defined by the character y_α is simply the one-dimensional subspace of all complex multiples of y_α. Thus the expansion formula becomes $f(x)=\sum c_y y(x)$ where $c_y=\int f(x)\overline{y(x)}dx$ and the summation is over all characters y. Convergence is of course in the sense of the $\mathcal{L}^2(G)$ norm, that is, "in the mean."

It was apparently first noticed by A. Weil [78] that these formulae could be given a much more symmetric form and generalized to arbitrary locally compact Abelian groups by making use of the Pontrjagin-van Kampen duality theory [62; 63; 39] for such groups. If G is locally compact and Abelian, then the set $G^\frown$ of all characters on G is itself an Abelian group under ordinary pointwise multiplication.

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[8] Cf. also Weyl [79].

[9] Cf. also Freudenthal [9], van Kampen [40], Pontrjagin [61] and Tannaka [74].

Moreover it is locally compact in a certain natural topology. Now if $y \in G^\frown$ and $x \in G$, then for fixed x, $y(x)$ looked at as a function of y is a character of $G^\frown$. The principal theorem of the Pontrjagin-van Kampen theory asserts that every character of $G^\frown$ is uniquely obtainable in this way from an element of G and that the topology of G regarded as $G^{\frown\frown}$ is the same as the given topology of G. Thus there is a natural pairing of locally compact Abelian groups such that each member of a pair is the character group of the other. Because of the symmetry of the relationship between G and $G^\frown$ it is customary to write (x, y) instead of $y(x)$. The resulting function of two variables is sometimes called the character function. If G is the additive group of the real line, $G^\frown$ is also isomorphic to the additive group of the real line and the character function may be taken to be such that $(x, y) = \exp(ixy)$. If G is the additive group of the reals modulo 2π, then G is isomorphic to the additive group of the integers and $(x, y) = \exp(inx)$ where n is an integer. In general the character group of a compact group is discrete and that of a discrete group compact.

Let us return to the formulae $f(x) = \sum_y c_y y(x)$ and $c_y = \int f(x)\overline{y(x)}dx$. Setting $c_y = f^\frown(y)$ to emphasize the fact that the set of coefficients is a function on $G^\frown$, we may rewrite these formulae in the form

$$f(x) = \int f^\frown(y)(x, y)dy \quad \text{and} \quad f^\frown(y) = \int f(x)\overline{(x, y)}dx$$

where the integrals are taken with respect to the Haar measures in G and $G^\frown$. Indeed, since $G^\frown$ is discrete, the Haar measure of a point may be taken to be one and summation over the group elements is the same as integrating over the group. In this form, however, the formulae are meaningful for arbitrary locally compact groups and A. Weil in [78] has established their validity for suitably restricted functions f. In particular the $f^\frown$ for $f \in \mathcal{L}^1(G) \cap \mathcal{L}^2(G)$ are dense in $\mathcal{L}^2(G^\frown)$ and (for suitable choice of the Haar measure in G) $\int |f(x)|^2 dx = \int |f^\frown(y)|^2 dy$ for all such f. Thus the map $f \to f^\frown$ may be uniquely extended by continuity to give a natural norm preserving linear map of all of $\mathcal{L}^2(G)$ onto all of $\mathcal{L}^2(G^\frown)$. This "Plancherel theorem" for locally compact Abelian groups reduces to the classical Plancherel theorem for Fourier transforms when G is taken to be the additive group of the real line and to the Parseval relation for Fourier series plus the Riesz-Fisher theorem when G is taken to be the real line modulo 2π. For arbitrary locally compact Abelian groups the generalized Fourier transform $f \to f^\frown$ makes possible a quite satisfactory solution of the invariant subspaces problem for $\mathcal{L}^2(G)$. In fact it is not hard to show that the Fourier transform of a translation invariant closed subspace of $\mathcal{L}^2(G)$ is the set of all members of $\mathcal{L}^2(G^\frown)$ which vanish (almost everywhere) outside of some measurable subset of $G^\frown$ and conversely. Thus there is a one-to-one inclusion preserving correspondence between the closed invariant subspaces of $\mathcal{L}^2(G)$ and the measurable subsets mod null sets of $G^\frown$.

7. **Normed rings and $\mathcal{L}^1(G)$.** The generalized Fourier transform also arises naturally when Gelfand's theory of normed rings (Banach algebras) is applied to the algebra $\mathcal{A}_G$ obtained by adjoining a unit to the algebra $\mathcal{L}^1(G)$. For an arbitrary Banach algebra $\mathcal{A}$ the Gelfand theory gives a canonical homomorphism $f \rightarrow f^0$ where $f \in \mathcal{A}$ and $f^0 = f^0(M)$ is a member of the ring of continuous complex-valued functions defined on the compact space $\mathfrak{M}$ of all maximal ideals in $\mathcal{A}$. When $\mathcal{A} = \mathcal{A}_G$, the maximal ideals other than $\mathcal{L}^1(G)$ itself are in one-to-one correspondence with the characters of G in such a manner that for each $f \in \mathcal{L}^1(G)$ and each $M \in \mathfrak{M} - \{\mathcal{L}^1(G)\}$, $f^0(M) = \int f(x) y_M(x) dx$ where y_M is the character corresponding to M. Thus functions on $\mathfrak{M}$ may be identified with functions on $G^\frown$ and the canonical mapping $f \rightarrow f^0$ reduces to the Fourier transform. The fact that $f^0 = 0$ implies that $f^0 = 0$ is equivalent to the fact that $\mathcal{A}_G$ is semisimple. The Fourier transform from this point of view seems to have been introduced independently by Gelfand and Raikov [20] on the one hand and I. E. Segal [68] on the other without knowledge of Weil's work. The application of the normed ring point of view to functions in $\mathcal{L}^2(G)$ and the proof of the Plancherel theorem seems to have been first carried out by M. Krein [43].

The invariant subspaces problem for $\mathcal{L}^1(G)$ is much more difficult than that for $\mathcal{L}^2(G)$ and so far is only very incompletely solved even when G is the real line. It is natural in view of the $\mathcal{L}^2$ theory to attempt to set up a correspondence between the closed ideals in $\mathcal{L}^1(G)$ and subsets of $G^\frown$. Following Segal,[10] who seems to have been the first to attack the problem for general groups [68; 69], let us define the hull of a closed ideal $\mathfrak{J}$ in $\mathcal{L}^1(G)$ as the set $h(\mathfrak{J})$ of all characters which define maximal ideals containing $\mathfrak{J}$. Let us define the kernel of a subset A of $G^\frown$ as the intersection $k(A)$ of the maximal ideals defined by members of A. Then $k(h(\mathfrak{J}))$ the kernel of the hull of $\mathfrak{J}$ contains $\mathfrak{J}$. If it were true that $\mathfrak{J} = k(h(\mathfrak{J}))$ for all $\mathfrak{J}$, then there would be a one-to-one inclusion inverting correspondence between the closed invariant subspaces of $\mathcal{L}^1(G)$ and the closed subsets of G and a satisfactory solution of the invariant subspaces problem would be at hand. Unfortunately L. Schwartz [67] has recently shown that this equality need not hold; specifically whenever G is Euclidean n space with $n \geqq 3$. I. E. Segal has shown on the other hand that it is always true that $\mathfrak{J}$ is "approximately" equal to $k(h(\mathfrak{J}))$. More precisely if O is any open set containing $h(\mathfrak{J})$,then $k(O) \subseteq \mathfrak{J}$. It follows that whenever $h(\mathfrak{J})$ is empty, then $\mathfrak{J} = k(h(\mathfrak{J}))$. This result for the real line implies Wiener's general Tauberian theorem and his theorem about spans of functions in $\mathcal{L}^1(G)$. In fact the nontrivial half of one form of the general Tauberian theorem (Theorem 4 of [80]) may be stated as follows. Let f be a bounded measurable function on the line and let c be a complex number. Let $\mathfrak{J}$ be the class of all functions K in $\mathcal{L}^1(-\infty, \infty)$ such that $\lim_{x\to\infty} \int_{-\infty}^{\infty} K(x-y)f(y)dy = c\int_{-\infty}^{\infty} K(x)dx$. Then if $\mathfrak{J}$ contains a function K_1 whose Fourier transform never vanishes, $\mathfrak{J}$ must coincide with $\mathcal{L}^1(-\infty, \infty)$. Now it is routine to verify that $\mathfrak{J}$ is a closed ideal. Moreover the fact that the Fourier transform of K_1 never vanishes is equivalent to the fact that K_1 is not contained in the maximal ideal defined by any character exp (ixy). This implies that the hull of $\mathfrak{J}$ is empty and hence that $k(h(\mathfrak{J})) = \mathcal{L}^1(-\infty, \infty)$. In Segal's paper, which also contains the developments necessary to obtain the other form of Wiener's theorem, this theorem is generalized in another direction: when G is the real line, then $\mathfrak{J} = k(h(\mathfrak{J}))$ whenever $h(\mathfrak{J})$ is the union of an open set and a closed set without limit points. I. Kaplansky [41] has recently extended this last result to a large class of locally compact Abelian groups and shown that for arbitrary locally compact Abelian groups $\mathfrak{J} = k(h(\mathfrak{J}))$ whenever $h(\mathfrak{J})$ has only a finite number of elements.

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[10] Results similar to those of Segal have been obtained independently by Godement [26].

8. **The Laplace transform and connections with the theory of functions of a complex variable.** Further connections of the theory of locally compact Abelian groups with classical analysis appear when other invariant classes of functions on a group are considered. Specifically it is possible in a great variety of ways to choose a measure μ on G which is different from Haar measure but absolutely continuous with respect to it in such a manner that $\mathcal{L}^1(\mu, G)$ is closed with respect to convolution.[11] The maximal ideals in the resulting Banach algebra are no longer in one-to-one correspondence with the characters of G but with what may be called generalized characters, that is, continuous homomorphisms of G into the multiplicative group of all nonzero complex numbers. Moreover, for particular such Banach algebras not all generalized characters need appear. Every

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[11] Convolution being defined as usual using Haar measure and not μ.

generalized character is uniquely a product of an ordinary character and a function of the form exp (l) where l is real-valued and has the property that $l(xy) = l(x) + l(y)$ for all x and y in G, that is, is a linear functional. If we let $\bar{G}$ denote the vector space of all continuous linear functionals on G, the maximal ideals of $\mathcal{L}^1(\mu, G)$ will be in one-to-one correspondence with the members of a subset of $G^\frown \times \bar{G}$. It can be shown that this subset is always of the form $G^\frown \times K$ where K is a suitable convex subset of the vector space $\bar{G}$. The canonical mapping taking each $f \in \mathcal{L}^1(\mu, G)$ into a function on the maximal ideals is then a mapping of functions on G into functions on $G^\frown \times K$. If f^0 is the map of f and y and l are members of G and k respectively, then

$$(*) \qquad f^0(y, l) = \int f(x)(x, y) \exp (l(x)) dx.$$

If G is the group of integers, then $G^\frown$ is the group of complex numbers of unit modulus and $\bar{G}$ is a one-dimensional vector space. A convex subset of $\bar{G}$ is an interval so that the functions f^0 are defined on the direct product of a circle with an interval, that is, an annulus. It is easy to see that the mapping from sequences to functions defined in an annulus thus obtained coincides with the corresponding mapping defined by the Laurent series. If G is the real line, then $G^\frown \times K$ is a strip and $f \rightarrow f^0$ is the two-sided Laplace transform. In the general case it is possible to introduce a notion of directional differentiation for functions defined on $G^\frown \times K$ and a relationship between directions of differentiation which leads to a notion of analyticity for functions on $G^\frown \times K$. The generalized Laplace transforms, that is, the functions f^0 defined by (*), then turn out to be analytic functions on $G^\frown \times K$. In this way at least some of the main notions in the theory of functions of a complex variable arise as special cases of more general notions occurring naturally in the theory of locally compact Abelian groups. One is led to conjecture that there will be general theorems about such groups which will unify and illuminate appropriate portions of the classical theory of functions of one or more complex variables. These matters have been discussed in outline by the author in [46]. A preliminary example of what can be done is the following (unpublished) generalization of Hadamard's three circles theorem. If f is analytic in $G^\frown \times K$ and is bounded in y for each fixed l in K, then $\log (\sup_{y \in G^\frown} |f(y, l)|)$ is a convex function of l on the convex subset K of the vector space $\bar{G}$. This theorem includes Hadamard's three circles theorem for several complex variables and hence Thorin's [75] generalization of the M. Riesz convexity theorem.

The ideal theory of the rings $\mathcal{L}^1(\mu, G)$ seems not to have been investigated at all in the general case, although Gelfand in [11] has defined such rings for the real line and integer groups and these rings have been further discussed in [22]. The fact that the functions f^0 are analytic instead of merely continuous can naturally be expected to produce many divergences from the theory of $\mathcal{L}^1(G)$. In particular the theorem of Kaplansky to the effect that an ideal whose hull is only a single point must be maximal certainly cannot be expected to carry over. In fact for each point y, l "interior" to $G^\frown \times K$ and each "direction of differentiation" v the set of all $f \in \mathcal{L}^1(G, \mu)$ for which $f_v(y, l) = f(y, l) = 0$ will be an ideal which will not be maximal but will be contained in only one maximal ideal. A number of questions suggest themselves. Is every ideal with a finite hull maximal provided that every point of the hull is an extreme point of the boundary of $G^\frown \times K$; that is, a point of the form y, l where l is an extreme point of K? Is every counter example to the proposed analogue of Kaplansky's theorem of the form just described (with higher order derivatives of course)? Do the extreme points of the boundary of $G^\frown \times K$ constitute the boundary in the sense of Silov [22] of the set of maximal ideals? These are questions which the author intends to investigate and discuss elsewhere. The last has been answered in the affirmative in special cases by Silov.

The work of Laurent Schwartz [66] on mean periodic functions has close connections with the foregoing. Schwartz considers the set of all continuous functions on the real line topologized so that convergence is uniform convergence on compact sets. The dual of this topological vector space is the set of all complex-valued Borel measures on the line which are concentrated in compact sets. These measures form a ring under convolution and there is a one-to-one inclusion inverting correspondence between the closed translation invariant subspaces of continuous functions and the closed ideals in this ring. This ring is of the same general nature as those considered above and its closed maximal ideals are in one-to-one correspondence with the points of the complex plane. The canonical homomorphism carries this ring into a multiplicative ring of entire functions. Schwartz shows essentially that every closed primary ideal in the original ring is determined by a complex number a and a non-negative integer k and consists of all members such that the corresponding entire function and its first k derivatives vanish at the point a. He shows furthermore that every closed ideal is an intersection of primary ones, thus determining all possible closed ideals in a very explicit fashion.

IV. GENERAL LOCALLY COMPACT GROUPS

9. **Preliminaries.** The theory of locally compact groups which are neither compact nor Abelian is much less well developed than the theories of either of these special classes. Even the $\mathcal{L}^2$ theory, which is more or less complete for compact and Abelian groups, is in a rather unsatisfactory state for more general groups. A part of the trouble may be ascribed to the fact that when the group is neither compact nor Abelian one must deal with "continuous" decompositions into "multi-dimensional parts." Thus neither the discreteness of the compact case nor the one-dimensionality of the Abelian case is available and there is a compounding of difficulties. A perhaps more serious source of difficulty is the fact that it is no longer possible to confine attention to finite-dimensional representations. There are locally compact groups—the group of all two-by-two complex matrices with determinant one is an example—which admit no nontrivial finite-dimensional unitary representations at all. It is thus necessary to turn to representations by unitary operators in infinite-dimensional Hilbert spaces and to analyze the more complicated phenomena which almost always appear when one makes a transition from the finite to the infinite.

10. **Positive definite functions and the existence of sufficiently many irreducible representations.** Let $\mathcal{L}^2(G)$ denote the Hilbert space of all complex-valued functions on G which are square summable with respect to right invariant Haar measure in G. For each $f \in \mathcal{L}^2(G)$ and each $s \in G$ let $U_s(f)$ be the function $g \in \mathcal{L}^2(G)$ such that $g(x) = f(xs)$ for all $x \in G$. Then $f \rightarrow U_s(f)$ is a unitary operator. Moreover the map $s \rightarrow U_s$ is a continuous representation of G in the sense[12] that it is a homomorphism of G into the group of all unitary operators in $\mathcal{L}^2(G)$ such that $(U_s(f), g)$ is continuous in s for all f and g in $\mathcal{L}^2(G)$. This representation is called the right regular representation of G. Its existence makes it clear that every locally compact group has sufficiently many representations to distinguish between any two group elements if infinite-dimensional representations are allowed. It does not follow at once that there need exist sufficiently many (or even any) nontrivial irreducible representations. This much less trivial existence theorem was first proved by Gelfand and Raikov [21]. Here irreducibility must be taken to mean that no closed invariant sub-

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[12] Actually $(U_s(f), g)$ is continuous for all f and g if and only if $U_s(f)$ is continuous for all f. In the sequel we shall use the terms representation and unitary representation interchangeably to denote a unitary representation which has one and hence both of these equivalent properties.

spaces exist. There are many examples showing that the theorem is false if irreducibility is taken to mean that no invariant subspaces at all exist.

In order to prove this theorem Gelfand and Raikov exploit a close relationship which exists between representations and those continuous functions on G which have the property of "positive definiteness." Let U ($x \rightarrow U_x$) be any representation of G. Let ϕ be any element in the associated Hilbert space and let f be the continuous complex-valued function defined on G by the equation $f(x) = (U_x(\phi), \phi)$. It is trivial to verify that f has the following property. If $x_1, x_2, \cdots, x_n$ is any finite subset of G and $c_1, c_2, \cdots, c_n$ is a set of complex numbers, then $\sum_{i,j=1}^{n} c_i \bar{c}_j f(x_i x_j^{-1}) \geqq 0$. Any function with this property is said to be positive definite. It is not difficult to show that conversely every continuous positive definite function is obtainable in this manner from a suitable representation of G. This representation may always be chosen so as to be cyclic in the sense that, for some ϕ, the $U_x(\phi)$ span a dense subspace of the associated Hilbert space. Moreover, if this is done, the correspondence is one-to-one in the following sense: $(U_x(\phi), \phi)$ and $(U'_x(\phi'), \phi')$ are the same for all x in G if and only if U and U' are unitary equivalent via a unitary transformation which carries ϕ into ϕ'. The representation associated with the continuous positive definite function f is irreducible if and only if f is "elementary" in the sense that it is not of the form $f_1 + f_2$ where f_1 and f_2 are linearly independent continuous positive definite functions. The existence of sufficiently many elementary positive definite functions on G and hence of sufficiently many irreducible representations is made to depend upon a theorem of Krein and Milman [44] asserting the existence of sufficiently many extreme points in certain convex sets. The convex set to which this theorem is applied is the set of all continuous positive definite functions f for which $f(e) = 1$. (Here e is the identity of G.)

Functions on G of the form $(U_x(\phi), \psi)$, where U ($x \rightarrow U_x$) is a representation of G and ϕ and ψ are in the Hilbert space in which the U_x act, are finite linear combinations with complex coefficients of functions of the form $(U_x(\phi), \phi)$. Thus if G is compact, the finite sums of elements from the ideals $\mathfrak{J}_\alpha$ are exactly the finite linear combinations of elementary positive definite functions. R. Godement in [27] has generalized one form of the Peter-Weyl theorem to arbitrary locally compact groups by applying the theory of Gelfand and Raikov to show that every continuous function on such a group can be uniformly approximated on every compact set by finite linear combinations of elementary positive definite functions. Actually Godement developed much of the theory of positive definite functions independently of Gelfand and Raikov. His paper [27] is a complete exposition of the work of Gelfand, Raikov, and himself, and contains many theorems about the behavior of positive definite functions on groups in addition to those described here.

11. The reducibility of representations. Let U be a finite-dimensional representation of a group G and let H be the underlying (finite-dimensional) Hilbert space. It is well known and easily proved that H is a direct sum of orthogonal subspaces in each of which U induces an irreducible representation. This decomposition is unique in the sense that the irreducible summands in any two such are unitary equivalent in some order. Actually the only nonuniqueness lies in the decomposition of direct sums of mutually unitary equivalent irreducible representations. In general the direct sum of all subspaces of H in which the "same" irreducible representation is induced is independent of the original decomposition. For infinite-dimensional unitary representations a similar theorem holds provided that G is compact [6; 37]. When G is not necessarily compact it is not difficult to see that no such decomposition need exist. The regular representation of a noncompact locally compact Abelian group is an example. Any nonzero closed invariant subspace of $\mathcal{L}^2(G)$ is readily seen to have proper closed invariant subspaces. On the other hand, the Plancherel theorem shows that $\mathcal{L}^2(G)$ may be mapped unitarily onto $\mathcal{L}^2(G^\frown)$ in such a manner that translation by s goes over into multiplication by (s, y) where y is a generic element of G. Roughly speaking the regular representation of G appears to be a sort of "continuous direct sum" of the one-dimensional (and hence irreducible) representations defined by the various characters.

Clearly what is needed is a generalization of the notion of direct sum which will include as a special case a precise notion of "continuous direct sum" and which can be used to decompose arbitrary representations into irreducible parts. A theory of such a notion has been developed by J. von Neumann [58] in order to decompose rings of operators into "factors" and the von Neumann theory has been applied to the theory of representations of separable locally compact groups of F. I. Mautner [48; 49]. Slightly reformulated,[13] the basic notion may be described as follows. Let S be a set and let $\mathcal{J}$ be a countably generated σ field of subsets of S. Let μ be a countably additive measure defined on $\mathcal{J}$. For each $n = 1, 2, \cdots, \infty$, let $\mathcal{H}_n$ be an n-dimensional separable Hilbert space. Let d be a measurable function from S to the positive integers and ∞. Let $\mathcal{H}(S, \mu, d)$ be the set of all functions from S to $\mathcal{H}_\infty \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots$ such that: (a) $f(x) \in \mathcal{H}_{d(x)}$ for all x, (b) for each $n = 1, 2, \cdots, \infty$ and each $v \in \mathcal{H}_n$, $(f(x), v)$ is a measurable function of x for x in $d^{-1}(n)$, (c) $\int (f(x), f(x)) d\mu(x) < \infty$. $\mathcal{H}(S, \mu, d)$ becomes a Hilbert space when the inner product of two members f and g is defined by the equation $[f, g] = \int (f(x), g(x)) d\mu(x)$. Now suppose that a representation $U^{(x)}$ $(s \to U_s^{(x)})$ of a group G in $\mathcal{H}_{d(x)}$ is given for each $x \in S$. If $U^{(x)}$ depends measurably on x in the sense that, for each n, $(U_s^{(x)}(v_1), v_2)$ is measurable as a numerical function on $d^{-1}(n)$ for each v_1 and v_2 in $\mathcal{H}_n$, then there is a unique representation U of G in $\mathcal{H}(S, \mu, d)$ such that for all f in $\mathcal{H}(S, \mu, d)$, the equation $(U_s f)(x) = U_s^{(x)}(f(x))$ holds for each s and almost all x. U (or any representation unitary equivalent to U) is said to be the generalized direct sum of the representations $U^{(x)}$ with respect to the measure μ. It is clear that this notion reduces to that of ordinary direct sum whenever S is a discrete measure space.

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[13] Cf. Godement [29] for an alternative approach.

This notion is adequate for the purpose indicated above. One of the principal results of [48] asserts[14] that an arbitrary representation (in a separable Hilbert space) of a separable locally compact group G is a generalized direct sum of irreducible representations of G. Unfortunately this result does not reduce the study of arbitrary representations to that of irreducible representations, for the decomposition in question can be rather badly nonunique. On the other hand, there is a generally possible unique reduction into components which, while not irreducible, nevertheless give evidence of being the proper fundamental building blocks. The details are as follows. Let U be a generalized direct sum of representations $U^{(x)}$ with the associated measure space S. For each measurable subset E of S, let P_E be the projection which takes $f \in \mathcal{H}(S, \mu, d)$ into $\phi_E f$ where ϕ_E is the characteristic function of E. Then the set $\mathcal{E}$ of all P_E is a σ-Boolean algebra of projections in the sense that the identity I is in $\mathcal{E}$, that whenever $P_1, P_2, \cdots$ are in $\mathcal{E}$ then $P_1 P_2 = P_2 P_1 \in \mathcal{E}$ and $I - P_1 \in \mathcal{E}$, and that if $P_i P_j = 0$ then $P_1 + P_2 + \cdots \in \mathcal{E}$. Moreover $\mathcal{E}$ commutes with U in the sense that $PU_s = U_s P$ for all $P \in \mathcal{E}$ and all $s \in G$. Conversely, given any σ-Boolean algebra $\mathcal{E}$ which commutes with a representation U, it is possible to decompose U as a generalized direct sum in such a manner that $\mathcal{E}$ is the σ-Boolean algebra defined by the decomposition. Moreover, modulo null sets and replacement of μ by an equivalent measure, the decomposition is uniquely determined. We may speak of the decomposition defined by $\mathcal{E}$. This decomposition will have almost all of its parts irreducible if and only if $\mathcal{E}$ is maximal; that is, if and only if whenever $\mathcal{E}' \supseteq \mathcal{E}$ where $\mathcal{E}'$ is also a σ-Boolean algebra commuting with U, then $\mathcal{E} = \mathcal{E}'$. In general there will be a number of distinct maximal $\mathcal{E}$'s commuting with a given U. This is so even for finite-dimensional representations, and occurs there when and only when some irreducible components are repeated. Consider now the σ-Boolean algebra $\mathcal{E}_0$ defined as follows. $E \in \mathcal{E}_0$ if and only if E commutes with all U_s and with all operators A which commute with all U_s. $\mathcal{E}_0$ is contained in every maximal $\mathcal{E}$ and is uniquely determined by U. The decomposition defined by $\mathcal{E}_0$ has the property that almost every component has as its $\mathcal{E}_0$ the set consisting of 0 and the identity. We shall call a representation V whose $\mathcal{E}_0$ is trivial in this sense a factor[15] representation. Thus every representation has an essentially unique decomposition as a generalized direct sum of factor representations.

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[14] Note that this furnishes an alternative proof of the existence of sufficiently many irreducible representations when G is separable.

In the finite-dimensional case a representation is a factor representation if and only if it is a direct sum of replicas of the same irreducible representation. In the infinite-dimensional case things are considerably more complicated. Given a representation U, let $\{U_s\}'$ denote the set of all bounded operators A in the underlying Hilbert space such that $AU_s = U_sA$ for all $s \in G$ and let $\{U_s\}''$ be similarly defined with $\{U_s\}$ replaced by $\{U_s\}'$. Then $\{U_s\}''$ is an operator ring in the sense of Murray and von Neumann [52] and a necessary and sufficient condition that U be a factor representation is that $\{U_s\}''$ be a factor, that is, that it contain only multiples of the identity in its center. Now Murray and von Neumann have studied factors in great detail and have classified them according to the behavior of a certain dimension function uniquely (up to a multiplicative constant) defined for the projections in any factor. The factor is said to be of type I_n, I_∞, II_1, II_∞, or III_∞ according as the (suitably normalized) dimension function takes on the values $1, 2, \cdots, n$; $1, 2, \cdots, \infty$, all real numbers between 0 and 1, all non-negative real numbers and ∞, or just 0 and ∞. Factors of all types exist. It is shown in [49] that a factor representation is a generalized direct sum of replicas of the same irreducible representation if and only if the corresponding factor is of type I_n or I_∞. Moreover, this irreducible representation is uniquely determined. For factor representations not of type I_n or I_∞, however, the behavior can be very different indeed. Mautner [50] has given an example of a group whose regular representation is a factor representation of type II_1 and has found a decomposition of it into irreducible components in which uncountably many inequivalent representations occur. The author has shown (unpublished) that this representation has another decomposition into irreducible parts such that (a) no two representations in the decomposition are unitary equivalent and (b) no representation in the decomposition is unitary equivalent to any representation in Mautner's decomposition. On the other hand, general factor representations do share the following property of factor representations of types I_n and I_∞. A decomposition as a direct sum of two other representations is always such that one component is unitary equivalent to a discrete part of the other. To summarize: Every unitary representation of a separable locally compact group is a unique generalized direct sum of factor representations. Every factor representation of type I_n or I_∞ is a generalized direct sum (actually an ordinary discrete direct sum) of replicas of a uniquely determined irreducible representation. A factor representation not of type I_n or I_∞ can be decomposed into irreducible parts, but the decomposition may be wildly nonunique. The structure of such representations is very little understood.

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[15] Because of the close connection with the Murray-von Neumann notion of factor for rings of operators. See below. Cf. Wigner [81].

There remains the question of the extent to which a representation is determined by the factor representations into which it may be decomposed. The first remark to be made is that it is not sufficient simply to list the representations which occur in the breakdown—the measure μ is itself of decisive importance. This is more or less clear from the fact that sets of representations of measure zero may be added and removed without altering the sum. A better appreciation of the situation may be obtained by considering what happens when G is Abelian. Let $G^{\frown}$ be the character group of G. Let μ be any Borel measure in $G^{\frown}$. Let V^{μ}_{s} be the operator which takes $f \in \mathcal{L}^2(G^{\frown}, \mu)$ into g where $g(y) = (s, y)f(y)$ for all $y \in G$. Then V^{μ} $(s \rightarrow V^{\mu}_{s})$ is a unitary representation of G and as a matter of fact every representation of G in which there are "no multiplicities" is unitary equivalent to some V^{μ}. Two such representations V^{μ} and V^{ν} are unitary equivalent if and only if μ and ν have the same null sets. When multiplicities are allowed, things are only slightly more complicated. Choose a Borel measure μ in $G^{\frown}$ and a Borel function d from $G^{\frown}$ to the positive integers and ∞. Let μ_n $(n = 1, 2, \cdots, \infty)$ be the unique Borel measure which agrees with μ on subsets of $d^{-1}(n)$ and is zero for subsets of the complement of $d^{-1}(n)$. Let $V^{\mu,d}$ be the direct sum of countably many replicas of V^{μ_∞}, one replica of V^{μ_1}, two replicas of V^{μ_2}, and so on. Then every representation of G is unitary equivalent to some $V^{\mu,d}$, and $V^{\mu,d}$ and $V^{\nu,d'}$ are unitary equivalent if and only if μ and ν

have the same null sets and d and d' are equal almost everywhere with respect to μ and ν. The regular representation of G has Haar measure for μ and $d(y) \equiv 1$. These facts about Abelian groups follow from the theory of multiplicity of spectra for normal operators in Hilbert space and the following spectral resolution theorem. If $s \rightarrow U_s$ is an arbitrary representation of the locally compact group G by operators in a Hilbert space $\mathcal{H}$, then there exists a mapping $E \rightarrow P_E$ of the Borel subsets E of G into projections in $\mathcal{H}$ such that the countable Boolean operations are preserved and such that for all ϕ and ψ in $\mathcal{H}(U_s(\phi), \psi) = \int (s, y) d[P_y(\phi), \psi]$ where integration is over G with respect to the completely additive set function $E \rightarrow (P_E\phi, \psi)$. This theorem was first proved by Stone [**73**] for the case in which G is the additive group of the real line. It was extended to arbitrary locally compact Abelian groups independently by Ambrose, Godement, and Neumark [**1**; **23**; **59**].

When G is not necessarily Abelian and U $(s \rightarrow U_s)$ is a representation of G, we have seen that U is an essentially unique generalized direct sum of factor representations $U^{(y)}$ with respect to a measure μ. In view of the foregoing it is natural to conjecture that two representations U and V are unitary equivalent if and only if almost all the $U^{(y)}$ can be mapped one-to-one onto almost all the $V^{(x)}$ in such a way that null sets and measurability are preserved and so that the image of each $U^{(y)}$ is unitary equivalent to it. So far as the author knows no proof of this theorem is at present available. It seems likely, however, that the methods of von Neumann and Mautner will be adequate to produce one. It is to be observed that the "only if" part of this conjecture cannot be expected to be true unless the $U^{(y)}$ and $V^{(x)}$ are known to arise from a canonical decomposition into factor representations defined by an $\mathcal{E}_0$ as described above. This follows from the fact noted above that a generalized direct sum of distinct irreducible representations can be a factor representation. The question: What measure spaces of factor representations "add up" to factor representations remains to be investigated. Until something substantial about its answer is known the theory of general representations can hardly be regarded as reduced to that of factor representations even when the above conjecture has been proved.

For a fully satisfactory theory of the decomposability of representations, even more than an answer to the questions of the last paragraph would be desirable. One would want a sort of "dual object"—a topological space $G^{\sim}$, the points of which would be classes[16] of factor representations. One would then hope for a theorem setting up a one-to-one correspondence between unitary equivalence classes of representations of G on the one hand, and on the other systems consisting of a family of null sets in $G^{\sim}$ and a function assigning one member of its class to each member of $G^{\sim}$. The prognosis for such a theorem for general locally compact groups is not encouraging. The author has studied in some detail the representations of semi direct products of locally compact Abelian groups.[17] For a large subclass of these groups a complete analysis of the representations can be given, and there is little doubt that for members of this subclass suitable dual objects can be found. However, for semi direct products not in this subclass, irreducible representations and a fortiori factor representations exist in a bewildering profusion which so far has defied exhaustive analysis. Certainly it is difficult to imagine their being all included in any sort of well behaved dual object. On the other hand, there are a number of groups other than the compact and Abelian ones for which dual objects and corresponding decomposition theorems are available. Gelfand and Neumark [15] have shown that this is the case for the group of all two-by-two complex matrices of determinant one. More generally, I. Kaplansky in an as yet unpublished work has obtained such a theorem for any "completely continuously representable" or "CCR" group; that is, for any group G which has the following property: If $f \in \mathcal{L}^1(G)$ and U is an irreducible representation of G, then $\int f(x) U_x dx$ is a completely continuous operator.[18] It can be shown that such groups admit only factor representations of type I_n and I_∞. It is not inconceivable that, conversely, every group all of whose factor representations are of type I is a CCR group. If so, Kaplansky's result may be regarded as solving the reducibility problem modulo the pathology of nontype I factors.

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[16] Given two factor representations U and V, it can be shown that either (a) no discrete direct summand of one is unitary equivalent to any discrete direct summand

12. $\mathcal{L}^2(G)$ and the Plancherel theorem for unimodular groups.

In this section we shall deal throughout with locally compact groups which are not only separable but also are "unimodular" in the sense

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of the other or (b) given any discrete direct summand of U and any discrete direct summand of V one is unitary equivalent to a discrete direct summand of the other. When (b) holds we may say that U and V are quasi unitary equivalent. The relation of quasi unitary equivalence is an equivalence relation in the technical sense and divides factor representations into classes of mutually quasi unitary equivalent elements.

[17] If G admits a closed normal subgroup G_1 and a second closed subgroup G_2 such that $G_1 \cap G_2 = e$ and $G_1 G_2 = G$, then G is said to be a semi direct product of G_1 and G_2.

[18] It is irrelevant whether left or right invariant Haar measure is used here.

that left and right Haar measures coincide. For such groups there is a single $\mathcal{L}^2(G)$ which is the underlying space for both the right and left regular representations. The family $\mathcal{J}$ of closed subspaces invariant under both right and left translation coincides with the family of closed subspaces which simultaneously reduce the right and left regular representations. A recent result[19] of Segal [**71**] analyzes the internal structure of this family in quite a satisfactory manner. This result (proved earlier by Murray and von Neumann [**53**] for discrete groups) asserts that the weakly closed self-adjoint operator rings R_l and R_r generated by the unitary operators in the left and right regular representations respectively are commutators of one another; that is, $A \in R_l$ if and only if it commutes with all members of R_r and vice versa. It implies in particular that the projections on the members of $\mathcal{J}$ are precisely the projections in the common center of R_r and R_l. Thus the projections on the members of $\mathcal{J}$ not only commute with one another, but form a σ-Boolean algebra of projections. More than this is true. The Segal result also implies that this Boolean algebra is identical with the one which defines the decomposition of the right (left) regular representation into factor components. Let S be a measure space of factor representations effecting this decomposition. It follows that the closed two-sided invariant subspaces of $\mathcal{L}^2(G)$ are in one-to-one inclusion preserving correspondence with the measurable sets mod null sets of S. Thus a theorem on the structure of the family of two-sided invariant closed subspaces of $\mathcal{L}^2(G)$ is available which is quite analogous to that which has already been discussed for compact and Abelian groups.

A more difficult problem is that of discussing the decomposition of individual functions in $\mathcal{L}^2(G)$; that is, proving a generalization of the Plancherel theorem. For compact groups, it will be recalled, the basic fact is that $\int |f(x)|^2 dx = \sum_\alpha d_\alpha \operatorname{Trace}\left(U_f^\alpha \, (U_f^\alpha)^*\right)$ where $f \in \mathcal{L}^2(G)$, d_α is the dimension of the irreducible representation indexed by α, and $U_f = \int U_{x^{-1}} f(x)dx$. In the general case it is natural to deal with factor representations and to replace the trace by the Murray-von Neumann relative trace.[20] In short, one is led to make the following conjecture. Let S be a measure space effecting the decomposition of the right regular representation of G into factor components. Then it is possible to choose a particular relative trace function Tr in the factor associated with each point y of S and a particular measure μ among those with the given null sets in S in such a way that for all $f \in \mathcal{L}^2(G) \cap \mathcal{L}^1(G)$ the formula

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[19] Godement implicitly announces this result in a footnote in [**28**].

[20] For suitably restricted operators in any factor there is defined [**57**] a linear functional which has the formal properties of a trace. This function is unique up to multiplication by a positive real number.

$$\int |f(x)|^2 dx = \int \mathrm{Tr}\,(U_f^y (U_f^y)^*) d\mu(y)$$

is valid. Such a formula has been established by F. I. Mautner [**49**] for discrete groups, by I. Kaplansky (unpublished) for CCR groups, by Gelfand and Neumark [**15**; **18**] for the group of all $n \times n$ complex matrices of determinant one, and by R. Godement [**24**; **25**] for central[21] groups.[22] Very recently I. E. Segal [**72**] has been able to take care of the general case.

For a really thorough going generalization of the Plancherel theorem one would demand a little more; namely (a) an "onto" theorem asserting that the mapping $f \rightarrow U_f$ can be extended by continuity so as to map $\mathcal{L}^2(G)$ onto all functions $y \rightarrow A_y$ from S to operators in the relevant factors which are measurable in a suitable sense and such that $\int \mathrm{Tr}\,[(A_y(A_y)^*]d(y) < \infty$, (b) an a priori dual object; that is, a topological space of factor representations given in advance and while not capable necessarily of decomposing arbitrary representations, at least adequate for decomposing the right regular representation, (c) some sort of a generalization of Haar measure for the measure μ; that is, a theorem asserting that μ can be chosen to have some invariance property with respect to Kronecker products of representations in the dual object and is uniquely defined by this property. The a priori dual object is at present available for central groups, CCR groups, and the group of all $n \times n$ complex matrices of determinant one. There is an onto theorem for the two-by-two case of the latter group. So far as the author is aware the invariant measure question has not been discussed for any non Abelian groups.

Just as for compact and Abelian groups, the Plancherel formula is connected with an expansion formula.[23] For suitably restricted f in $\mathcal{L}^2(G)$

$$f(x) = \int \mathrm{Tr}(U_x^y U_f^y) d\mu(y)$$

for all x in G. This formula expresses f as an integral with respect to y of a family of functions, the "yth" of which is a limit of finite linear combinations of "matrix elements" of the "yth" factor representation. For Abelian groups the expansion formula not only sets up a one-to-one correspondence between functions in $\mathcal{L}^2(G)$ and functions in $\mathcal{L}^2(G^\frown)$, but also sets up such a correspondence between the positive definite functions on G and the finite Borel measures on G. Specifically the formula $f(x)=\int(x, y)g(y)dy$ where $g\in\mathcal{L}^1(G)$ and $g(y)\geqq 0$ for all y may be generalized by substituting a finite Borel measure α for g and writing $f(x)=\int(x, y)d\alpha(y)$. The theorem that this formula sets up a one-to-one correspondence between measures and positive definite functions was proved by Herglotz when G is the group of integers, by Bochner [**5**] when G is the real line, and by Weil [**78**] and Raikov [**64**; **65**] for locally compact Abelian groups in general. The reader will find the close connection between this theorem and the Plancherel theorem discussed from various points of view in these papers and in a recent work of Cartan and Godement [**7**]. The paper of Cartan and Godement presents an elegant and concise exposition of a large part of the theory of locally compact Abelian groups starting with a simple derivation of the generalized Herglotz-Bochner theorem from the Krein-Milman theorem.

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[21] Some of the statements in [**24**] are incorrect but have been corrected in a letter to I. Kaplansky. Presumably a corrected version of [**24**] will be published in due course.

[22] A central group is one whose group of inner autmorphisms has a compact closure in the suitably topologized group of all automorphisms. This notion is due to Godement.

[23] Cf. Lewitan [**40**].

Because of the connection between positive definite functions and representations one can expect a close connection between expansion theorems for positive definite functions and decomposition theorems for unitary representations. For locally compact Abelian groups, for example, it is easy to deduce the Ambrose-Godement-Neumark generalization of Stone's theorem from the generalized Bochner-Herglotz theorem and vice versa. For non Abelian groups, on the other hand, the lack of a satisfactory dual object for decomposing general representations is reflected in the lack of a satisfactory generalization of the Bochner-Herglotz theorem. One can, however, guess at the form such a theorem would take; namely: If G is a group with a dual object S and f is a continuous positive definite function on G, then there exists a measure α on S and a function $A(y\rightarrow A_y)$ from S to positive definite operators such that A_y lies in a factor associated with y and $f(x)=\int \operatorname{Tr}(U_x^y A_y)d\mu(y)$ where U^y $(x\rightarrow U_x^y)$ is the corresponding factor representation.

13. One-sided invariant subspaces. Very little has been done toward studying the one-sided invariant subspaces of $\mathcal{L}^2(G)$. Possibly the generalized direct sum theory can be applied to prove that each right invariant closed subspace of $\mathcal{L}^2(G)$ is defined by a suitably defined "measurable family" of right invariant closed subspaces—one at each point of the measure space effecting the decomposition of the right regular representation into factor components. If so, one is reduced to studying the right invariant subspaces in a Hilbert space under the action of a ring of operators which is a factor. The structure of such a family of subspaces is clear when the factor is of type I_n or I_∞. It is then order isomorphic to the family of all closed subspaces of a Hilbert space. For factors of other types it is not understood at all. W. Ambrose [2] has inaugurated a program which may throw some light on the situation. He proposes to regard $\mathcal{L}^2(G)$ as an "algebra" with a partially defined multiplication and to study its structure from the point of view of the Wedderburn theory. The left and right "ideals" in this algebra are the left and right closed invariant subspaces of $\mathcal{L}^2(G)$.

14. **$\mathcal{L}^1(G)$ and other group algebras.** There is also little that can be said about the ideal theory of $\mathcal{L}^1(G)$ when G is neither compact nor Abelian. Such results as are available are primarily of a negative character. The representations of $\mathcal{L}^1(G)$ by bounded operators in Hilbert space, for example, are only loosely related to the ideal theory of $\mathcal{L}^1(G)$. Many non unitary equivalent such representations may have a common kernel ideal. Moreover it is possible for an irreducible representation to have a kernel ideal which is not maximal—there is an example in which such an ideal is the null ideal and at the same time an intersection of maximal ideals with finite-dimensional quotients. Perhaps the most significant positive result is that of I. E. Segal [69] asserting that in general the zero ideal is an intersection of closed primitive ideals.

There are other ways, however, of generalizing the classical notion of group algebra and some of these lead to objects with a somewhat more tractable structure theory. I. E. Segal [70], for example, starts with the ring of all bounded linear operators in $\mathcal{L}^2(G)$ defined by convolution on the left with members of $\mathcal{L}^1(G)$ and proceeds to its uniform closure. A closely related algebra has been considered by Gelfand and Neumark [19]. Both of these algebras are isomorphic to uniformly closed algebras of operators in Hilbert space and such algebras (called C^* algebras) have given rise to a rapidly growing literature. The connection between a group and its attached C^* algebra is close enough so that many results about the one can be interpreted as results about the other, and vice versa, and in some cases the approach via the algebra has certain advantages. In particular the results of Kaplansky on CCR groups have been obtained in this way.

15. **The determination of representations.** In understanding the

phenomena which arise in generalizing the Plancherel theorem, as well as for various other reasons, it is useful to have a more or less complete analysis of the irreducible and factor representations of particular groups which are neither compact nor Abelian. Such an analysis has been carried out by Bargmann [**3**] for the three- and four-dimensional Lorentz groups[24] and by Gelfand and Neumark [**13**; **14**; **15**; **16**; **17**] for the group of $n\times n$ complex matrices of determinant one[25] and the group of affine transformations of the line [**12**]. The author, as has already been indicated, has studied the representations of semi direct products of Abelian groups. The complete analysis which is possible in certain cases has led to a theorem [**47**] about "imprimitive" representations of arbitrary separable locally compact groups which promises to be useful in studying representations in general. In particular it turns out that every decomposition of the regular representation of a subgroup G_0 of a separable locally compact group G as a generalized direct sum defines in a natural way a similar decomposition of the regular representation of G. This phenomenon when G_0 is Abelian has been discussed by Godement [**28**].

APPENDIX

(*Added June* 1950.) Since this article was written a number of further developments in the theory it describes have come to our attention. We sketch some of these below.

(1) Godement [**30**; **31**; **32**; **33**] has extended his theory of central groups to the more extensive family of groups of finite class. The definition of this notion is such that a group is of finite class if and only if every neighborhood of the identity admits a subneighborhood which is invariant under inner automorphisms. All discrete, compact, and locally compact Abelian groups are of finite class. Via a theory of "characters" for these groups Godement obtains a Plancherel theorem for them based on a topological a priori dual object. His work is closely related to and in part dependent on that of J. Dixmier [**8**] on rings of finite class.

(2) In a continuation of [**49**] Mautner has obtained independently of Segal a proof of the "weak" form of the general Plancherel theorem (§12, paragraph 2 of this paper). Mautner's proof leans heavily on von Neumann's theory of weight functions for operator rings. Among other results in Mautner's paper is the theorem that if a factor repre-

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[24] Cf. also Harish-Chandra [**35**].

[25] The quotient of the 2×2 group by its center is the four-dimensonal Lorentz group.

sentation is not of type I, then any decomposition into irreducible parts has noncountably many components no two of which are unitary equivalent.

(3) Via a notion of dual semi-groups in a group G and its "real character group" $\overline{G}$ the author has found a natural way of extending portions of the classical theory of the one-sided Laplace transform and of power series in a manner paralleling the generalization of the "two-sided" theory described in §8. In particular one can prove a theorem about "completely monotonic" functions in semi-groups which generalizes the solution of the Hausdorff moment problem and the Hausdorff-Bernstein-Widder theorem on completely monotonic functions. The proof makes use of the Krein-Milman theorem in much the same way that Cartan and Godement used it to prove the Plancherel theorem.

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SELF-DUAL CONFIGURATIONS AND REGULAR GRAPHS

H. S. M. COXETER

1. **Introduction.** A configuration (m_c, n_d) is a set of m points and n lines in a plane, with d of the points on each line and c of the lines through each point; thus $cm=dn$. Those permutations which preserve incidences form a group, "the group of the configuration." If $m=n$, and consequently $c=d$, the group may include not only *symmetries* which permute the points among themselves but also *reciprocities* which interchange points and lines in accordance with the principle of duality. The configuration is then "self-dual," and its symbol (n_d, n_d) is conveniently abbreviated to n_d. We shall use the same symbol for the analogous concept of a configuration in three dimensions, consisting of n points lying by d's in n planes, d through each point.

With any configuration we can associate a diagram called the *Menger graph* [**13**, p. 28],[1] in which the points are represented by dots or "nodes," two of which are joined by an arc or "branch" whenever the corresponding two points are on a line of the configuration. Unfortunately, however, it often happens that two different configurations have the same Menger graph. The present address is concerned with another kind of diagram, which represents the configuration uniquely. In this *Levi graph* [**32**, p. 5], we represent the points and lines (or planes) of the configuration by dots of two colors, say "red nodes" and "blue nodes," with the rule that *two nodes differently colored are joined whenever the corresponding elements of the configuration are incident.* (Two nodes of the same color are never joined.) Thus the Levi graph for (m_c, n_d) has m red nodes and n blue nodes, with each red node joined to c blue nodes and each blue node joined to d red nodes, so that there are $cm=dn$ branches altogether.

As simple instances in two dimensions we have the triangle 3_2, whose Levi graph is a hexagon with red and blue vertices occurring

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[1] I take this opportunity to correct an error on the third line of page 31 of [**13**]: The words "the direct product . . . namely" should be deleted. This was kindly pointed out by Robert Frucht, whom I would also thank for his constructive criticism of the present address in manuscript. Numbers in brackets refer to the bibliography at the end of the paper.

alternately; and the complete quadrangle (4_3, 6_2), whose Levi graph may be regarded as a tetrahedron with a red node at each vertex and a blue node at the midpoint of each edge.

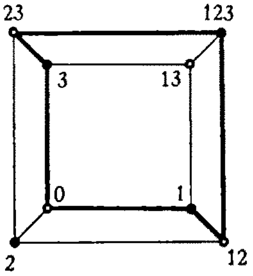

FIG. 1. The cube γ_3.

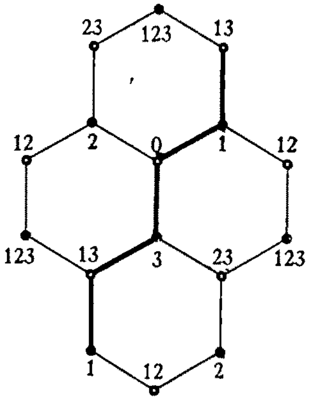

FIG. 2. The regular map $\{6, 3\}_{2,0}$.

As instances in three dimensions we have the tetrahedron 4_3, whose Levi graph (Fig. 1 or 3) consists of the vertices and edges of a cube (with the vertices colored alternately red and blue); and Möbius's configuration 8_4, whose Levi graph (Fig. 5) consists of the vertices and edges of a four-dimensional hypercube γ_4 [12, p. 123], as we shall see in §10. This 8_4 may be regarded as a pair of tetrahedra so placed that each vertex of either lies in a face-plane of the other.

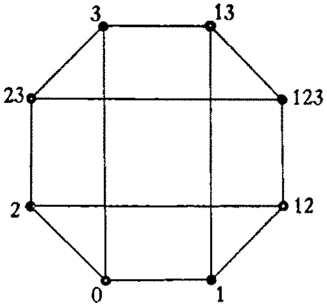

FIG. 3. A distorted cube.

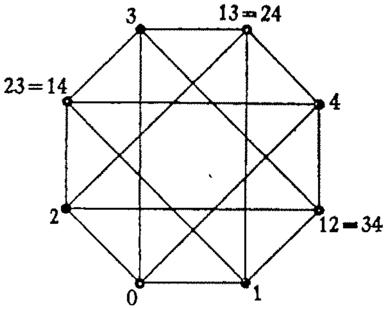

FIG. 4. $\gamma_4/2$.

The symbols at the vertices in Figs. 1 and 5 represent Cartesian coordinates in the following manner: 0 is the origin, and every other symbol indicates which of the three or four coordinates have the value 1, while the rest are zero. Thus two vertices are joined by an edge whenever their symbols are derived from each other by inserting or removing a single digit. (Here 0 does not count as a digit, but rather as the absence of digits.) A vertex is red or blue according as its symbol contains an even or odd number of digits (thus 0 is red). The group of symmetries and reciprocities of Möbius's configuration is of order $4!\,2^4=384$; for it is isomorphic with the symmetry group of the hypercube, and this consists of the permutations of the digits 1, 2, 3, 4, combined with reflections R_1, R_2, R_3, R_4 which act on the symbols as follows: R_i adds a digit i whenever that digit is not yet present, and removes i whenever it is present; for example, R_3 transforms the square 0 1 13 3 into the same square with its vertices named in the reverse order.

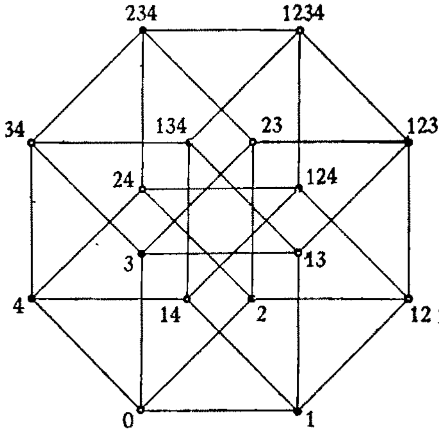

FIG. 5. The hypercube γ_4.

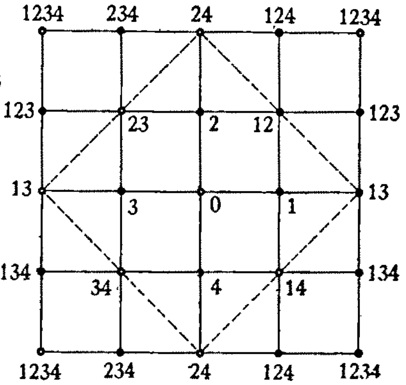

FIG. 6. The regular maps $\{4, 4\}_{4,0}$ and $\{4, 4\}_{2,2}$.

It is possible (in three ways) to select 16 of the 24 square faces of the hypercube so as to form the regular skew polyhedron $\{4, 4|4\}$ [11, p. 43]. Topologically, this is a regular map on a torus, as in Fig. 6 (with peripheral elements identified as marked). Fig. 4 is named $\gamma_4/2$ because it can be derived from the hypercube γ_4 by identifying opposite elements, as in the passage from spherical to elliptic space. When applied to Fig. 6, this identification yields a map of eight (instead of sixteen) squares on a torus, given by identifying opposite sides of the square drawn in broken lines. Comparing Fig. 4 with Fig. 3, we see that this graph of 8 nodes and 16 branches may be regarded as a cube with opposite vertices joined, though its group (of order $2\cdot 4!^2=1152$) is much larger than that of the cube itself. It is the Levi graph for a rather trivial configuration consisting of four points on a line and four planes through the same line.

Fig. 2 shows that the Levi graph for the tetrahedron (Fig. 1) forms another regular map on a torus: it decomposes the surface into four hexagons [13, p. 25]. If we look for these hexagons (such as 0 1 12 123 23 3) in the original cube, we find that they are Petrie polygons: regular skew polygons whose various pairs of adjacent sides belong to different faces. Conversely, the map has its own Petrie polygons (such as 0 1 13 3), which are the faces of the cube.

This investigation is justified by the general principle that interesting configurations are represented by interesting graphs. In practice, either subject is liable to throw light on the other. A known property of Kummer's configuration 16_6 led to the discovery of a new regular skew polyhedron in six dimensions, as we shall see in §13. On the other hand, a certain graph, described long ago by an electrical engineer, was found to represent a highly symmetrical configuration 12_3 in the real projective plane (§8).

2. **Regular graphs.** To be precise, a *graph* is a collection of N_0 objects called *nodes*, N_1 pairs of which satisfy a certain symmetric relation. Such a pair of nodes is called a *branch* (or rather, the two nodes are said to be *joined* by a branch). Since we shall never allow two branches to join the same two nodes, it is convenient to regard the N_0 nodes as points and the N_1 branches as line-segments joining pairs of the points. If the number of branches at a node is the same for all nodes, say d, the graph is said to be of *degree d*, and we see that

$$2N_1 = dN_0.$$

An ordered sequence of s distinct branches, consecutively adjacent (forming a continuous path from one node to another in a definite direction) is called an *s-arc*; for example, the 1-arcs are the $2N_1$ directed branches. A graph is said to be connected if every two nodes are joined by some s-arc. We shall be concerned solely with connected graphs. A closed sequence of m distinct branches, consecutively adjacent (forming a continuous path from a node to itself in m steps, $m>2$) is called an *m-circuit*. The smallest value of m for which an m-circuit occurs is called the *girth* of the graph; for example, the graph of vertices and edges of the cube has degree 3 and girth 4. It is often possible to find an N_0-circuit that includes all the nodes of the graph (and N_0 of the N_1 branches); this is called a *Hamiltonian* circuit [12, p. 8], and enables us to draw the graph as an N_0-gon with N_1-N_0 diagonals inserted. (See Figs. 3, 4, 9, 12, 15, 18, 22, 23.)

A graph of degree d evidently has $d-1$ times as many s-arcs as $(s-1)$-arcs, provided $1<s<m$. Thus the number of s-arcs is

$$(d-1)^{s-1}2N_1 = (d-1)^{s-1}dN_0 \qquad (1 < s < m).$$

Those permutations of the nodes which leave them joined as before form a group, "the group of the graph." A graph is said to be *s-regular* if its s-arcs are all alike while its $(s+1)$-arcs are not all alike, that is, if the group is transitive on the s-arcs but not on the $(s+1)$-arcs. Thus the cube is 2-regular (not 3-regular, since a 3-arc may belong either to a face or to a Petrie polygon). For an s-regular graph of degree 3, the two ways in which a given s-arc may be extended to an $(s+1)$-arc are essentially different; so the subgroup leaving the s-arc invariant is of order 1, and the order of the whole group is equal to the number of s-arcs, namely

$$2^s N_1.$$

But when $d>3$, the final branch of an $(s+1)$-arc may sometimes be chosen in several equivalent ways; so we can only say that the order is *divisible* by $(d-1)^{s-1}2N_1$.

The above definitions differ only very slightly from those proposed by Tutte [**47**], who proved that, for every s-regular graph of girth m,

$$s \leqq m/2 + 1.$$

He called the graph an *m-cage* in the case of maximum regularity

$$s = [m/2 + 1].$$

Restricting consideration to the case when $d=3$, he found a cage for each value of m up to 8, except 7, and proved that there are no other cages (of degree 3).

The 3-cage is the complete graph with four nodes, that is, the graph of vertices and edges of a tetrahedron. The 4-cage is the Thomsen graph, which has six nodes 1, 2, 3, 4, 5, 6, and nine branches ij, where i is even and j odd [**22**, p. 403; **5**, p. 35]. The 5-cage is the Petersen graph, whose ten nodes may be denoted by the unordered pairs of digits 1, 2, 3, 4, 5, with the rule that two nodes are joined whenever their symbols have no common digit [**28**, p. 194]. The 6-cage and 8-cage are more complicated graphs which we shall describe in §§4 and 9.

The Petersen graph may conveniently be drawn as a large pentagon surrounding a small pentagram, with corresponding vertices joined [**19**, p. 222]. Accordingly, let us denote it by

$$\{5\} + \left\{\frac{5}{2}\right\}$$

and try to generalize this procedure. The symbol $\{n\}+\{n/d\}$ will denote an n-gon $A_0A_1 \cdots A_{n-1}$ and a star n-gon $B_0B_dB_{2d}B_{3d} \cdots$, with A_iB_i joined. The star polygon $\{n/d\}$ exists whenever n/d is a fraction in its lowest terms with $n>2d$ [**12**, p. 93]. Thus the number of different kinds of n-gon is $\phi(n)/2$, and the values of n for which there is just one star n-gon are given by

$$\phi(n) = 4,$$

namely $n=5$, 8, 10, 12. On examining Figs. 11, 17, and 21, we find that $\{8\}+\{8/3\}$ and $\{12\}+\{12/5\}$ are 2-regular, while $\{10\}+\{10/3\}$ is 3-regular. The last is the 3-regular graph described by Tutte [**47**, p. 460]. But all three of these graphs were described earlier by Foster [**18**, p. 315].

The regularity of such a graph is easily computed by writing the mark 0 at one node, 1 at each adjacent node, 2 at any further nodes adjacent to these, and so on. Then all the s-arcs emanating from the initial node are marked $012 \cdots s$.

The complete graph with $n+1$ nodes is evidently 2-regular. This graph, whose group is symmetric, may be described as the graph of vertices and edges of an n-dimensional simplex. More generally, the graph of vertices and edges of an n-dimensional regular polytope ($n>2$) is 2-regular whenever the vertex figure is a simplex [**12**, p. 128] but only 1-regular otherwise. However, the regularity is occasionally increased by identifying opposite vertices so as to obtain a regular "honeycomb" covering elliptic $(n-1)$-space. The case of the four-dimensional hypercube γ_4 has already been mentioned. The other instance is the dodecahedron, which reduces to the Petersen graph [**30**, p. 69; **12**, p. 51, Fig. 3.6E with $ij=ji$].

We saw in §1 how every configuration can be represented in a unique manner by a "Levi graph." Clearly, the configuration and graph are isomorphic: the group of the configuration (including reciprocities as well as symmetries) is the same as the group of the graph. In particular, *if a configuration n_3 has an s-regular Levi graph* ($s>1$), *the order of its group is*

$$2^s N_1 = 2^s 3n.$$

3. **Regular maps.** It was proved by Petersen [**38**, p. 420] that every graph can be embedded into a surface so as to cover the surface with a map of N_2 non-overlapping simply-connected regions (polygons) whose boundaries consist of the N_1 branches and N_0 nodes of the graph. Since a polyhedron is such a map, it is natural to call the regions *faces*, the branches *edges*, and the nodes *vertices*. The em-

bedding is by no means unique, but there must be at least one for which the characteristic

$$-N_0+N_1-N_2$$

in minimum. When the minimum characteristic is -2, the graph is said to be *planar*, since it can be embedded into the ordinary plane or sphere. The simplest graphs of characteristic -1 are the Thomsen graph and the complete 5-point (or four-dimensional simplex). The corresponding maps are derived from the hexagonal prism and pentagonal antiprism by identifying antipodal elements. A famous theorem of Kuratowski, elegantly proved by Whitney [50], states that a necessary and sufficient condition for a graph to be planar is that it have no part homeomorphic to either of these two special graphs. (In the case of a graph of degree 3, only the Thomsen graph needs to be mentioned.)

Those permutations of the elements of a map which preserve the incidences form a group $\mathfrak{g}$, "the group of the map." It is interesting to compare this with $\mathfrak{G}$, the group of the graph formed by the vertices and edges alone. Clearly, $\mathfrak{g}$ is a subgroup of $\mathfrak{G}$, and any operation of $\mathfrak{G}$ not belonging to $\mathfrak{g}$ yields another map of the same kind in which the same vertices and edges are differently distributed into faces. Hence the graph can be embedded into the same kind of surface in a number of ways equal to the index of $\mathfrak{g}$ in $\mathfrak{G}$. In particular, if $\mathfrak{g}$ is the whole of $\mathfrak{G}$, the embedding is unique.

A map is said to be *regular* if its group includes the cyclic permutation of the edges belonging to any one face and also the cyclic permutation of the edges that meet at any one vertex of this face [7]. Thus the group must be transitive on the vertices, and on the edges, and on the faces. Such a map is "of type $\{p, q\}$" if p edges belong to a face, and q to a vertex. The *dual* map, whose edges cross those of the original map, is then of type $\{q, p\}$.

The above cyclic permutations or "rotations," R and S, satisfy the relations

$$\mathrm{R}^p = \mathrm{S}^q = (\mathrm{RS})^2 = 1,$$

and generate a group of order $4N_1/\epsilon$, where $\epsilon=2$ or 1 according as the surface is orientable or unorientable ("two-sided" or "one-sided") [8, p. 408]. When $\epsilon=1$ this is the whole group of the map, but when $\epsilon=2$ it may be either the whole group or a subgroup of index 2. The latter case arises when the map is symmetrical by reflection as well as by rotation, that is, when there is an operation that interchanges two adjacent faces without altering their common vertices.

The *Petrie polygon* of a regular map is an h-circuit of edges, such that every two consecutive edges, but no three, belong to a face [**12**, pp. 24, 61, 90, 108]. A particular specimen of this polygon is determined by any two adjacent sides of any face; therefore the various specimens are all alike, and we are justified in speaking of *the* Petrie polygon of the map. The number h is easily seen to be ϵ times the period of the "translation" $\mathrm{R}^2\mathrm{S}^2$ [**13**, Fig. xi]. Since every edge belongs to two such circuits, the Petrie polygons themselves form a regular map of type $\{h, q\}$ having the same N_1 edges, the same N_0 vertices, and the same group, though covering a different surface. The relation between the two maps is evidently symmetric: the faces of either are the Petrie polygons of the other. (See Figs. 1 and 2.)

Every map of type $\{p, q\}$ can be derived by making suitable identifications in a simple-connected "universal covering map" [**46**, p. 8] which we call simply "the map $\{p, q\}$." This is, in general, infinite, and we naturally think of it as covering the Euclidean or hyperbolic plane [**43**, p. 162]. In fact, since the Euler-Poincaré characteristic

$$K = -N_0 + N_1 - N_2 = \left(1 - \frac{2}{p} - \frac{2}{q}\right) N_1$$
$$= [(p-2)(q-2) - 4] \frac{N_1}{pq}$$

cannot be negative unless $(p-2)(q-2) < 4$, the only finite universal regular maps $(K = -2)$ are

$$\{p, 2\}, \quad \{2, p\}, \quad \{3, 3\}, \quad \{4, 3\}, \quad \{3, 4\}, \quad \{5, 3\}, \quad \{3, 5\}.$$

The first of these is the partition of the sphere into two hemispheres by a p-gon along the equator; and the corresponding graph is merely the p-gon, $\{p\}$. The second, being the partition of the sphere into p lunes by p meridians, is irrelevant to our discussion, as we are not interested in graphs having two nodes joined by more than one branch. (However, Tutte includes $\{2, 3\}$ in his treatment [**47**], calling it the 2-cage.)

The remaining spherical $\{p, q\}$'s are the five Platonic solids, for which

$$N_1 = 2pq/[4 - (p-2)(q-2)], \qquad h = (4N_1 + 1)^{1/2} - 1$$

[**12**, pp. 5, 13, 19]. Of these, all save the tetrahedron $\{3, 3\}$ have pairs of antipodal elements [**12**, p. 91]. Identifying such pairs, we obtain the four regular maps in the elliptic plane $(K = -1)$:

$$\{4, 3\}/2, \qquad \{3, 4\}/2, \qquad \{5, 3\}/2, \qquad \{3, 5\}/2.$$

When regarded as graphs, the first is the complete 4-point, like $\{3, 3\}$; the second is merely a triangle with repeated sides; the third is the Petersen graph, as we saw in §2; and the fourth is the complete 6-point (or five-dimensional simplex).

Regular maps on the torus ($K=0$) are more interesting [**13**]. There is a map $\{4, 4\}_{b,c}$ for any non-negative integers b and c (with $b \geqq c$ to avoid repetition). This is derived from the ordinary plane tessellation $\{4, 4\}$, whose vertices have integral Cartesian coordinates, by identifying all points of the lattice generated by the vectors (b, c) and $(-c, b)$. Similarly, $\{3, 6\}_{b,c}$ is derived from the infinite triangular tessellation $\{3, 6\}$, whose vertices have integral coordinates referred to oblique axes inclined at $\pi/3$, by identifying all points of the lattice generated by the vectors (b, c) and $(-c, b+c)$. Thus $\{4, 4\}_{b,c}$ has b^2+c^2 vertices, while $\{3, 6\}_{b,c}$ has b^2+bc+c^2. Finally $\{6, 3\}_{b,c}$ is the dual of $\{3, 6\}_{b,c}$, and has $2(b^2+bc+c^2)$ vertices. These maps are symmetrical by reflection whenever $b=c$ or $bc=0$ (as in Figs. 2, 6, 14, 20), but only by rotation in other cases (for example, Fig. 8).

It often happens that the map of minimum characteristic for a given graph is not regular, although a regular map is obtained by allowing a higher characteristic. For instance, the Thomsen graph and the complete 5-point, which form irregular maps on the elliptic plane, form regular maps on the torus, namely

$$\{6, 3\}_{1,1} \quad \text{and} \quad \{4, 4\}_{2,1}$$

[**13**, Figs. iv and ii].

Regarded as a graph, $\{4, 4\}_{b,c}$ is 1-regular if $b+c>4$, since there are two obviously different types of 2-arc, one of which belongs to a 4-circuit while the other does not. For the same reason, $\{4, 4\}_{3,0}$ is 1-regular. But the straight 2-arc does belong to a 4-circuit if $b+c=4$; so it is not surprising to find that $\{4, 4\}_{4,0}$ and $\{4, 4\}_{3,1}$ are 2-regular, while $\{4, 4\}_{2,2}$ is 3-regular (Fig. 6). Also $\{4, 4\}_{2,1}$, being the complete 5-point, is 2-regular. The girth of $\{4, 4\}_{b,c}$ is 4 if $b+c>3$, but only 3 if $b+c=3$.

All the graphs $\{3, 6\}_{b,c}$ are 1-regular, of girth 3 and degree 6. But *there are infinitely many 2-regular graphs of degree* 3. In fact, $\{6, 3\}_{b,c}$ is 2-regular whenever $b+c>3$; but $\{6, 3\}_{2,1}$ is 4-regular (Fig. 8), while $\{6, 3\}_{1,1}$ and $\{6, 3\}_{3,0}$ are 3-regular (Fig. 14). The girth of $\{6, 3\}_{b,c}$ is 6 if $b+c>2$, but only 4 if $b+c=2$ (as in Fig. 2 or the Thomsen graph).

When b and c are coprime, we obtain a Hamiltonian circuit for $\{4, 4\}_{b,c}$ by taking a "straight path" along the edges [13, p. 27]. When this circuit is drawn as a regular polygon $\{b^2+c^2\}$, the remaining branches of the graph form a star polygon

$$\left\{\frac{b^2+c^2}{bb'+cc'}\right\}$$

having the same b^2+c^2 vertices. Here b' and c' are given by the congruence

$$bc' - cb' \equiv \pm 1 \pmod{b^2+c^2}$$

with whichever sign makes $bb'+cc'<(b^2+c^2)/2$. In particular, $\{4, 4\}_{b,1}$ can be drawn as a $\{b^2+1\}$ with its inscribed $\{(b^2+1)/b\}$ (given by $b'=1$, $c'=0$).

Foster has drawn a pentagram $\{5\}$ with its inscribed pentagram $\{5/2\}$, and a decagon $\{10\}$ with its inscribed decagram $\{10/3\}$, as instances of "symmetrical geometrical circuits" [18, p. 316, Fig. 10]. These 2-regular graphs are now seen to be $\{4, 4\}_{2,1}$ and $\{4, 4\}_{3,1}$.

Similarly, the graph $\{3, 6\}_{b,c}$ can be drawn, when b and c are coprime, as a $\{b^2+bc+c^2\}$ with two inscribed star polygons of different densities.

The Petrie polygon of $\{4, 4\}_{b,c}$ is an h-circuit, where h is twice the period of R^2S^2 in the group

$$R^4 = S^4 = (RS)^2 = (RS^{-1})^b(R^{-1}S)^c = 1$$

[13, p. 25]. Since $(RS^{-1})^b(R^{-1}S)^b=(RS^{-1}R^{-1}S)^b=(R^2S^2)^b$, we have $h=2b$ if $b=c$; but whenever $b\neq c$,

$$h = 2(b^2+c^2)/(b, c).$$

(The denominator is the greatest common divisor.) Similarly, for $\{3, 6\}_{b,c}$ or $\{6, 3\}_{b,c}$, h is twice the period of R^2S^2 in

$$R^3 = S^6 = (RS)^2 = (RS^{-2})^b(R^{-1}S^2)^c = 1,$$

namely

$$h = 2(b^2+bc+c^2)/(b, c)$$

for all values of b and c [8, p. 418]. In particular, when $(b, c)=1$, $\{4, 4\}_{b,c}$ and $\{3, 6\}_{b,c}$ have the peculiarity that the number of sides of the Petrie polygon is twice the number of vertices of the whole map, which means that the Petrie polygon is *singular* (as Petrie himself noticed in 1931); for example, the Petrie polygon of $\{4, 4\}_{2,1}$ is a singular decagon such as 0213243041 [13, Fig. i].

The Petrie polygon of $\{6, 3\}_{b,c}$ has the same number of sides as that of the dual $\{3, 6\}_{b,c}$, but is never singular. In fact, since $\{6, 3\}_{b,c}$ has $2(b^2+bc+c^2)$ vertices, its Petrie polygon is Hamiltonian when $(b, c)=1$. In this case the graph can be drawn as a regular $2(b^2+bc+c^2)$-gon with certain pairs of vertices joined; for example, the Thomsen graph $\{6, 3\}_{1,1}$ can be drawn as a hexagon with opposite vertices joined, and $\{6, 3\}_{2,1}$ can be drawn as in Fig. 9.

On the other hand, $\{4, 4\}_{b,c}$ and $\{3, 6\}_{b,c}$ have Hamiltonian Petrie polygons when $(b, c)=2$, except in the single case of $\{4, 4\}_{2,2}$ (Fig. 6). The "red" vertices of Fig. 20 (that is, those marked with capital letters) form a $\{3, 6\}_{2,2}$ whose Petrie polygons are dodecagons such as

$$A_1B_2C_3A_4B_1C_2A_3B_4C_1A_2B_3C_4.$$

Thus the graph of vertices and edges of $\{3, 6\}_{2,2}$ can be drawn as a dodecagon $\{12\}$, a dodecagram $\{12/5\}$, and a pair of hexagons, all having the same twelve vertices. (This is the Menger graph for Fig. 19.)

The three maps $\{4, 4\}_{2,2}$, $\{6, 3\}_{3,0}$, and $\{6, 3\}_{1,1}$ have $h=p$; so it is not surprising to find that in each case the derived map formed by the Petrie polygons is isomorphic to the original map. For instance, the Thomsen graph forms a map $\{6, 3\}_{1,1}$ having faces

$$123456, \qquad 143652, \qquad 163254$$

and Petrie polygons

$$123654, \qquad 143256, \qquad 163452,$$

or vice versa.

Two regular maps of positive characteristic will be considered in §§5 and 7.

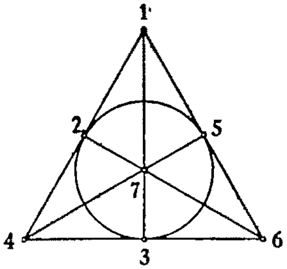

FIG. 7. Fano's 7_3.

4. **Fano's** 7_3. The regular map $\{6, 3\}_{2,1}$ first arose in the work of Heawood on generalizations of the four-color problem [21, p. 333]. Since it consists of seven hexagons, each contiguous to all the others,

it demonstrates the fact that as many as seven colors may be needed for coloring a map on a torus. But Heawood proved also that seven colors will suffice for every map on a torus; so the "seven-color problem" was completely solved.

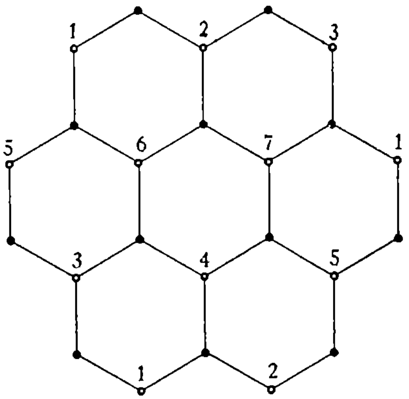

FIG. 8. The regular map $\{6, 3\}_{2,1}$.

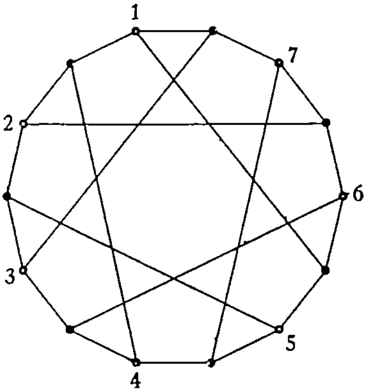

FIG. 9. Another view of the 6-cage.

Let alternate vertices of $\{6, 3\}_{2,1}$ be numbered 1, 2, 3, 4, 5, 6, 7, as in Fig. 8. Each of the remaining seven vertices is then determined by the three to which it is joined; so we have a system of seven triples

$$124, \quad 235, \quad 346, \quad 457, \quad 561, \quad 672, \quad 713,$$

derived from one another by cyclic permutation of the digits. If the two types of vertices are the red and blue nodes of a Levi graph, representing the points and lines of a configuration 7_3, the seven triples show which sets of points are collinear. Every three collinear points are the diagonal points of the complete quadrangle formed by the remaining four points. This state of affairs is indicated in Fig. 7, where the circle is to be regarded as a seventh line. Its impossibility in the usual systems of geometry is often taken as an axiom [**17**, p. 115; **49**, p. 45].

But such a configuration occurs in many finite geometries; for example, in $PG(2, 2)$ where it is the whole plane [**17**, p. 114; **49**, p. 202]. This means that each point has three coordinates (not all zero) belonging to the field of residue-classes modulo 2, namely 0 and 1 with the rule

$$1 + 1 = 0,$$

and three points are collinear whenever their coordinates in each position have sum zero. Such a coordinate symbol may be regarded as a mark in the Galois field $GF[2^3]$, and the simple rule is that three points are collinear whenever their marks have sum zero. Let the modulus of the Galois field be the irreducible cubic

$$\lambda^3+\lambda+1$$

or 1011. Then the successive powers of the primitive root λ or 10 are

$$\lambda=010,\ \lambda^2=100,\ \lambda^3=011,\ \lambda^4=110,\ \lambda^5=111,\ \lambda^6=101,\ \lambda^7=001.$$

These are the proper coordinate symbols for the points previously named 1, 2, 3, 4, 5, 6, 7; and the rule for collinearity is a consequence of the identity $\lambda^r(1+\lambda+\lambda^3)=0$.

The Levi graph for $PG(2, 2)$ may be drawn as in Fig. 9 by placing the numbers 1, 2, $\cdots$, 7 at alternate vertices of a regular 14-gon, joining the vertex between 1 and 2 to 4, that between 2 and 3 to 5, and so on. The peripheral 14-gon appears in $\{6, 3\}_{2,1}$ as a Petrie polygon, which is Hamiltonian since 2 and 1 are coprime.

The blue node 124 belongs to six 2-arcs (as defined in §2): two through each of the red nodes 1, 2, 4. The only nodes not used up in this manner are the four red nodes 3, 5, 6, 7. This remark suggests a third construction for the graph. Regard these four red nodes as forming a tetrahedron. Take a blue node on each of the six edges; for example, a blue node 137 on the edge 37. Join the two blue nodes on each pair of opposite edges, and take a red node on each join; for example, a red node 1 on the join of 137 and 156. Finally, join these three red nodes 1, 2, 4 to a seventh blue node 124.

The relation between the finite projective geometry and its Levi graph shows at once that the graph is a 6-cage (that is, 4-regular, of girth 6). For, each 6-circuit of the graph represents a triangle of the geometry, and each 4-arc represents a sequence

point-line-point-line-point or line-point-line-point-line,

consecutively incident. Since every such sequence is part of a triangle, all the 4-arcs are alike, and the graph is 4-regular. The group of the graph, being the group of collineations and correlations of $PG(2, 2)$, is of order 336 with a simple subgroup of index 2. The order may also be computed by setting $s=4$ and $n=7$ in the expression $2^s 3n$ at the end of §2.

Most of these ideas can be generalized at once to $PG(2, p^n)$ for any prime-power p^n. Singer [42, p. 378] observed that the

$$q=p^{2n}+p^n+1$$

points in this finite plane may be so ordered that the ith point is represented by the mark λ^i, where λ is a primitive root of $GF[p^{3n}]$ while λ^q is a primitive root of the subfield $GF[p^n]$ to which the coordinates of the points belong. The condition for three points λ^a, λ^b, λ^c to be collinear is

$$\alpha\lambda^a + \beta\lambda^b + \gamma\lambda^c = 0,$$

where α, β, γ belong to the $GF[p^n]$; therefore multiplication by λ represents a collineation.

For the same reason as before, the Levi graph for this configuration q_{p^n+1} is 4-regular, of grith 6; so we may call it a 6-cage of degree p^n+1. Singer's cyclic rule provides a Hamiltonian circuit, enabling us to draw the graph as a $\{2q\}$ with certain diagonals inserted.

Since there are $N_0=2q$ nodes, the number of 4-arcs is

$$(d-1)^3 dN_0 = 2p^{3n}(p^n+1)q,$$

whereas the order of the group of collineations and correlations of $PG(2, p^n)$ is known to be

$$2np^{3n}(p^{2n}-1)(p^{3n}-1) = 2n(p^n-1)^2p^{3n}(p^n+1)q$$

[48, p. 368]. Since a 4-arc represents the three sides and two vertices (or vice versa) of a triangle, it follows that the collineations leaving a triangle entirely invariant form a subgroup of order $n(p^n-1)^2$. The factor

$$(p^n-1)^2 = q - 3p^n$$

is simply the number of points not lying on any side of a given triangle, that is, the number of ways in which the triangle can be completed into a quadrangle. This leaves just n for the order of the subgroup leaving a quadrangle entirely invariant, namely, the group of automorphisms of the Galois field, which is the cyclic group of order n generated by the transformation

$$x \to x^p.$$

Generalizing 7_3 in another direction, we construct a graph of n red and n blue nodes, such that the red nodes are numbered $0, 1, \cdots, n-1$ while the blue nodes are joined to the triples

$$0\ 1\ r, \qquad 1\ 2\ r+1, \qquad \cdots \pmod{n}$$

with $2<r<n/2$. This inequality ensures that no two triples shall contain more than one common member. The case $n=8$ $(r=3)$ will be considered in §5.

When $n=9$ or 10, it is immaterial whether we take $r=3$ or 4; for the transformation $x \to 5x+4$ changes the triple 013 (mod 9) into 401, while $x \to 3x+1$ changes 013 (mod 10) into 140. These systems form configurations 9_3 and 10_3 in the real projective plane: in the notation of Kantor [26; 27] they are the 9_3 of type (C) and the 10_3 of type (A). The Levi graphs can be drawn in a manner analogous to Figs. 9 and 12. But these cases, where the graphs are not even 1-regular, seem less interesting than the configurations of Pappus and Desargues (Kantor's 9_3 of type (A) and 10_3 of type (B)) which we shall discuss in §§6 and 7.

On the other hand, an infinite family of 2-regular graphs of this cyclic nature can be obtained by considering the maps $\{6, 3\}_{b,c}$, where b and c are coprime. When the vertices of any map $\{6, 3\}_{b,c}$ are colored alternately red and blue, those of either color evidently form a map $\{3, 6\}_{b,c}$. (When the $\{6, 3\}_{b,c}$ arises as the Levi graph for a configuration, the $\{3, 6\}_{b,c}$ is the corresponding Menger graph.) The two colors occur alternately along any Petrie polygon of the $\{6, 3\}_{b,c}$, and the red vertices of the Petrie polygon form a "straight path" in the red $\{3, 6\}_{b,c}$. When b and c are coprime, the Petrie polygon of $\{6, 3\}_{b,c}$ is Hamiltonian; so also is the straight path in $\{3, 6\}_{b,c}$, as we remarked in §3. Assigning consecutive numbers to the vertices of this straight path, we obtain a triple system of the above kind with

$$n = b^2 + bc + c^2$$

[13, p. 27]. In terms of b' and c', given by the congruence

$$bc' - cb' \equiv 1 \pmod{n},$$

we find $r=bb'+bc'+cc'$ or $n+1-(bb'+bc'+cc')$, whichever is smaller. Thus, in the case of $\{6, 3\}_{b,1}$ we have simply

$$n = b^2 + b + 1, \quad b' = b - 1, \quad c' = 1, \quad r = n + 1 - (b^2 + 1) = b + 1;$$

so the triples are

$$0\ 1\ b+1, \qquad 1\ 2\ b+2, \qquad \cdots \pmod{b^2+b+1}.$$

5. **The Möbius-Kantor** 8_3. The triple system

$$0\ 1\ 3, \qquad 1\ 2\ 4, \qquad 2\ 3\ 5, \qquad \cdots \pmod{2p}$$

provides a combinatorial solution for Möbius's problem of finding a pair of simple p-gons, each inscribed in the other [36, p. 446]. In fact, the vertices of the p-gon 0 2 4 · · · would lie on the respective sides 13, 35, · · · of the p-gon 1 3 5 · · · , and the vertices of the latter would lie on the sides 24, 46, $\cdots$ of the former. For a realization of this combinatorial scheme in the projective plane, Kantor considered an arbitrary p-gon 0 2 4 $\cdots$ and a tentative position for the point 1 on the side 24. This determines further points on the remaining sides, connected by a chain of perspectivities

$$1 \overset{0}{\barwedge} 3 \overset{2}{\barwedge} 5 \cdots$$

which leads eventually to a point $1'$ on the original side 24. The problem is solved if this $1'$ coincides with 1; so the given p-gon 0 2 4 $\cdots$ provides two solutions whenever the projectivity $1 \barwedge 1'$ is hyperbolic [**26**, pp. 916–917; **27**, pp. 1291–1291]. Thus a solution is always possible in the complex plane, though not necessarily in the real plane.

In the case of quadrangles ($p=4$), writing 8 instead of 0, we have the triple system

1 2 4, 2 3 5, 3 4 6, 4 5 7, 5 6 8, 6 7 1, 7 8 2, 8 1 3

and the perspectivities

$$1 \overset{8}{\barwedge} 3 \overset{2}{\barwedge} 5 \overset{4}{\barwedge} 7 \overset{6}{\barwedge} 1'.$$

Möbius proved that in this simplest case the projectivity $1 \barwedge 1'$ is necessarily elliptic; hence the configuration 8_3 cannot be constructed in the real plane. However, in the complex plane, in terms of the cube roots of unity

$$\omega = (-1 + 3^{1/2}i)/2 \quad \text{and} \quad \omega^2 = (-1 - 3^{1/2}i)/2,$$

the eight points may be taken to be

1 $(1, 0, 0)$,	2 $(0, 0, 1)$,	3 $(\omega, -1, 1)$,	4 $(-1, 0, 1)$,
5 $(-1, \omega^2, 1)$,	6 $(1, \omega, 0)$,	7 $(0, 1, 0)$,	8 $(0, -1, 1)$,

so that the eight lines are their polars with respect to the conic

$$x^2 + y^2 = z^2,$$

namely

782 $[1, 0, 0]$,	671 $[0, 0, 1]$,	568 $[-\omega, 1, 1]$,	457 $[1, 0, 1]$,
346 $[1, -\omega^2, 1]$,	235 $[1, \omega, 0]$,	124 $[0, 1, 0]$,	813 $[0, 1, 1]$.

The configuration 8_3 cannot constitute the whole of a finite geometry as 7_3 does; for the eight points occur in pairs of *opposites* (such as 1 and 5) which do not belong to lines, and the eight lines occur in pairs of opposites (such as 124 and 568) which have no common point. Nevertheless, eight points and eight lines having the desired incidences can be found in some finite geometries such as $PG(2, 3)$ or, more conveniently, $EG(2, 3)$. This is the affine geometry in which the two coordinates of a point belong to the field of residue-classes modulo 3, namely 0, 1, 2 with the rules

$$1 + 2 = 0, \qquad 2 + 2 = 1, \qquad 2 \times 2 = 1,$$

and three points are collinear whenever their coordinates in both positions have sum zero. Such a coordinate symbol may be regarded as a mark in the Galois field $GF[3^2]$, and the simple rule is that three points are collinear whenever their marks have sum zero.[2] Let the modulus of the Galois field be the irreducible quadratic

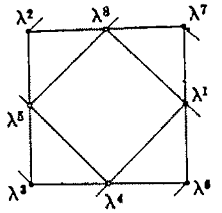

FIG. 10. The Möbius-Kantor 8_3.

$$\lambda^2 + \lambda + 2$$

or 112. Then the successive powers of the primitive root λ or 10 are

$$\lambda = 10, \lambda^2 = 21, \lambda^3 = 22, \lambda^4 = 02, \lambda^5 = 20, \lambda^6 = 12, \lambda^7 = 11, \lambda^8 = 01.$$

These are the proper coordinate symbols for the points previously named 1, 2, 3, 4, 5, 6, 7, 8; and the rule for collinearity is a consequence of the identity $\lambda^r(1+\lambda+\lambda^3) = 0$, which holds since

$$\lambda^3 + \lambda + 1 = (\lambda + 2)(\lambda^2 + \lambda + 2).$$

The only point of the geometry not present in the configuration is the origin 00. (See Fig. 10, where the points are represented in the Euclidean plane as if the coordinate residue 2 were the ordinary number -1. This representation naturally obscures the collinearity of such points as λ^4, λ^5, λ^7.)

The corresponding Levi graph, of 16 nodes and 24 branches, has a Hamiltonian circuit whose alternate vertices are 1 2 3 4 5 6 7 8, enabling us to draw it as in Fig. 12 and to see at a glance that it is 2-regular.

---

[2] The generalization to $EG(2, p^n)$ has been worked out by Bose [6, pp. 3–5].

Each node of either color is joined by 2-arcs to six of the remaining seven nodes of that color; for example, the red node 1 is joined to 2, 4, 6, 7, 8, 3, but not to 5. Thus the nodes (and consequently also the branches) occur in pairs of opposites, such as 1 and 5. The operation of interchanging opposites is indicated in the notation by adding or subtracting 4, or multiplying by $\lambda^4=2$, or making a half-turn about the origin of the finite affine geometry. In the complex projective plane, with the above choice of coordinates, it is the harmonic homology with center $(\omega, \omega^2, 1)$ and axis $[\omega, \omega^2, -1]$.

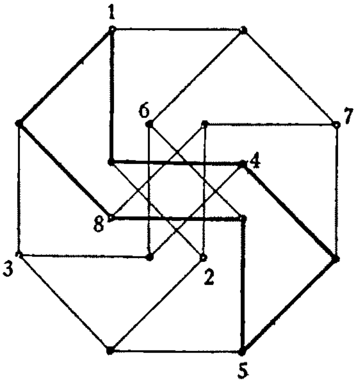

FIG. 11. The 2-regular graph $\{8\}+\{8/3\}$.

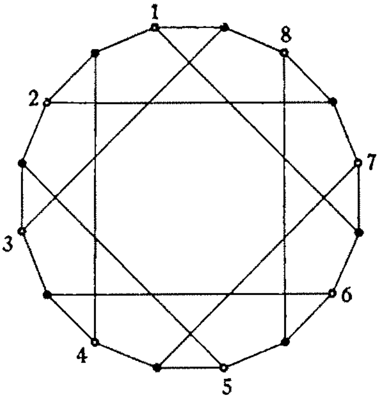

FIG. 12. Another view of $\{8\}+\{8/3\}$.

The same graph has already been mentioned in §2 qua $\{8\}+\{8/3\}$. In Fig. 11, the central $\{8/3\}$ has been twisted to exhibit the fact that the branches of such a graph can be found among the edges of the four-dimensional hypercube (Fig. 5). This $\{8/3\}$ and the peripheral $\{8\}$ are projections of two Petrie polygons of the hypercube [**12**, p. 223], and it is easy to pick out four further Petrie polygons, one of which has been drawn in heavy lines. Naming their alternate vertices, we thus find six 8-circuits:

$$1\ 3\ 5\ 7,\qquad 1\ 2\ 5\ 6,\qquad 1\ 4\ 5\ 8,$$

$$2\ 4\ 6\ 8,\qquad 3\ 4\ 7\ 8,\qquad 2\ 3\ 6\ 7.$$

They correspond to the three ways in which the 8_3 can be regarded as a pair of simple quadrangles, each inscribed in the other. They form a regular map of six octagons, each sharing two opposite sides with another. The characteristic is

$$-N_0+N_1-N_2=-16+24-6=2.$$

Referring to Threlfall's catalogue of such maps [**46**, p. 44] we find this listed as "Nr. 2." Since it is symmetrical by reflection, its group is of order $2\cdot 6\cdot 8=96$. But the group of the graph (which is also the group of symmetries and reciprocities of the configuration) has order $2^s3n=96$. So in this case the configuration and the map are isomorphic: no symmetry is lost by embedding the graph into a surface. In other words, the embedding is unique, unlike the embedding of the 6-cage into the torus (which can be done in eight different ways).

Returning to the complex projective plane, we see that the conic

$$x^2 + y^2 = z^2$$

determines a polarity which appears in the map as a half-turn about the mid-point of either of the two opposite edges 4 457, 8 813. (The conic touches line 457 at point 4, and line 813 at point 8.) Since there are altogether twelve pairs of opposite edges, we can immediately infer the existence of twelve such conics, each touching two opposite lines of the configuration at two opposite points, while the remaining six points and six lines form two self-polar triangles.

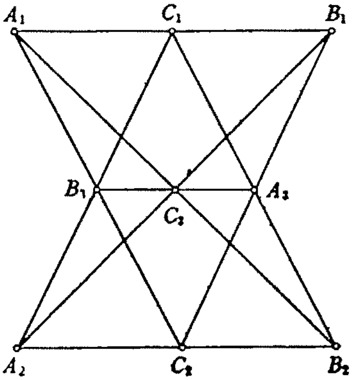

FIG. 13. Pappus's 9_3.

6. Pappus's 9_3. In the analytic foundations of general projective geometry of two dimensions we define a *point* as a triple of numbers (x, y, z), not all zero, with the understanding that $(x\lambda, y\lambda, z\lambda)$ is the same point for any $\lambda\neq 0$ [**24**, p. 176]. Similarly we define a *line* $[X, Y, Z]$ which is the same as $[\lambda X, \lambda Y, \lambda Z]$; and we say that the point (x, y, z) and line $[X, Y, Z]$ are *incident* if

$$Xx + Yy + Zz = 0.$$

These "numbers" may be elements of any division ring. It follows from the above definition that the general point collinear with (x, y, z) and (x', y', z') may be expressed as $(x\lambda+x', y\lambda+y', z\lambda+z')$. One of the most striking passages in the development is Hilbert's observation that the commutativity of multiplication is equivalent to Pappus's Theorem, which may be stated as follows:

If two triangles are doubly perspective they are triply perspective.

We say that two triangles $A_1A_2A_3$ and $B_1B_3B_2$ are ***perspective*** if the lines A_1B_1, A_2B_3, A_3B_2 all pass through one point, say C_1. We say that they are ***doubly*** perspective if the same kind of relation holds after the vertices of one triangle have been cyclically permuted, namely if either

$$A_1B_3,\ A_2B_2,\ A_3B_1 \text{ all pass through one point } C_2$$

or

$$A_1B_2,\ A_2B_1,\ A_3B_3 \text{ all pass through one point } C_3$$

[**49**, p. 100]. Finally, they are ***triply*** perspective if all three of these relations hold, as in Fig. 13.

Clearly, we can choose a coordinate system so that the points

$$A_1, \qquad A_2, \qquad A_3, \qquad C_1, \qquad B_1, \qquad B_3, \qquad B_2$$

are

$$(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),\ (1, 1, 1),\ (x, 1, 1),\ (1, y, 1),\ (1, 1, z).$$

Then the lines A_1B_3, A_2B_2, A_3B_1, A_1B_2, A_2B_1, A_3B_3 are

$$[0, -1, y],\ [z, 0, -1],\ [-1, x, 0],\ [0, z, -1],\ [-1, 0, x],\ [y, -1, 0].$$

Thus

A_3B_1 passes through the point $C_2 = A_1B_3 \cdot A_2B_2 = (1, yz, z)$ if $xyz = 1$,

and

A_3B_3 passes through the point $C_3 = A_1B_2 \cdot A_2B_1 = (xz, 1, z)$ if $yxz = 1$.

This shows that Pappus's Theorem holds if and only if

$$xy = yx.$$

The two triply perspective triangles and their centers of perspective form the *Pappus configuration* 9_3, whose incidences are summarized in the statement that the points A_i, B_j, C_k are collinear whenever

$$i + j + k \equiv 0 \pmod 3$$

[**31**, p. 106; **29**, p. 17].

In its more orthodox form, Pappus's Theorem asserts the collinearity of the cross joins

$$A_3 = B_1C_2 \cdot B_2C_1, \qquad B_3 = C_1A_2 \cdot C_2A_1, \qquad C_3 = A_1B_2 \cdot A_2B_1$$

of three points A_1, B_1, C_1 on one line with three points A_2, B_2, C_2 on another. In Möbius's configuration 8_3, which we discussed in §5, we find three points 2, 8, 7 on one line, and three points 3, 4, 6 on another, while two of the intersections of cross joins are 1 and 5. According to Pappus's Theorem, the three lines 15, 26, 37 must be concurrent. Since the two triads may be rearranged as 8, 7, 2 and 6, 3, 4, the three lines 15, 26, 48 are likewise concurrent. Hence:

In any geometry satisfying Pappus's Theorem, the four pairs of opposite points of 8_3 *are joined by four concurrent lines.*

The result is a special Pappus configuration in which the two "triply perspective triangles" have collapsed to form sets of *collinear* points: 235 and 671. The centers of perspective form a third degenerate triangle: 480, So altogether we have a configuration

$$(9_4,\ 12_3)$$

of nine points lying by threes on twelve lines, four through each point. Collinear points are indicated by the rows, columns, diagonals, and "broken diagonals" of the square matrix

$$\begin{array}{ccc} 2 & 8 & 7 \\ 5 & 0 & 1 \\ 3 & 4 & 6 \end{array}$$

[**35**, p. 335].

Returning to the general case, we find that the Pappus configuration 9_3 may be regarded in six ways as a cycle of three "Hessenberg" triangles, each inscribed in the next. Three of the ways are

$$\begin{array}{c|c|c} A_3B_1C_1 & A_1B_3C_1 & A_1B_1C_3 \\ A_1B_2C_2 & A_2B_1C_2 & A_2B_2C_1 \\ A_2B_3C_3 & A_3B_2C_3 & A_3B_3C_2 \end{array}$$

and the other three can be derived from these by the consistent interchange of two of the suffix numbers, say 1 and 2. The corresponding Levi graph, of 18 nodes and 27 branches, turns out to be 3-regular. Each of the eighteen Hessenberg triangles appears as a 6-circuit, and the nine of them displayed above form a map of nine hexagons on a torus, as in Fig. 14, which we recognize as $\{6, 3\}_{3,0}$. Thus the graph can be embedded into the torus in two distinct ways, in agreement with the fact that the group of the graph has order $2^3 3\cdot 9=216$, while that of the map has order $2\cdot 6\cdot 9=108$.

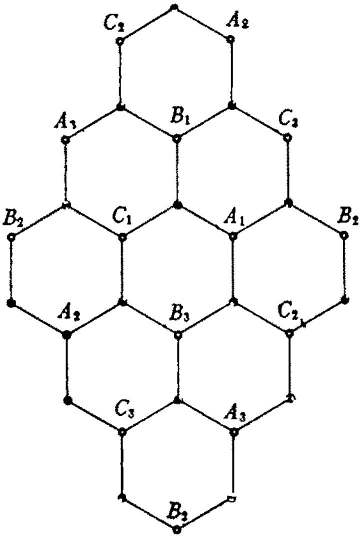

FIG. 14. The regular map $\{6, 3\}_{3,0}$.

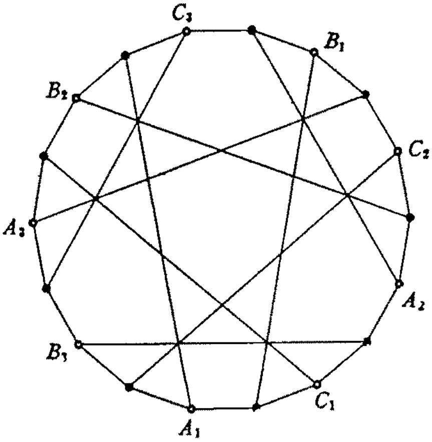

FIG. 15. Another view of the Pappus graph.

Kommerell [29, p. 19] observed that each Hessenberg triangle determines an involutory reciprocity for which that triangle is "self-polar" while the other two triangles in the same cycle are interchanged. The reciprocity is not, in general, a polarity; but it will be if these other two triangles are in perspective [14, p. 66]. In one of the two maps $\{6, 3\}_{3,0}$, the Hessenberg triangle appears as a hexagon and the reciprocity appears as a half-turn about the center of that hexagon; in the other map the same triangle appears as a Petrie polygon (zig-zag) and the reciprocity appears as a reflection.

After locating the Hamiltonian circuit

$$A_3\ B_3\ A_1\ C_1\ A_2\ C_2\ B_1\ C_3\ B_2,$$

we can draw the graph as in Fig. 15.

7. **Desargues' 10_3.** As long ago as 1846, Cayley remarked that the ten lines and ten planes determined by five points of general position in projective 3-space meet an arbitrary plane in a *Desargues configuration* (Fig. 16) consisting of ten points

12 23 34 45 15 13 24 35 14 25

and ten lines

345 145 125 123 234 245 135 124 235 134

with the rule that the point *ij* is incident with the line *ijk* [9, p. 318].

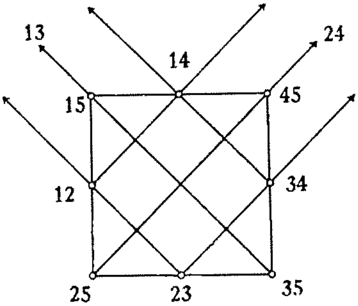

FIG. 16. Desargues' 10_3.

This configuration can be regarded in ten ways as a pair of triangles in perspective; for example, the triangles

14 24 34 and 15 25 35

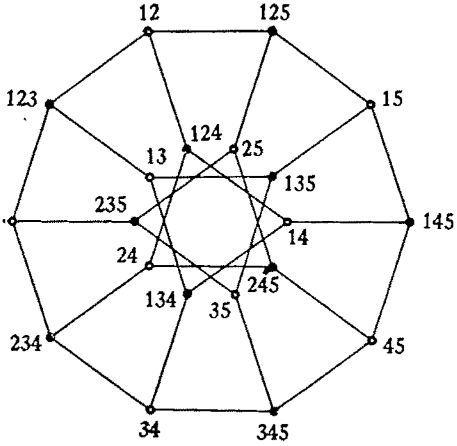

FIG. 17. The 3-regular graph $\{10\}+\{10/3\}$.

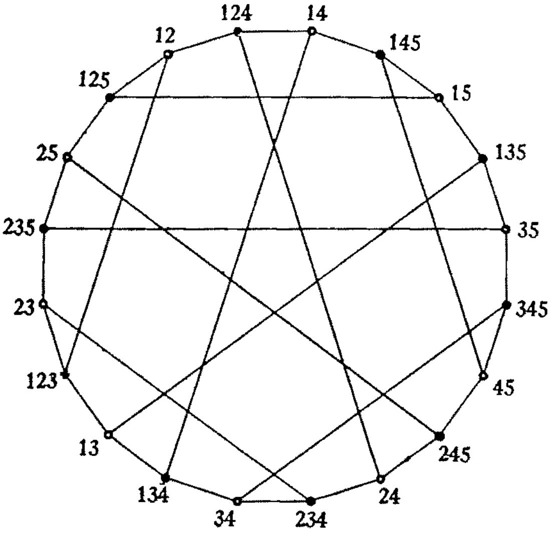

FIG. 18. Another view of the Desargues graph.

are in perspective from the point 45 and from the line 123, and are interchanged by a homology which appears in this notation as the transposition (4 5). Since the symmetric group of order 120 is generated by four transpositions, the collineation group of the configuration is generated by four such homolgies. G. K. C. von Staudt [44, p. 134] proved that the ten points and ten lines are interchanged by a certain polarity. Adjoining this to the collineation group of order 120, we obtain a group of collineations and correlations of order 240, which is abstractly the direct product of the symmetric group with the group of order 2.

We see in Fig. 17 that the Levi graph, of twenty nodes and thirty lines, is

$$\{10\} + \left\{\frac{10}{3}\right\},$$

in the notation of §2. Since this is 3-regular, its group has order

$$2^3\,30 = 240.$$

The 10_3 can be regarded in six ways as consisting of a pair of simple pentagons, consecutive vertices of either lying on alternate sides of the other; for example

12 23 34 45 51
14 42 25 53 31

[49, p. 51]. This instance may be associated with the cycles (12345), (14253), and we can use in a similar fashion five other pairs of cycles. Corresponding to these twelve pentagons, the graph contains twelve 10-circuits; for example, the cycle (12345) provides the 10-circuit

12 123 23 234 34 345 45 451 51 512.

From the cycle (12345) we derive five others by shifting each digit in turn two places on (or back). This systematic rule gives us one cycle from each of the six pairs, namely

(12345), (23145), (13425), (12453), (14235), (12534).

The six corresponding 10-circuits form a regular map of six decagons, each sharing two opposite sides with another; for example, the 10-circuits arising from (12345) and (23145) share the sides

123 23 and 145 45.

This map has characteristic $-20+30-6=4$; but the surface is unorientable. The remaining six 10-circuits form another such map, and the faces of either map are the Petrie polygons of the other.

In either map, the cyclic permutations of three digits, which generate the icosahedral (alternating) group, appear as trigonal rotations about vertices, while von Staudt's polarity appears as a half-turn about the center of any face (or rather, about the centers of all faces simultaneously).

It is interesting to compare this with another regular map of six decagons on the same surface, namely the map formed by the six Petrie polygons of the regular dodecahedron $\{5, 3\}$. Again each decagon shares two opposite sides with another, and again the group is the direct product of the icosahedral group with the group of order 2. We have here a remarkably perfect instance of maps that are not isomorphic but "allomorphic" [**43**, p. 101]. The difference is seen by considering the girth of the graph of vertices and edges. In the last case, where the vertices and edges are simply those of the dodecahedron itself, the girth is 5; but the girth of $\{10\}+\{10/3\}$ is 6.

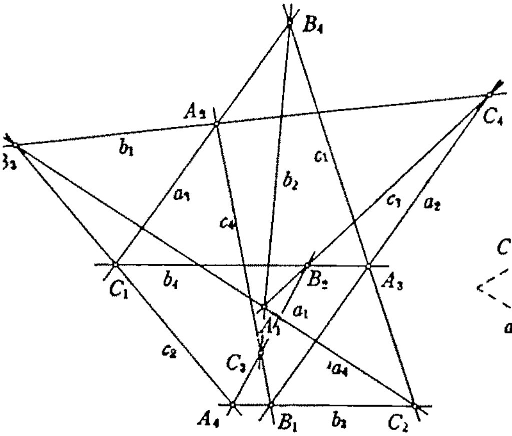

FIG. 19. A regular 12_3.

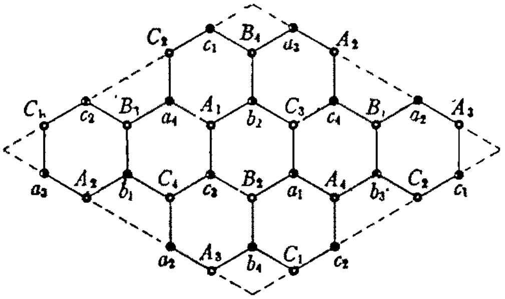

FIG. 20. The regular map $\{6, 3\}_{2,2}$.

8. **A regular 12_3.** After proving the impossibility of a pair of real quadrangles, each inscribed in the other, Möbius remarked: "Auf Vielecke von mehreren Seiten habe ich die Untersuchung nicht ausgedehnt" [**36**, p. 446]. As we observed in §5, the problem of finding a pair of mutually inscribed p-gons was solved for the complex projective plane by Kantor in 1881, using the invariant points of a certain projectivity $1 \barwedge 1'$ on the side 24 of a given p-gon $024 \cdots$. Möbius had shown, in 1828, that this projectivity has no real invariant points when $p=4$. Since all quadrangles are projectively equivalent, we cannot remedy this state of affairs by modifying the initial quadrangle. But two p-gons with $p>4$ are not, in general, projectively equivalent; so it is natural to expect a suitable choice of the

initial p-gon to render the above projectivity hyperbolic. When $p=5$ this expectation is realized by the "non-Desarguesian" 10_3: the (A) of Kantor [27, pp. 1292, 1314] which is the VII of Martinetti [**33a**, p. 17]. It can hardly be doubted that greater values of p will serve just as well. However, the corresponding Levi graphs are only 0-regular. The higher regularity of the graph for 8_3 is a consequence of the fact that this special configuration can be regarded *in three different ways* as a pair of mutually inscribed quadrangles. To achieve this refinement when $p>4$ we must abandon the cyclic scheme.

Fig. 19 shows a 12_3 which can be regarded in three ways as a pair of hexagons, each inscribed in the other:

$$A_1\ B_2\ C_1\ A_2\ B_1\ C_2 \quad \text{and} \quad A_3\ B_4\ C_3\ A_4\ B_3\ C_4,$$
$$A_1\ B_3\ C_1\ A_3\ B_1\ C_3 \quad \text{and} \quad A_2\ B_4\ C_2\ A_4\ B_2\ C_4,$$
$$A_1\ B_4\ C_1\ A_4\ B_1\ C_4 \quad \text{and} \quad A_2\ B_3\ C_2\ A_3\ B_2\ C_3.$$

The configuration is self-dual; for, if the twelve lines are denoted by

$$\begin{array}{lll} a_1 = A_4B_2C_3, & b_1 = A_2B_3C_4, & c_1 = A_3B_4C_2, \\ a_2 = A_3B_1C_4, & b_2 = A_1B_4C_3, & c_2 = A_4B_3C_1, \\ a_3 = A_2B_4C_1, & b_3 = A_4B_1C_2, & c_3 = A_1B_2C_4, \\ a_4 = A_1B_3C_2, & b_4 = A_3B_2C_1, & c_4 = A_2B_1C_3, \end{array}$$

then the twelve points are

$$\begin{array}{lll} A_1 = a_4\cdot b_2\cdot c_3, & B_1 = a_2\cdot b_3\cdot c_4, & C_1 = a_3\cdot b_4\cdot c_2, \\ A_2 = a_3\cdot b_1\cdot c_4, & B_2 = a_1\cdot b_4\cdot c_3, & C_2 = a_4\cdot b_3\cdot c_1, \\ A_3 = a_2\cdot b_4\cdot c_1, & B_3 = a_4\cdot b_1\cdot c_2, & C_3 = a_1\cdot b_2\cdot c_4, \\ A_4 = a_1\cdot b_3\cdot c_2, & B_4 = a_3\cdot b_2\cdot c_1, & C_4 = a_2\cdot b_1\cdot c_3. \end{array}$$

The triads of collinear points comprise twelve of the twenty-four possible arrangements $A_iB_jC_k$ with i, j, k all different. The remaining twelve triads form triangles; for instance $A_2B_4C_3$ is a triangle with sides b_2, c_4, a_3. Hence the Levi graph (consisting of twelve red nodes $A_1, \cdots$, twelve blue nodes $a_1, \cdots$, and thirty-six branches $A_1b_2, \cdots$) contains twelve 6-circuits such as

$$A_2a_3B_4b_2C_3c_4,$$

and can be embedded into the torus to make a map of twelve hexagons which we recognize as $\{6, 3\}_{2,2}$ (Fig. 20).

The symmetries and reciprocities of the configuration form a group of order 144. We can obtain this number by considering either the graph or the map. Since the graph is 2-regular, its group has order

$$2^2 36 = 144;$$

and since the map is symmetrical by reflection, its group has order four times the number of edges.

The Petrie polygons of $\{6, 3\}_{2,2}$ are dodecagons such as

$$A_1 b_2 C_3 c_4 B_1 a_2 A_3 b_4 C_1 c_2 B_3 a_4 \quad \text{and} \quad a_1 B_2 c_3 C_4 b_1 A_2 a_3 B_4 c_1 C_2 b_3 A_4,$$

which can be put together to form $\{12\} + (12/5\}$, as in Fig. 21.

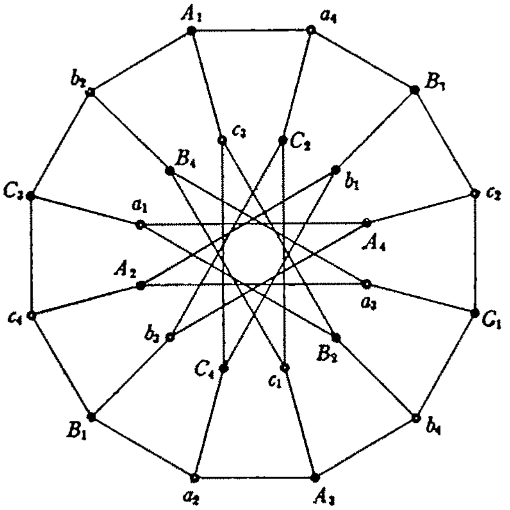

FIG. 21. The 2-regular graph $\{12\} + \{12/5\}$.

When Tutte was told about this graph, he discovered a Hamiltonian circuit, enabling us to draw it as a regular 24-gon with appropriate diagonals inserted (Fig. 22).

The configuration 12_3 may be regarded as a set of three triangles

$$A_1 A_2 A_3, \qquad B_2 B_3 B_1, \qquad C_3 C_1 C_2$$

in perspective by pairs from centers A_4, B_4, C_4, while the vertices lie on three lines

$$A_1 B_3 C_2, \qquad A_3 B_2 C_1, \qquad A_2 B_1 C_3.$$

A special case was described by Hesse, Salmon, and Zacharias [51, p. 149], namely the case when the vertices also lie on the three lines

$$A_1 B_1 C_1, \qquad A_2 B_2 C_2, \qquad A_3 B_3 C_3,$$

while the centers of perspective lie on a line $A_4B_4C_4$. We now have a configuration $(12_4, 16_3)$ of twelve points lying by threes on sixteen lines, four through each point. Calling the four new lines

$$d_1, \quad d_2, \quad d_3, \quad d_4,$$

we see that the twelve points are expressible in the form

$$a_i \cdot b_j \cdot c_k \cdot d_l,$$

where the suffixes $ijkl$ run over the twelve odd permutations of 1234.

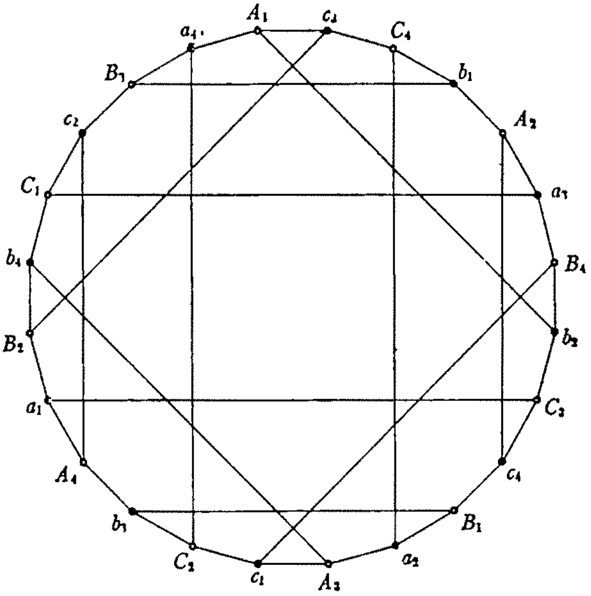

FIG. 22. Another view of $\{12\}+\{12/5\}$.

For comparison with Zacharias's notation, we observe that he numbered the lines from 1 to 16 in the order

$$d_1 \quad b_2 \quad c_3 \quad a_4 \quad c_4 \quad a_3 \quad d_2 \quad b_1 \quad a_2 \quad c_1 \quad b_4 \quad d_3 \quad b_3 \quad d_4 \quad a_1 \quad c_2.$$

Hesse constructed this $(12_4, 16_3)$ by taking the points of contact of the twelve tangents to an elliptic cubic curve that can be drawn from three collinear points of the curve. To make it real we must use a bipartite cubic with the three collinear points all on the odd branch. This is most easily seen by regarding the cubic as the locus of the point

$$(\wp u, \wp' u, 1)$$

where the elliptic function has the real period 2ω and the imaginary period $2\omega'$, so that points on the odd branch are given by real values of the parameter u while points on the oval are given by real values plus the imaginary half-period ω'. Three collinear points on the odd branch may be taken to have real parameters -2α, -2β, -2γ, where

$$\alpha + \beta + \gamma \equiv 0 \pmod{2\omega},$$

and then the parameters of the twelve points of contact are as follows:

A_1	α	A_2	$\alpha + \omega$	A_3	$\alpha + \omega'$	A_4	$\alpha + \omega + \omega'$
B_1	β	B_2	$\beta + \omega'$	B_3	$\beta + \omega + \omega'$	B_4	$\beta + \omega$
C_1	γ	C_2	$\gamma + \omega + \omega'$	C_3	$\gamma + \omega$	C_4	$\gamma + \omega'$

The sets of collinear points are now evident. (In one way of drawing the figure, we find the points $A_1B_1C_1A_2B_4C_3$ in that order on the odd branch, and $A_4B_3C_2A_3B_2C_4$ on the oval.) Feld [**17a**, p. 553] considered the case when α, β, γ are 0 and $\pm 2\omega/3$, so that A_1, B_1, C_1 are the three real inflexions while the rest are the nine real sextactic points.

Zacharias discovered that a Euclidean specialization of this $(12_4, 16_3)$ is provided by Morley's celebrated figure of a triangle $A_1A_2A_3$ with its angle trisectors and their intersections. Morley's theorem states that the trisectors

$$A_1B_1C_1, \quad A_2B_1C_3, \quad A_2B_2C_2, \quad A_3B_2C_1, \quad A_3B_3C_3, \quad A_1B_3C_2$$

form an equilateral triangle $B_2B_3B_1$ by their first intersections and another equilateral triangle $C_3C_1C_2$ by their second intersections. Zacharias observed that these two equilateral triangles are in perspective with $A_1A_2A_3$ and with each other, from collinear centers C_4, B_4, A_4.

9. **The Cremona-Richmond** 15_3. Six points of general position in real projective 4-space—say 1, 2, 3, 4, 5, 6—determine fifteen further points such as P_{12}: the intersection of the line 12 with the hyperplane 3456. These lie by threes on fifteen lines such as $P_{12}P_{34}P_{56}$: the common line of the three hyperplanes

$$3456, \qquad 1256, \qquad 1234$$

[**39**, p. 131]. This configuration 15_3 in four dimensions was chosen by Baker as a frontispiece for both volumes II and IV of his *Principles of geometry*. A 15_3 in two or three dimensions can be derived by projection. Dualizing the three-dimensional version, we obtain a set of

15 lines and 15 planes which can be identified with the configuration formed by the tritangent planes of a nodal cubic surface [**15**].

The 15_3 in two dimensions is self-dual. For, instead of associating the points with the fifteen pairs of six symbols 1, 2, 3, 4, 5, 6 and the lines with the fifteen *synthemes* such as (12, 34, 56) we can just as well associate the lines with the fifteen pairs of six other symbols

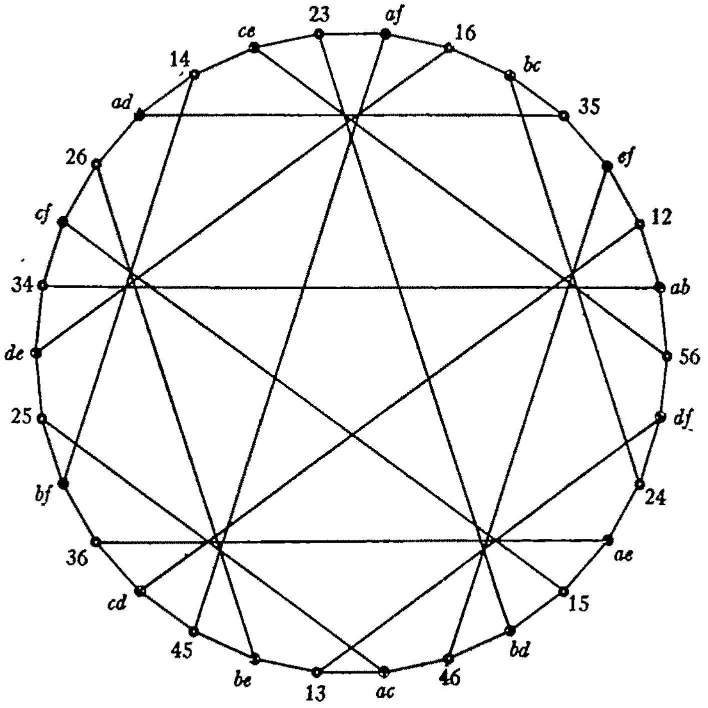

FIG. 23. Tutte's 8-cage (the most regular of all graphs).

a, b, c, d, e, f and the points with fifteen synthemes such as (ab, cd, ef). Richmond's way of doing this is indicated on the Levi graph in Fig. 23. Thus the group of the configuration is the group of automorphisms of the symmetric group of degree six [**8**, p. 210] and its order is

$$2\cdot 6! = 1440.$$

Martinetti cited this 15_3 as the simplest instance of a configuration containing *no triangles* [**33**, p. 174; cf. **41**, p. 44]. The corresponding property of the Levi graph is that its girth is not 6 but 8. Since the order of the group is

$$1440 = 2^5\, 3\cdot 15,$$

the regularity is $s=5$. Hence this graph is Tutte's 8-cage, the most regular of all graphs [**47**, p. 460]. Tutte's own description of it can be reconciled with ours as follows. Regard the eight red nodes

$$\begin{matrix} 15 & 36 & 25 & 46 \\ 26 & 45 & 16 & 35 \end{matrix}$$

as forming a cube (or two interpenetrating tetrahedra: 15 25 35 45 and 16 26 36 46). Take a blue node on each of the twelve edges; for instance, a blue node *cf* on the edge 15 26, and a blue node *de* on the opposite edge 25 16. Join the blue nodes on each pair of opposite edges, and take a red node on each join; for instance, a red node 34 on the join of *cf* and *de*, and a red node 12 on the join of *cd* and *ef*. Join the two red nodes thus associated with each set of four parallel edges, and take a blue node on each join; for instance, a blue node *ab* on the join of 12 and 34. Finally, join the three blue nodes thus associated with the three dimensions, namely *ab*, *ce*, *df*, to a fifteenth red node 56.

There is apparently no regular embedding for this graph. We must be content with a map of one decagon and ten octagons. Naming alternate vertices, we find a decagon

$$13\ 24\ 35\ 14\ 25$$

surrounded by octagons 16 24 36 45, 12 35 16 34, and others derived from these two by cyclic permutation of the digits 1, 2, 3, 4, 5 (leaving 6 unchanged). The surface is easily seen to be unorientable, of characteristic

$$-30+45-(1+10)=4.$$

The decagon represents, in the configuration, a pentagon that is self-polar for the reciprocity $(1\ d)(2\ b)(3\ a)(4\ f)(5\ c)(6\ e)$. Thus Richmond's notation could be improved by writing

$$a,\ b,\ c,\ d,\ e,\ f \quad \text{in place of his} \quad d,\ b,\ a,\ f,\ c,\ e.$$

On the other hand, the configuration contains another pentagon that is self-polar for the "natural" reciprocity

$$(1\ a)(2\ b)(3\ c)(4\ d)(5\ e)(6\ f)$$

without any change of notation.

The 8-cage is not only the Levi graph for the 15_3, but also represents, in a different way, the 60_{15} known as *Klein's configuration* [**45**, p. 447; **25**, p. 42]. Klein's sixty points and sixty planes are the vertices and faces of fifteen tetrahedra whose pairs of opposite edges are the directrices of the fifteen linear congruences determined by pairs of six mutually apolar linear complexes (or six mutually commutative null polarities) in complex projective 3-space. The fifteen red nodes of the 8-cage represent the pairs of directrices (or the fifteen congruences) while the fifteen blue nodes represent the tetrahedra.

10. **Möbius's** 8_4. Before resuming the discussion of Möbius's mutually inscribed tetrahedra (§1), let us mention a few other self-dual configurations of points and planes in real projective 3-space. The simplest kind is the n_n consisting of n points on a line and n planes through the same line. Since each point is incident with every plane, the Levi graph consists of n red nodes all joined to each of n blue nodes. Since this is 3-regular, of girth 4, we call it a 4-cage of degree n. When $n=2$ it is the square $\{4\}$. When $n=3$ it is the Thomsen graph $\{6, 3\}_{1,1}$. When $n=4$ it is $\gamma_4/2$ (Fig. 4) or $\{4, 4\}_{2,2}$ (Fig. 6).

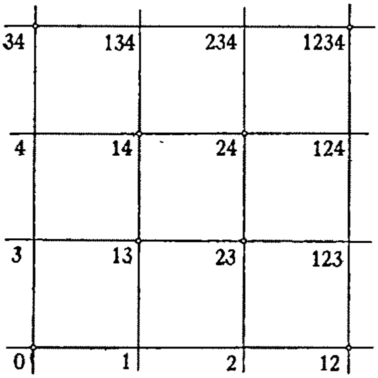

FIG. 24. Gallucci's version of Möbius's 8_4.

Let 1, 2, 3 denote three planes forming a trihedron with vertex 0, and let 23, 31, 12 be arbitrary points on the three edges, forming a fourth plane 123. We now have a tetrahedron 4_3, whose Levi graph is the cube γ_3 (Fig. 1). Continuing in the same manner, let 4 denote a fourth plane through 0, meeting planes 1, 2, 3 in three lines on which we take further arbitrary points 14, 24, 34. In this way we obtain three new planes 234, 341, 412, which intersect in an eighth point 1234. By a theorem of Möbius [36, p. 443], this point 1234 lies on the plane 123, so that we have altogether a configuration 8_4, whose Levi graph is the four-dimensional hypercube γ_4 (Fig. 5).

Möbius deduced his theorem from the observation that the quadrangular relation $Q(A'B'C', F'G'H')$ for six collinear points implies $Q(F'G'H', A'B'C')$. This is a corollary of Pappus's Theorem, which

Baker showed to be equivalent to the theorem that if three skew lines all intersect three other skew lines, any transversal to the first set of three intersects any transversal to the second set [1, pp. 47–51]. The following argument relates Möbius's theorem to the transversal theorem directly [cf. **20**, p. 76; **4**, p. 144].

The line joining points 13 and 14 is the line of intersection of planes 1 and 134; likewise the join of 23 and 24 is the intersection of 2 and 234 (see Fig. 24). Hence the transversal from 0 to these two lines is the intersection of planes 1 and 2, and the transversal from 34 is the intersection of 134 and 234. Similarly, the transversal from 0 to the two lines 13 23 and 14 24 is 3·4, and the transversal from 12 is 123·124. The "transversal theorem" asserts that the two lines 134·234 and 123·124 intersect, while Möbius's theorem asserts that the four planes 234, 134, 124, 123 are concurrent; these two statements are clearly equivalent.

The following analytic proof of Möbius's theorem indicates its dependence on the commutative law. Choose the tetrahedron of reference so that the four points and four planes

$$\begin{array}{cccc} 23, & 31, & 12, & 0, \\ 1, & 2, & 3, & 123, \end{array}$$

are

$$\begin{array}{cccc} (1, 0, 0, 0), & (0, 1, 0, 0), & (0, 0, 1, 0), & (0, 0, 0, 1), \\ [1, 0, 0, 0], & [0, 1, 0, 0], & [0, 0, 1, 0], & [0, 0, 0, 1]. \end{array}$$

Take the unit plane [1, 1, 1, 1] to pass through the line 123·4, so that the plane 4 is [1, 1, 1, 0]. Then the points 14, 24, 34 are of the form

$$(0, 1, -1, a), \qquad (-1, 0, 1, b), \qquad (1, -1, 0, c),$$

and the planes 234, 314, 124 are

$$[0, c, -b, 1], \qquad [-c, 0, a, 1], \qquad [b, -a, 0, 1].$$

These and 123 all pass through the point $(a, b, c, 0)$, which is the desired point of concurrence 1234. (The incidence of 124 and 1234 requires $ba = ab$.)

Benneton [**3**, p. 30] makes a different choice of unit plane so as to exhibit the four planes and four points

$$\begin{array}{cccc} 234, & 314, & 124, & 4, \\ 14, & 24, & 34, & 1234 \end{array}$$

in the form

$$[0,\ c,\ -b,\ a],\quad [-c,\ 0,\ a,\ b],\quad [b,\ -a,\ 0,\ c],\quad [a,\ b,\ c,\ 0],$$
$$(0,\ c,\ -b,\ a),\quad (-c,\ 0,\ a,\ b),\quad (b,\ -a,\ 0,\ c),\quad (a,\ b,\ c,\ 0).$$

Representing both the point (x, y, z, t) and the plane $[x, y, z, t]$ by the quaternion $t+xi+yj+zk$, he neatly expresses all the points and planes of the configuration in the form

$$1,\quad i,\quad j,\quad k,\quad V,\quad iV,\quad jV,\quad kV,$$

where V denotes the pure quaternion $ai+bj+ck$.

In §1 we described the reflections R_1, R_2, R_3, R_4 of the hypercube, which add a digit to every symbol lacking it, and remove the same digit from every symbol containing it. We now see that the corresponding reciprocities of Möbius's 8_4 may be described as follows: R_1, R_2, R_3 have the effect of multiplying every quaternion on the left by i, j, k, respectively, while R_4 has the effect of multiplying on the right by V. It follows that these four mutually commutative reciprocities can be extended to the whole space as *null polarities* [**25**, p. 38].

When the 8_4 is represented by its Levi graph, γ_4, the two mutually inscribed tetrahedra appear as two opposite cells γ_3, transformed into each other by the reflection R_4. Since γ_4 has four pairs of opposite cells, we see clearly that the 8_4 can be regarded *in four different ways* as a pair of mutually inscribed tetrahedra, transformed into each other by a null polarity.

11. **Cox's** $(2^{d-1})_d$. After constructing Möbius's 8_4 as above, let 5 be a fifth plane through 0, meeting planes 1, 2, 3, 4 in four lines on which we take further arbitrary points 15, 25, 35, 45. Leaving out the five planes, one by one, we obtain five points

$$2345,\qquad 3451,\qquad 4512,\qquad 5123,\qquad 1234.$$

By a theorem of Homersham Cox [**10**, p. 67], these five points lie in one plane 12345, so that we have altogether a configuration 16_5, whose Levi graph is the five-dimensional hypercube γ_5 [**12**, p. 244]. By symmetry, we merely have to prove that the first four of the five points are coplanar; and this follows from the dual of Möbius's theorem as applied to the four points 15, 25, 35, 45 in the plane 5. (We simply add the digit 5 to all the symbols occurring in Möbius's theorem; that is, we apply the reciprocity R_5. Cf. Richmond [**39a**].)

Continuing in this manner we obtain Cox's general theorem, to the effect that d concurrent planes $1, 2, \cdots, d$, with arbitrary points on their lines of intersection, determine a configuration of 2^{d-1} points and 2^{d-1} planes, with d points on each plane and d planes through each point, so that the Levi graph is the d-dimensional hypercube γ_d. Thus the group of symmetries and reciprocities is of order

$$2^d\ d!.$$

Instead of *arbitrary* points (12, 13, and so on) on the lines of intersection of pairs of the d given planes, we may take the points determined on these lines by an arbitrary sphere through the point of concurrence 0. The 2^{d-1} planes intersect the sphere in 2^{d-1} circles [**37**, p. 271]. By stereographic projection we obtain Clifford's configuration of 2^{d-1} points and 2^{d-1} circles, with d points on each circle and d circles through each point. These again are represented by the 2^d vertices of γ_d.

12. **Kummer's 16_6.** Let the five initial planes of Cox's 16_6 be represented by the quaternions

$$i,\quad j,\quad k,\quad V = ai + bj + ck,\qquad V' = a'i + b'j + c'k.$$

If the reciprocity R_1 were a null polarity, it would transform the plane V' into iV' or $(0,\ c',\ -b',\ a')$, which is a definite point on the line of intersection of planes 1 and 5, whereas Cox would take his point 15 to be arbitrary on that line, say $(0,\ c',\ -b',\ \alpha)$. Hence, for the general 16_6 the reciprocities $R_1, \cdots, R_5$ are *not* null polarities; and the same remark holds for any greater value of d.

On the other hand, the special case when $R_1, \cdots, R_5$ *are* null polarities is interesting in a different way. R_1, R_2, R_3 are the operations of multiplying on the left by i, j, k, while R_4 and R_5 are the operations of multiplying on the right by two anticommutative pure quaternions (or orthogonal vectors) V and V'. Writing

$$V = qiq^{-1},\qquad V' = qjq^{-1},$$

we can use a coordinate transformation to change the symbols of the planes

$$1,\qquad 2,\qquad 3,\qquad 4,\qquad 5$$

from

$$i,\qquad j,\qquad k,\qquad qiq^{-1},\qquad qjq^{-1}$$

to

$$iq,\qquad jq,\qquad kq,\qquad qi,\qquad qj.$$

Then the initial point 0 is not 1 but q, and the reciprocities R_1, R_2, R_3

have the same effect as before, while R_4 and R_5 are the operations of multiplying on the right by i and j. Applying the product $R_1R_2R_3R_4R_5$, we find that the plane 12345 is $kjiqij=qk$, which passes through the initial point q. (This is in marked contrast with the general 16_5, where the point 0 and plane 12345 are not incident.)

Applying the null polarities $R_1, \cdots, R_5$ in turn, we find that the incidence of 0 and 12345 implies the incidence of 1 and 2345, of 12 and 345, and so on. Thus the special configuration is really not a 16_5 but a 16_6; and it is natural to use the symbol 6 for 12345, 16 for 2345, 126 as an alternative for 345, and so on, in agreement with the classical notation for the sixteen *nodes* and sixteen *tropes* of Kummer's quartic surface [**25**, p. 16]. We now have six mutually commutative null polarities $R_1, \cdots, R_6$: the first three are left-multiplications by i, j, k, and the last three are right-multiplications by i, j, k [see **1a**, p. 138; **3**, p. 38]. The essential difference between this 16_6 and Cox's 32_6 is that now $R_1R_2R_3R_4R_5R_6$ is the identity.

We have seen that the Levi graph for Cox's 16_5 is the five-dimensional hypercube γ_5. The effect of the new incidences is to insert new branches joining the pairs of opposite vertices (as in the derivation of Fig. 4 from Fig. 3). More symmetrically, we can take the six-dimensional hypercube γ_6 (which represents 32_6) and identify pairs of opposite elements. Thus

The Levi graph for Kummer's 16_6 *is* $\gamma_6/2$.

In other words, after coloring the sixty-four vertices of γ_6 alternately red and blue, we can say that the sixteen pairs of opposite red vertices represent the sixteen nodes of Kummer's surface, while the sixteen pairs of opposite blue vertices represent the sixteen tropes. The thirty-two red vertices by themselves belong to the semi-regular polytope $h\gamma_6$ [**12**, p. 158] ,whose connection with Kummer's 16_6 has already been pointed out by Du Val [**16**, p. 65].

The most convenient coordinates for the vertices of γ_6 (of edge 2) are

$$(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1),$$

with an even number of minus signs for a red vertex and an odd number of minus signs for a blue vertex. The 240 squares such as

$$(\pm 1, \pm 1, 1, 1, 1, 1)$$

represent the 120 *Kummer lines* [**25**, p. 77]; the 160 cubes such as

$$(\pm 1, \pm 1, \pm 1, 1, 1, 1)$$

represent the 80 *Rosenhain tetrads* [**25**, p. 78]; the sixty γ_4's such as

$$(\pm 1, \pm 1, \pm 1, \pm 1, 1, 1)$$

represent thirty[3] Möbius configurations 8_4; and the twelve cells γ_5 represent the six null polarities.

Each of the 160 cubes is obtained by fixing the signs of three of the six coordinates and leaving the remaining three ambiguous. Since the fixing of signs can be done in 8 ways (for the same choice of three coordinates), we thus obtain 8 cubes which together exhaust the 64 vertices of γ_6, and which occur in four pairs of opposites. Any two of these four pairs of opposite γ_3's belong to a pair of opposite γ_4's. Finally, the three coordinates whose signs were fixed can be chosen in

$$\binom{6}{3} = 20$$

ways. We thus verify the well known fact that there are twenty ways of regarding the 16_6 as a set of four Rosenhain tetrahedra any two of which form a Möbius configuration 8_4 [see **34** or **2**].

Besides these peripheral elements of γ_6, there are also 120 large inscribed cubes (of edge $2^{3/2}$ instead of 2) such as

$$\begin{gathered}(1, 1, 1, 1, 1, 1)(1, 1, 1, 1, -1, -1)\\ (-1, -1, 1, 1, 1, 1)(-1, -1, 1, 1, -1, -1)\\ (1, 1, -1, -1, 1, 1)(1, 1, -1, -1, -1, -1)\\ (-1, -1, -1, -1, 1, 1)(-1, -1, -1, -1, -1, -1),\end{gathered}$$

which represent the 60 Göpel tetrads of points and the 60 Göpel tetrads of planes [**25**, p. 79].

13. **The skew icosahedron.** In the above list of significant subsets of Kummer's 16_6, we have not mentioned the *Weber hexads* [**25**, p. 80] such as 0, 12, 23, 34, 45, 51. The corresponding vertices of γ_6 are

$$\begin{gathered}0,\\ 12, \qquad 23, \qquad 34, \qquad 45, \qquad 51,\\ 1236, \qquad 2346, \qquad 3456, \qquad 4516, \qquad 5126,\\ 123456;\end{gathered}$$

or, in terms of coordinates,

---

[3] Blaschke [4, p. 151] found only fifteen Möbius configurations; but that was surely a mistake. After the removal of any one 8_4, the remaining points and planes of the 16_6 form another 8_4.

$$(1, 1, 1, 1, 1, 1),$$
$$(-1, -1, 1, 1, 1, 1), (1, -1, -1, 1, 1, 1), \cdots, (-1, 1, 1, 1, -1, 1),$$
$$(-1, -1, -1, 1, 1, -1), (1, -1, -1, -1, 1, -1), \cdots,$$
$$(-1, -1, 1, 1, -1, -1),$$
$$(-1, -1, -1, -1, -1, -1).$$

These twelve points in Euclidean 6-space form twenty equilateral triangles of side $2^{3/2}$, such as

0 12 23, 12 23 1236, 23 1236 2346, and 1236 2346 123456,

which are the faces of a *regular skew icosahedron* (Fig. 25). This remarkable surface is just as symmetrical as the Platonic icosahedron [12, p. 52], but its angles are different. In fact, the "vertex figure"

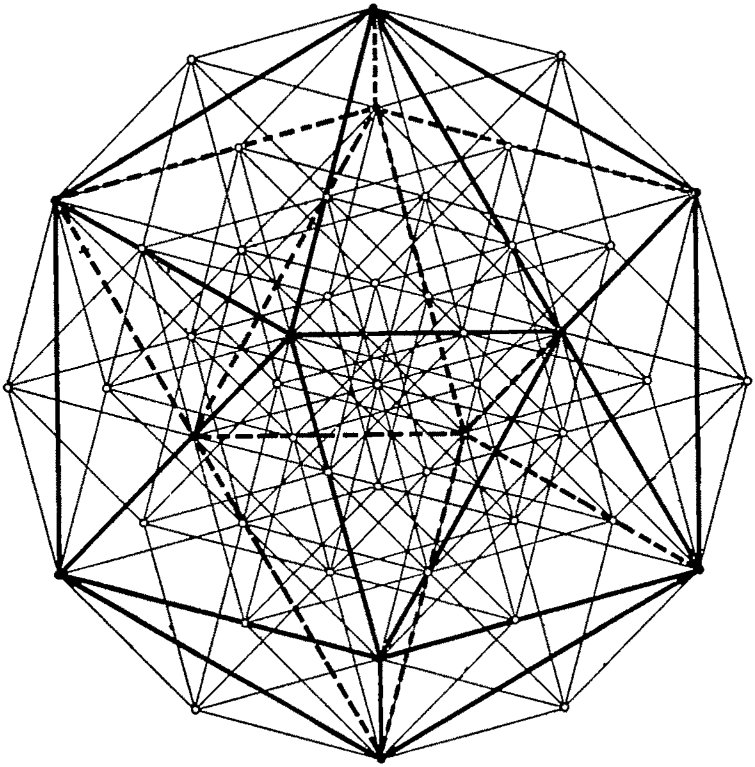

FIG. 25. γ_6 and the skew icosahedron.

12 23 34 45 51,

instead of being a regular plane pentagon of angle 108°, is a regular skew pentagon of angle 90° (since 12 23 34 41 is a square). Similarly, the dihedral angle is reduced from about 138° 12′ to 109° 28′, since the two adjacent faces

$$0\ 12\ 23 \quad \text{and} \quad 0\ 23\ 34$$

belong to the octahedron 0 12 23 34 41 1234.

From the fifteen points 12, 13, $\cdots$, 56 we can pick out a rectangular skew pentagon such as 12 23 34 45 51 in seventy-two ways; hence there are seventy-two skew icosahedra for each vertex of γ_6, and

$$72\cdot 64/12 = 384$$

of them altogether, representing the 192 Weber hexads of nodes and the 192 Weber hexads of tropes.

14. Self-reciprocal polytopes in the Euclidean 4-space.

Although the Levi graph has been defined only for configurations of points and lines, or of points and planes, the idea can evidently be extended, for example, to configurations of points and hyperplanes in more than three dimensions. We shall be content to describe two instances: the configurations 5_4 and 24_6 formed by the vertices and cells of the four-dimensional polytopes $\{3, 3, 3\}$ and $\{3, 4, 3\}$ [**12**, pp. 120, 148].

The former is the four-dimensional simplex α_4. In the Levi graph its five vertices are represented by five red nodes A, B, C, D, E, and the respectively opposite cells by five blue nodes a, b, c, d, e. Red and blue nodes are joined whenever their letters disagree. Thus the group is of order 240: the direct product of the symmetric group of degree 5 and the group of order 2.

The five letters can be arranged as a cycle in twelve ways, corresponding to the twelve Petrie polygons of the simplex [**12**, p. 225]. Every four consecutive vertices of the Petrie polygon belong to a cell; and by placing the symbol for the cell between the middle two of the four vertices, we obtain a symmetrical Hamiltonian circuit of the graph. In this manner the Petrie polygon $ABCDE$ yields the circuit

$$A\ d\ B\ e\ C\ a\ D\ b\ E\ c,$$

which we may rewrite as

$$0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9,$$

so that the even and odd digits represent the red and blue nodes, or the vertices and cells of the simplex, and the symbols for a vertex and its opposite cell differ by 5. Thus the graph can be drawn as a regular decagon 0123456789 with its inscribed decagram 0369258147 [see **18**, p. 315, Fig. 10, IV-11-A]. This 2-regular graph of degree 4 and girth 4 can be embedded into the torus to form the map $\{4, 4\}_{3,1}$ (see §3), which has ten square faces

$$0143,\ 1254,\ 2365,\ \cdots,\ 9032.$$

The embedding can be done in 6 ways, since the group of the map is of order 40. This order arises as

$$10\cdot 2\cdot 2,$$

since the ten symbols can be cyclically permuted or reversed, or multiplied by 3 (mod 10) so as to interchange the decagon and decagram.

A more symmetrical way of drawing the graph is in Euclidean 4-space, where we may take two equal regular simplexes in reciprocal positions and join each vertex of either to the four nearest vertices of the other. (This is analogous to the two tetrahedra of Kepler's *stella octangula* which can be joined by the edges of a cube.) Since the regular four-dimensional simplex has the Schläfli symbol $\{3, 3, 3\}$, an appropriate notation for this aspect of the graph is

$$2\{3, 3, 3\}.$$

Similarly, $2\{3, 4, 3\}$ denotes the 1-regular graph of degree 6 and girth 4 whose 48 nodes are the vertices of two reciprocal 24-cells $\{3, 4, 3\}$, each vertex of either being joined to the six vertices of an octahedral cell of the other. This is the Levi graph for the polytope $\{3, 4, 3\}$ qua configuration 24_6. Robinson [**40**, p. 44] called it $\delta_4+\delta_4'$, and remarked that its 144 branches are the sides of 18 concentric octagons in different planes. By identifying opposite elements we derive

$$2\{3, 4, 3\}/2:$$

the Levi graph for Stephanos's configuration 12_6 consisting of three "desmic" tetrahedra in real projective 3-space [**45**; **25**]. In fact, the Cartesian coordinates for the 24 vertices of $\{3, 4, 3\}$, being the constituents of Hurwitz's 24 quaternion units

$$\pm i,\ \pm j,\ \pm k \quad \text{and} \quad (\pm 1 \pm i \pm j \pm k)/2,$$

are the same as the homogeneous coordinates for the 12 points of the 12_6 [see **23**, p. 170; **3**, p. 35].

By Pappus's Theorem as stated in §6, if a triangle $A_1A_2A_3$ is in perspective with $B_1B_2B_3$ and with $B_3B_2B_1$, it is also in perspective with $B_2B_1B_3$. Stephanos's desmic tetrahedra come from the analogous theorem that if a tetrahedron $A_1A_2A_3A_4$ is in perspective with $B_2B_1B_4B_3$ and $B_3B_4B_1B_2$ and $B_4B_3B_2B_1$, it is also in perspective with $B_1B_2B_3B_4$. However, the resemblance is superficial, as the permutations of the B's are no longer cyclic. In one respect this three-dimensional theorem is simpler: instead of depending on Pappus's Theorem, it can be deduced from the "propositions of incidence" alone. In fact, the given perspectivities imply that the line A_iB_j intersects A_jB_i, whence A_iB_i intersects A_jB_j. Thus the four lines A_iB_i all intersect one another, and since they are not all coplanar they must be concurrent.

By a simple change of notation, from $B_1B_2B_3$ to $B_2B_3B_1$, we obtain a rule for sets of collinear points that is identical with the rule for Hesse's $(12_4, 16_3)$ which we considered in §8. Hence, as Zacharias observed [51, p. 152], this plane configuration can be derived from Stephanos's 12_6 by projection from an arbitrary point in the 3-space.

SUMMARY

Configuration	n_d	Space	Group order	Graph or map	Girth m	Regularity s	Number of s-arcs $2(d-1)^{s-1}dn$
Fano's	7_3	$PG(2, 2)$	336	$\{6, 3\}_{2,1}$	6	4	336
Möbius-Kantor	8_3	Complex plane	96	$\{8\}+\{8/3\}$	6	2	96
Pappus's	9_3	Real plane	216	$\{6, 3\}_{3,0}$	6	3	216
Desargues'	10_3	Real plane	240	$\{10\}+\{10/3\}$	6	3	240
The new	12_3	Real plane	144	$\{12\}+\{12/5\}$ or $\{6, 3\}_{2,2}$	6	2	144
Cremona-Richmond	15_3	Real plane	1440	Tutte's 8-cage	8	5	1440
	3_3	Real 3-space	72	$\{6, 3\}_{1,1}$	4	3	72
	4_4	Real 3-space	1152	$\{4, 4\}_{2,2}$	4	3	288
Tetrahedron	4_3	Real 3-space	48	$\{6, 3\}_{2,0}$	4	2	48
Möbius's	8_4	Real 3-space	384	$\{4, 4\}_{4,0}$	4	2	192
Cox's	$(2^{d-1})_d$	Real 3-space	$2^d d!$	γ_d	4	2	$2^d d(d-1)$
Kummer's	16_6	Real 3-space	23040	$\gamma_6/2$	4	2	960
Stephanos's	12_6	Real 3-space	1152	$2\{3, 4, 3\}/2$	4	1	144
Simplex	5_4	Real 4-space	240	$\{4, 4\}_{3,1}$	4	2	120

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THE JUNE MEETING IN SEATTLE

The four hundred sixtieth meeting of the American Mathematical Society was held at the University of Washington, Seattle, Washington, on Friday and Saturday, June 16–17, 1950, in conjunction with the Pacific Northwest Section of the Mathematical Association of America. Some 80 persons were in attendance, including the following 55 members of the Society:

T. M. Apostol, B. H. Arnold, S. P. Avann, R. W. Ball, Constance Ballantine, J. P. Ballantine, R. A. Beaumont, Z. W. Birnbaum, J. L. Brenner, D. G. Chapman, Harold Chatland, Paul Civin, C. M. Cramlet, D. B. Dekker, Worthie Doyle, F. E. Ehlers, Howard Eves, K. S. Ghent, F. L. Griffin, S. G. Hacker, Mary Haller, Edwin Hewitt, J. L. Hildebrand, J. W. Hurst, R. D. James, S. A. Jennings, L. G. Jones, J. M. Kingston, M. S. Knebelman, M. Z. Krzywoblocki, H. B. Mann, Rhoda Manning, L. H. McFarlan, A. S. Merrill, M. R. Moore, A. F. Moursund, D. C. Murdoch, Ivan Niven, Gloria Olive, D. B. Owen, T. S. Peterson, A. R. Poole, John Raymond, Louise Rosenbaum, R. A. Rosenbaum, E. C. Schlesinger, W. M. Stone, N. Y. Tang, F. H. Tingey, G. E. Uhrich, D. V. V. Wend, L. B. Williams, R. M. Winger, Fumio Yagi, H. S. Zuckerman.

On Friday evening there was a joint dinner with the Association at the banquet room of the University Commons. Following the dinner the visitors were guests of the Department of Mathematics at an informal evening at the Faculty Club.

On Saturday morning there was a general session for the invited address by Professor Edwin Hewitt of the University of Washington, entitled *Ideal theory in algebras of linear functionals*. This was followed by a session on Algebra for contributed papers. Professor M. S. Knebelman presided at both sessions. In the afternoon there was a session on analysis and geometry, Professor R. D. James presiding.

During the morning session, the members present voted unanimously to request the Council of the Society to schedule a meeting for Saturday, June 16, 1951, to be held in conjunction with a meeting of the Pacific Northwest Section of the Mathematical Association of America at the State College of Washington, Pullman, Washington, and expressed the hope that these joint meetings might become annual events. At the afternoon meeting the members voted a resolution of appreciation to the University of Washington, the Department of Mathematics, and the committees on arrangements and entertainment.

In the absence of the Associate Secretary, Professor R. M. Winger acted as secretary to the meeting.

Abstracts of papers presented at the meeting follow. Those abstracts whose numbers are followed by the letter "*t*" were presented by title. Paper number 419 was read by Professor Beaumont.

ALGEBRA AND THEORY OF NUMBERS

408. B. H. Arnold: *Birkhoff's Problem* 20.

Birkhoff (*Lattice theory*, 1948, p. 57, Problem 20) gives a relation which he suggests may generate a polarity between the classes of closed sets and the classes of valid statements of convergence in various topologizations of a set. In the present paper, an example is given to show that Birkhoff's relation does not give the desired polarity. Using a slightly different relation, the polarity is obtained. Abstract conditions are given which are necessary and sufficient for (Birkhoff's notation) $\Delta = (\Delta^*)^+$ and $T = (T^+)^*$. (Received March 31, 1950.)

409. J. L. Brenner: *Polynomial parametrizations.*

Let x, y, t be indeterminates, R a domain of integrity; $f = f(x, y)$ a homogeneous polynomial, $f \in R[x, y]$. If nonconstant polynomials $P_i = \sum a_{ij} t^j$ ($i = 1, 2$) exist, $a_{ij} \in R$, such that $G(t) = f(P_1, P_2)$ is a nonzero constant K, then $f(x, y) = c(a_{2n}x - ya_{1n})^m$ for some c, n; $c \in R$. Thus if R is also a unique factorization domain, then all polynomial parameterizations, if there are any, of $f = K$, $K \neq 0$, are given by $x = a_{1n}S(t) + d$, $y = a_{2n}S(t)$, $d \neq 0$. As an application, the 2×2 unitary matrixes over $R[t]$ are found. (Received April 7, 1950.)

410*t*. J. L. Brenner: *Matrices of quaternions.* Preliminary report.

Let A be an $n \times n$ matrix with quaternion elements. A nonsingular matrix T exists such that $TAT^{-1} = (b_{ij})$, where $b_{ij} = 0$ whenever $i > j+1$. The elements of T are rational functions of the elements of A. Using this last semi-canonical form, it is proved that *every matrix of quaternions has a characteristic root* λ, such that $Ax = x\lambda$ for some vector x. The methods used appeared in Ann. of Math. vol. 39 (1938) p. 476; also used is the result that every polynomial of degree n and type 1 has a root (Eilenberg-Niven, Bull. Amer. Math. Soc. vol. 50 (1944) pp. 246–248). (Received May 31, 1950.)

411*t*. Frank Harary: *A generalization of ring duality.*

Several results of A. L. Foster's papers *The idempotent elements of a commutative ring form a Boolean algebra*; *ring duality and transformation theory* and *Maximal idempotent sets in a ring with unit* are generalized to the class of all rings having a nonzero idempotent element. In particular, (i) an analogous duality theory is found, and (ii) if an idem-set is a subset of a ring in which each element is idempotent and any two elements commute, then any maximal idem-set is a generalized Boolean algebra. (Received April 6, 1950.)

412*t*. G. P. Hochschild: *Automorphisms of simple algebras.*

A theory of automorphisms of a simple algebra, A, is established by combining the Galois theory for fields with the theory of commutators in simple algebras. The results concern the usual relationships between groups of automorphisms, subjected to appropriate regularity and completeness conditions, and simple subrings B of A, such that the center of A is finite over its intersection with B, and B generates a simple algebra over the center of A. (Received May 3, 1950.)

413. D. C. Murdoch: *Isolated component ideals in a noncommutative ring.*

Let R be a noncommutative ring. N. H. McCoy (Amer. J. Math. vol. 71 (1949) pp. 823–833) has defined an m-system as a set M of elements such that if $a \in M$ and $b \in M$ then $axb \in M$ for some x in R. The complement in R of a prime ideal is an m-system. Corresponding to a fixed m-system M, a right M-n-system N is defined as a set of elements containing M such that $m \in M$ and $n \in N$ imply $nxm \in N$ for some x in R. If $\mathfrak{a}$ is an ideal and M an m-system, the right upper M-component of $\mathfrak{a}$ is an ideal $\mathfrak{u}(\mathfrak{a}, M)$ consisting of all elements x such that every right M-n-system containing x meets $\mathfrak{a}$. The right lower M-component $\mathfrak{l}(\mathfrak{a}, M)$ is the ideal consisting of all x such that $xRm \subseteq \mathfrak{a}$ for some m in M. Both $\mathfrak{u}(\mathfrak{a}, M)$ and $\mathfrak{l}(\mathfrak{a}, M)$ are generalizations of the isolated component ideals defined by Krull in the commutative case. Properties of $\mathfrak{u}(\mathfrak{a}, M)$ and $\mathfrak{l}(\mathfrak{a}, M)$ and relations between them are investigated. In certain cases principal components of $\mathfrak{a}$, in the sense of Krull, can be defined and $\mathfrak{a}$ is then the crosscut of its right lower principal components. (Received April 3, 1950.)

414*t*. D. C. Murdoch: *Isolated primary components of an ideal in a noncommutative ring.*

An ideal $\mathfrak{q}$ in a noncommutative ring R is defined to be right primary if, when a is not in $\mathfrak{q}$, $aRb \subseteq \mathfrak{q}$ implies $b \in \mathfrak{r}(\mathfrak{q})$ where $\mathfrak{r}(\mathfrak{q})$ is the radical of $\mathfrak{q}$ in the sense of McCoy (Amer. J. Math. vol. 71 (1949) pp. 823–833). If either the ascending or descending chain condition holds in all quotient rings $R/\mathfrak{a}$, with $\mathfrak{a}$ not prime, then the radical of a right primary ideal is prime. In rings with ascending chain condition not every ideal is representable as the crosscut of a finite number of right primary ideals. However if an ideal is so representable, then each upper M-component $\mathfrak{u}(\mathfrak{a}, M)$ (see preceding abstract), where M is the complement of a minimal prime divisor of $\mathfrak{a}$, is right primary. These are called the isolated primary components of $\mathfrak{a}$ and they must all occur among the right primary components in any minimal representation of $\mathfrak{a}$ as a finite crosscut of right primary ideals. (Received March 6, 1950.)

415. Ivan Niven: *A binary operation on sets of positive integers.*

Let A and B denote the sets of increasing positive integers $\{a_i\}$ and $\{b_i\}$. Define the "product" AB to be the set $\{b_{a_i}\}$. Under this operation, sets of positive integers form an associative, non-commutative groupoid. Basic properties are determined, including the relations of the densities of A, B, and AB, for both limit and number theoretic densities. (Received May 5, 1950.)

416*t*. R. M. Robinson: *Arithmetical definability of field elements.*

It is shown that an element of an algebraic field is arithmetically definable if and only if it is fixed for all automorphisms of the field. (Received April 27, 1950.)

417*t*. A. R. Schweitzer: *Grassmann's extensive algebra and modern number theory.* II.

The author considers Grassmann's algebra in relation to investigations of Cauchy and Kronecker. In C. R. Acad. Sci. Paris (1847) p. 1120 Cauchy obtained a "real" interpretation of the complex number $a+bi$ ($i^2=-1$) by specializing a polynomial generalization of Gauss's relation of congruence. Subsequently (C. R. Acad. Sci. Paris (1853) p. 75) Cauchy construed the units of the complex number $a+bi$ (and also the units of Hamilton's quaternion) as instances of his "algebraic keys" (clefs algébriques) and showed that his theory of the latter is applicable to his method of congruence (C. R. Acad. Sci. Paris (1853) p. 168). Grassmann noted (C. R. Acad. Sci. Paris (1854)) that Cauchy's algebraic keys correspond to his "extensive magnitudes." The two standpoints are essentially distinct, as Grassmann has suggested (*Gesammelte Werke*, vol. 1, part 2, pp. 9, 10, 399). Kronecker extended, in effect, Cauchy's method of congruence to representations of negative numbers and rationals (J. Reine Angew. Math. vol. 101 (1887) pp. 345, 346) as a contribution to a plan of "arithmetization" of mathematical disciplines (ibid. pp. 338, 339, 355) which included replacement of algebraic numbers (ibid. p. 347) with the aid of modular systems (J. Reine Angew. Math. vol. 92 (1882) pp. 1–122). (Received May 3, 1950.)

418t. A. R. Schweitzer: *On the place of the algebraic equation in Grassman's extensive algebra.*

The author discusses the following problem: Given the points $x_1, x_2, \cdots, x_{n+1}, y_1, y_2, \cdots, y_n, t_1, t_2, \cdots, t_n$ in Grassmann space, to find the point y_{n+1} such that $f(u_1, u_2, \cdots, u_{n+1})=f(v_1, v_2, \cdots, v_{n+1})$ where $u_i=f(x_i, t_1, t_2, \cdots, t_n)$, $v_i=f(y_i, t_1, t_2, \cdots, t_n)$ $(i=1, 2, \cdots, n+1)$; $f(x_1, x_2, \cdots, x_{n+1})$ is a linear function of the x's with numerical coefficients a_i different from zero and belonging to a field F; $f(u_1, u_2, \cdots, u_{n+1})=f(\lambda_1 x_2, \lambda_2 x_3, \cdots, \lambda_{n+1} x_1)$ where the numerical coefficients λ_i belong to F. For $n=1$ the preceding generalizes a problem equivalent to a problem considered by Grassmann, *Gesammelte Werke*, vol. 1, part 1, pp. 158, 159. The author shows that the solution of the preceding problem requires that $a_1^n(x_1-y_1)+\lambda_1 a_1^{n-1}(x_2-y_2)+\cdots+(\lambda_1\lambda_2\cdots\lambda_n)(x_{n+1}-y_{n+1})=0$, where the λ's are different from zero but otherwise arbitrary; the sum of the coefficients of the differences x_i-y_i is an algebraic equation in a_1 and of the nth degree. Also λ_{n+1} must satisfy a vanishing determinant whose rows are respectively: $1, \lambda_1, \lambda_1\lambda_2, \cdots, (\lambda_1\lambda_2\cdots\lambda_n)$; $1, \lambda_2, \lambda_2\lambda_3, \cdots, (\lambda_2\lambda_3\cdots\lambda_{n+1})$; $\cdots 1, \lambda_{n+1}, \lambda_{n+1}\lambda_1, \cdots, (\lambda_{n+1}\lambda_1\cdots\lambda_{n-1})$. Reference is made to Grassmann, ibid. pp. 165, 373. (Received May 3, 1950.)

419. H. S. Zuckerman and R. A. Beaumont: *A characterization of the subgroups of the additive rationals.*

All of the subgroups of the additive group of rational numbers are determined by means of a simple characterization. Many of their properties are readily obtained. For example, all isomorphisms between the subgroups are determined. Also, a complete survey of the rings which have these subgroups as additive groups is given. (Received April 28, 1950.)

ANALYSIS

420. T. M. Apostol: *Asymptotic series related to the partition function.*

Let $G_p(x)$ be the Lambert series $\sum_{n=1}^{\infty} n^{-p}x^n/(1-x^n)$ and let $f_p(x)=\exp(G_p(x))$. If $f_p(x)$ is developed into a power series $f_p(x)=1+\sum_{n=1}^{\infty} a_p(n)x^n$, then for $p=1$ the coefficients reduce to $p(n)$, the number of unrestricted partitions of n. For $p>1$, the coefficients are no longer integers but they still have some properties resembling those of $p(n)$. For odd $p>1$ the author obtains an asymptotic formula for $a_p(n)$ analogous to Rademacher's famous convergent series for $p(n)$. The series for $a_p(n)$ is obtained with bounded error term. The method used is the classic Farey-dissection method and use is made of transformation formulae of the author (see Bull. Amer. Math. Soc. vol. 55 (1949) p. 715) which give the behaviour of $f_p(x)$ near the singularities on the unit circle. It is possible that the error term can be replaced by an error which tends to zero, thus giving a convergent series for $a_p(n)$, but this would require improving the estimate on a complicated type of exponential sum. (Received April 25, 1950.)

421. D. G. Chapman: *A note on quasi-analytic functions.*

Let M_n be a sequence of positive numbers and C_{m_n} be the class of infinitely differentiable functions over the closed interval (0, 1) such that to each function f there exists a real number $k>0$ such that $|f^n(x)| \leqq k^n M_n$ for $0 \leqq x \leqq 1$, $n \geqq 1$. If for any f in C_{m_n}, $f^n(x_0)=0$ for some x_0 in (0, 1) and all n implies $f(x)=0$, then C_{m_n} is quasi-analytic. A simple new proof is here given for the theorem that a necessary condition that C_{m_n} be quasi-analytic is that $\sum_{n=1}^{\infty} 1/M_n^*$ diverge where $M_n^* = \inf_{r \geqq n} M_r^{1/r}$ (this condition is also known to be sufficient). Some extensions to abstract spaces are considered. (Received May 5, 1950.)

422. Paul Civin: *Approximation to conjugate functions.*

Suppose the trigonometric polynomials $P_n(x)$ and a function $f(x)$ satisfy the relation $f(x)-P_n(x)=O(n^{-\alpha})$, $0<\alpha<\infty$, uniformly in x. The conjugate trigonometric polynomials and the conjugate function then satisfy the relation $\bar{f}(x)-\overline{P}_n(x) =O(n^{-\alpha}\log n)$ uniformly in x, and this order is the best possible for arbitrary $P_n(x)$. For the Fourier partial sums Salem and Zygmund [Trans. Amer. Math. Soc. vol. 59 (1946) pp. 11–22] showed that the order of the conjugate approximation was $n^{-\alpha}$. (Received April 26, 1950.)

423t. V. L. Klee: *Some remarks concerning reflexive spaces and adjoint spaces.*

With the aid of a characterization of weak compactness due to Šmulian, there is established a characterization of reflexivity which has the following corollary: (I) *A normed linear space is reflexive if and only if in each of its isomorphs, every continuous linear functional attains its maximum on the unit sphere.* For spaces having a basis, this result was announced by R. C. James (Bull. Amer. Math. Soc. Abstract 56-1-80). Also proved is (II) *The space (m) has an isomorph which is not equivalent to any adjoint space.* This answers a question of J. Dixmier (Duke Math. J. vol. 15 (1948) p. 1070). (Received May 5, 1950.)

424t. B. O. Koopman: *A probabilistic generalization of matric Banach algebras.*

Let X be any nonempty abstract set and S any σ-algebra of subsets of X. Let $\mathfrak{B}$ be the normed algebraic system whose elements are the complex-valued functions $\phi(x, E)$, defined and bounded over $(x, E) \in (X, S)$, measurable (S) in x (E fixed), and countably additive in E (x fixed). Linear combinations of elements of $\mathfrak{B}$ with complex coefficients are defined as usual; the (noncommutative) product $(\phi\psi)(x, E)$ is defined as the integral of $\psi(y, E)$ with respect to the measure $\phi(x, F_y)$, where x and E remain fixed while y, F_y are variables of integration. The norm $N(\phi)$ in $\mathfrak{B}$ is defined as the least upper bound over $x \in X$ of the total absolute variation over X of $\phi(x, E)$ regarded as a set function. It is shown that the system $\mathfrak{B}$ is a Banach algebra over the complex field, possessing the identity $\epsilon = \epsilon(x, E) = \chi_E(x)$ the characteristic function of E. This $\mathfrak{B}$ is evidently a generalization of matric algebra. It arises in the study of stochastic processes, where transition probabilities $P(x \to E) = \pi(x, E)$ are elements of B. (Received May 24, 1950.)

425*t*. E. R. Lorch: *Differentiable inequalities*. II.

The present report constitutes a continuation of Bull. Amer. Math. Soc., Abstract 55-7-356. The subject here dealt with is that of *approximation by differentiable surfaces*. A natural class $\mathfrak{C}$ of differentiable surfaces is one in which the surfaces and their adjoints have a common domain Ω and a common domain Ω^* respectively. Furthermore, $\mathfrak{C}$ is closed under positive scalar multiplication and addition (if $G(x) = c$ and $H(x) = c$ are in $\mathfrak{C}$, so is $\lambda G(x) + \mu H(x) = c$). For such a class a separation theorem may be established: If the surface $\mathfrak{S}_0$ separates $\mathfrak{S}_1$ from the origin, then $\mathfrak{S}_1^*$ separates $\mathfrak{S}_0^*$ from the origin. It is now possible to introduce a metric in a natural class. The distance between two surfaces is measured by the amount of magnification and contraction necessary to produce separation. This metric space is completed by means of convergent sequences. For every surface of the completed space there is a unique adjoint surface and these two give rise to a bilinear inequality $\sum x_i^* y_i \leqq \psi(x^*) \cdot \phi(y)$. The theory is applied to the natural class of closed convex surfaces. It is shown that any surface bounding a body convex in the classical sense may be approximated by surfaces convex in the sense of this paper. (Received April 25, 1950.)

426*t*. E. R. Lorch: *Differentiable inequalities*. III.

This section gives an application of the theory of convexity outlined in two previous reports to the theory of volumes and mixed volumes. A large number of formulas are developed for these, including all well known classic equations. If $\phi(x)$ is positively homogeneous of degree 1, if $rG(x) = \phi^r(x)$, $r > 1$, and if $G(x) = c$ is a closed convex surface $\mathfrak{S}$, then the volume V^* bounded by the adjoint surface $\mathfrak{S}^*$ satisfies $V^* = \lim_{r \to 1} [n(r-1)]^{-1} \int_\Omega |G_{ij}(x)| d\omega$ where the integration is over the unit sphere. If Φ_{ij} is the cofactor ϕ_{ij} in $|\phi_{ij}|$, then $V^*(i, j) = n^{-1} \int_\Omega \phi \Phi_{ij} x_i x_j d\omega$ is called a partial volume. One has $\sum V^*(i, j) = V^*$. The theory of mixed volumes is developed. If $\mathfrak{K}_0$ is a convex body given by $rG(x) \leqq 1$ and $\mathfrak{K}_1$ by $rH(x) \leqq 1$, then $V(\lambda \mathfrak{K}_0^* + \mu \mathfrak{K}_1^*) = \lim_{r \to 1} [n(r-1)]^{-1} \int_\Omega | \lambda G_{ij} + \mu H_{ij} | d\omega$. The mixed volume of order ν, V_ν^*, may be represented as the integral of the trace of a matrix. Given a matrix $\|a_{ij}\|$, let $\|a_{ij}\|^{(\nu)}$ represent the νth compound matrix of $\|a_{ij}\|$. It is the matrix of the transformation induced by $\|a_{ij}\|$ and operating on the determinants defining the Grassmanian coordinates. Then $C_{n,\nu} V_\nu^* = \lim_{r \to 1} [n(r-1)]^{-1} \int_\Omega |G_{ij}| \cdot T(\|H_{ij}\| \ \|G_{ij}\|^{-1}) d\omega$ where $T\|a_{ij}\|$ denotes a trace. (Received April 25, 1950.)

427*t*. G. M. Wing: *Cesàro averages of the coefficients of Schlicht functions*.

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ $(a_1 = 1)$ be analytic and schlicht in $|z| < 1$. Let $S_N(k) = \sum_{n=0}^{N-1} C_{n+k-1,\, k-1} a_{N-n}$ and $\sigma_N(k) = S_N(k)/C_{N+k+1,k+1}$. If the Bieberbach conjecture,

$|a_n| \leqq n$, holds, then $|\sigma_N(k)| \leqq 1$. By means of an integral representation of $\sigma_N(k)$ numbers $A(k)$ are determined such that $\lim\sup_{N\to\infty}|\sigma_N(k)| \leqq A(k)$, and it is proved that $\lim_{k\to\infty} A(k) = 1$. For this result, only the classical distortion theorem of Koebe is needed. Use is also made of recent results on the area of the map of $|z| \leqq r < 1$ by $f(z)$ (I. E. Basilevitch, Dokladii Akademii Nauk USSR. vol. 65 (1949) pp. 253–255) to show that $A(1) \leqq 1.31$, $A(2) \leqq 1.14$, $A(3) \leqq 1.11$. These compare favorably with the best known estimate of the coefficients themselves, $\lim\sup_{n\to\infty}|a_n|/n \leqq e/2$ (ibid.). (Received May 1, 1850.)

APPLIED MATHEMATICS

428. M. Z. Krzywoblocki: *On the latest developments in the integral operator method in compressible flow.*

After a brief representation of the principal ideas of Bergman's integral operator method as applied to the solution of partial differential equations of a compressible fluid, the author discusses the main three types of flow equations: subsonic (elliptic), supersonic (hyperbolic), transonic (mixed). Next, the author represents the application of the method to the solution of problems in a three-dimensional space, and particularly to the axially symmetric flow patterns of a compressible fluid. (Received May 4, 1950.)

429. Howard Eves: *A note on Schick's theorem.*

This paper gives a new and simple proof, based upon a construction in the Gauss plane, of Schick's theorem to the effect that the pedal triangle of one of four points with respect to the other three is invariant in form under a direct circular transformation. Some known associated theorems are reestablished in a simple way, and then some probably new theorems are proved. The following may be mentioned as an example of the latter: *Let $A'B'C'$, $A''B''C''$ be the pedal triangles of a pair of isogonal conjugate points M' and M'' of a given triangle ABC. Then the pedal triangle of M' for triangle $A'B'C'$ and the pedal triangle of M'' for triangle $A''B''C''$ are, respectively, directly similar to triangles $A''B''C''$ and $A'B'C'$.* (Received April 12, 1950.)

430t. Howard Eves: *Some consequences of a fundamental construction in the Gauss plane.*

This paper gives a simple construction for finding the point D of the Gauss plane such that $(ABCD) = (A'B'C'D')$, where A, B, C, A', B', C', D' are any given points. As special cases simple constructions are given for finding the limit and double points of the homography $(ABCZ) = (A'B'C'Z')$. Known theorems concerning the situations where $(ABCD)$ is real, harmonic, equianharmonic, or orthocyclic are reestablished as corollaries to the above constructions. The geometry of the situation where $(ABCD)$ is pure imaginary is considered in detail. As an example of a new theorem the following is proved: *Let M be the Miquel point for triangle ABC of the triad A', B', C', and let O' be the circumcenter of triangle $A'B'C'$. Then M and O' are corresponding points under the direct circular transformation set up by A, A'; B, B'; C, C'.* H. F. Sandham has shown that under the same transformation M' and M are corresponding points, where M' is the isogonal conjugate of M for triangle ABC. (Received April 12, 1950.)

431. M. S. Knebelman: *Spaces of relative parallelism.*

Parallelism with respect to a continuous curve joining two, not necessarily distinct, points in an n-dimensional space is defined by means of a nonsingular matrix, the structure matrix of the space. The elements of this matrix are functionals from the space of directed continuous arcs to the space of real numbers. If $\lambda(P)$ is a contravariant vector, the vector obtained by parallel displacement with respect to the arc C_P^Q is given by $\lambda(Q)=M(C_P^Q)\lambda(P)$. For covariant vectors the relation is $\mu(Q)=\mu(P)\cdot M^{-1}(C_P^Q)$. The structure matrix is subject to a number of conditions, Fréchet differentiability being one and $M(C_P^R)=M(C_P^R)\cdot M(C_P^Q)$ another. From this one obtains the affine structure matrix $L(C(\alpha))=dM(C_{\alpha_0}^x)\cdot M^{-1}(C_{x_0}^x)$, L being independent of x_0 and thus defining the local properties of the structure. Covariant differentiation is defined in terms of L, and some fundamental invariants, such as scalar and vector torsions, of the curvature matrix are obtained. The projective structure is defined by the matrix $\mathcal{M}=M/(\det M)^{1/n}$ and from it one obtains the projective connection matrix $\Pi=d\mathcal{M}\cdot\mathcal{M}^{-1}$. Regarding $\mathcal{M}$ as a mapping of the tangent space at P on the tangent space at Q one obtains a generalization of Cartan's results dealing with the holonomy group of the space. (Received April 13, 1950.)

432*t*. Yeh Mo: *Foci of spherical conics.*

The intersection of a sphere with a quadric cone whose vertex lies at the center of the sphere is called a spherical conic. A real non-circular spherical conic consists of two real disjoint closed parts. Within each part there are two real foci. Using elementary analytic methods, the author proves the following theorem: The sum of the spherical distances of any point of a part of a real non-circular spherical conic from two real foci within this part is constant. The tangent to a real non-circular spherical conic at any point makes equal angles with the tangents to the focal arcs of that point with respect to two real foci within any one part. (Received April 27, 1950.)

433*t*. Yeh Mo: *Foci of plane curves.*

In this paper the author proves the following theorems: (1) The point F and the line D are respectively a focus and a corresponding directrix of the curve which is the locus of a point P such that $(FP^2)^m=eMP^n$ where MP is the perpendicular distance from P to the line D, and m, n are positive integers such that $n\geq 2$, $n>m$, and $(n, m)=1$. (2) The two points F' and F are two foci of the curve which is the locus of a point P such that $\pm\lambda F'P\pm\mu FP=a$ where λ, μ are two nonzero constants with absolute values both different from $a/|F'F|$; and if $\lambda F'P+\mu FP=a$ is the real part of the curve, and if θ and ϕ are respectively the angles which the tangent line to this part at P makes with lines PF' and PF, then $\lambda\cos\theta+\mu\cos\phi=0$. (3) The conformal transform of a focus of a given curve is, in general, a focus of the conformal transform of the given curve. (4) If a curve has at least one axis of symmetry and at least one focus, then, with respect to each axis, there is at least one focus situated on it and one of the corresponding directrices is perpendicular to it; and any remaining foci are situated symmetrically with respect to this axis. (Received April 27, 1950.)

434*t*. A. R. Schweitzer: *On the derivation of the regressive product in Grassmann's geometrical calculus.*

In his *Ausdehnungslehre* of 1844 Grassmann derives the regressive product (eingewandtes Produkt) by means of his calculus of systems or spaces (compare Gebietslehre: *Gesammelte Werke*, vol. 1, part 2, pp. 16–23; see also R. Grassmann, *Die Ausdehnungslehre*, Stettin, 1891, pp. 13–25, 50) including concepts applicable to the algebra of logic such as "gemeinschaftliches System," "nächstumfassendes System," "Unterordnung," "Ueberordnung," and so on. In this paper regressive products associated with an n-simplex ($n=2, 3$) are derived with the aid of elementary properties of determinants. Reference is made to Grassmann, *Gesammelte Werke*, vol. 1, part 2, p. 43, paragraphs 62, 63, and p. 400; F. Caspary, J. Reine Angew. Math. vol. 92 (1882) pp. 123–144. (Received May 3, 1950.)

435*t*. A. R. Schweitzer: *A metric generalization of Grassmann's geometrical calculus.*

The relation between n-simplexes, $\alpha_1\alpha_2 \cdots \alpha_{n+1} = k\cdot\beta_1\beta_2\cdots\beta_{n+1}$ ($n=1, 2, 3, \cdots$) fundamental for Grassmann's calculus (Amer. J. Math. vol. 35 (1913) pp. 37–56) is generalized by the author to the relation $\alpha_1\alpha_2\cdots\alpha_{n+1} \equiv k\cdot\beta_1\beta_2\cdots\beta_{n+1}$ (mod $\xi_1\xi_2\cdots\xi_{n+1}$) with meaning: There exists the n-simplex $\xi_1\xi_2\cdots\xi_{n+1}$ such that $\alpha_1\alpha_2\cdots\alpha_{n+1} \equiv \xi_1\xi_2\cdots\xi_{n+1}$ and $\xi_1\xi_2\cdots\xi_{n+1} \equiv k\cdot\beta_1\beta_2\cdots\beta_{n+1}$. Then Grassmann's relation $\alpha_1\alpha_2\cdots\alpha_{n+1} \equiv k\cdot\beta_1\beta_2\cdots\beta_{n+1}$ is replaced by taking $\alpha_1\alpha_2\cdots\alpha_{n+1}$ for the modulus and the relation of geometric congruence $\alpha_1\alpha_2\cdots\alpha_{n+1} \equiv \beta_1\beta_2\cdots\beta_{n+1}$ is replaced by taking $k=1$ and $\beta_1\beta_2\cdots\beta_{n+1}$ for the modulus. Metrical properties corresponding to Hilbert's theory in his *Grundlagen der Geometrie* (7th ed., Leipzig and Berlin, 1930, pp. 11–14) are developed postulationally in terms of an undefined relation of congruence between n-simplexes. Relatively to the author's system in terms of his relation K_n (Amer. J. Math. vol. 31 (1909) pp. 365–410) congruence between n-simplexes is expressed in terms of an undefined relation E_n. Reference is made to the author's note, Bull. Amer. Math. Soc. vol. 15 (1908) pp. 79–81. (Received May 3, 1950.)

436*t*. S. K. B. Stein: *Projection of 2 polygons.*

Let P and P' be two simple polygons in the plane. We prove the following theorem. It is possible to move P' rigidly onto P in such a manner that the area of intersection of the two interiors is greater than zero and that the directed projection in every direction of the portion of P' interior to P is zero. Indeed this can be done in an infinity of manners. The theorem holds for all dimensions and for almost all curves. (Received May 3, 1950.)

437*t*. Fred Supnick: *On the perspective deformation of polyhedra.* II. *Solution of the convexity problem.*

In the first part of this paper the question raised by S. S. Cairns whether every geodesic triangulation of the sphere is a central projection of a convex polyhedron is answered. (Geodesic triangulation means edges $<\pi$, and faces $<$hemisphere in area.) It is shown that certain geodesic triangulations of the sphere of octahedral type cannot be central projections of convex polyhedra. In the second part of this paper the problems of determining the necessary and sufficient conditions for any geodesic triangulation of the sphere to be a central projection of a convex polyhedron and of determining an algorithm for finding the solutions are solved. This is done by reducing the characterization problem to that of solving systems of linear inequalities, thus permitting the use of known results. (Received May 29, 1950.)

LOGIC AND FOUNDATIONS

438*t*. F. B. Jones: *An elementary two color problem*. II.

Consider a unit square region in the number plane with (0, 0) and (1, 1) as the ends of a diagonal. It is possible to color certain of the points of this region red and certain others green such that (1) no point on $y=x$ is colored, (2) all other points are colored either red or green, and (3) if H_x is any uncountable set of straight line arcs spanning the square parallel to the y-axis and H_y is the image of H_x with respect to $y=x$, then the intersection of H_x with H_y contains both a red and a green point. This settles the question raised in a previous paper (Bull. Amer. Math. Soc. Abstract 56-2-212) and may be used to establish a characterization of the first transfinite cardinal based on the decomposition of the pairs of a set into sets. (Received May 5, 1950.)

439*t*. Hao Wang: *The non-finitizability of impredicative principles.*

Let A be an ordinary second-order functional calculus with number theory as its theory of individuals and B be the system obtained from it by omitting classes which cannot be defined without using bound class variables. The principles of class formation in both A and B embody infinitely many special cases. We know that the principle of B can be replaced by a finite number. In this paper it is shown that this is impossible for that of A. An enumeration of all the classes required to exist by the axioms of B can actually be expressed in A. So that by the diagonal method, there exists a class of A not enumerated and therefore some case of the impredicative principle of A which is independent of the axioms of B. Similar things can be proved for any subsystem B' of A which contains finitely many axioms. Hence A is non-finitizable. It also follows that there exists a sequence of classes each containing its predecessors such that each of them can be proved to exist in B' but their union cannot. Similarly impredicative principles in functional calculi of higher orders up to ω are non-finitizable. (Received May 22, 1950.)

440*t*. Hao Wang: *Translatability and relative consistency.*

With an arithmetization of syntax we can express the consistency of a system S by an arithmetic proposition Con (S). Two systems S and S' are said to be of similar strength if Con (S) is derivable from Con (S') in number theory and vice versa. Therefore, for every system S, if Con (S') is provable in S and S contains number theory then S and S' are not of similar strength. S is said to be translatable into S' if there is a recursive function mapping propositions and theorems of S respectively into those of S'. It is proved that if S and S' are translatable into each other, then they are of similar strength. All consistent systems with decision procedures are of similar strength because their consistency can all be proved in number theory. However, the consistency of quantification theory can be proved in number theory too, although it is known to be an undecidable system. It is further shown by extending a result of Hilbert-Bernays that each system S is translatable into the system obtained from number theory by adding Con (S) as a new axiom. (Received May 22, 1950.)

441*t*. Hao Wang: *On the relative strength of certain ordinary systems.*

L is an ordinary set theory with finitely many axioms, being adequate for developing the second-order functional calculus founded on integers. L'' is related to L as an nth-order functional calculus is to an $(n-1)$th, and L' is like L'' except that variables of the highest type are not allowed in defining classes of lower types. $L\#$ is L plus the proposition Con (L) as a new axiom. The following things are proved. $L\#$ and L' are of similar strength. Although a Tarski truth set for $L\#$ is obtainable in L', Con $(L\#)$ is not provable in L'. If we use in L' and L the same definition for the class of natural numbers, then the proposition that every class of natural numbers has a least member is independent of the axioms of L' and Con (L) is an unprovable proposition of L' of the form $(k)\phi k$ for which we can prove $\phi 0$ and $\phi n \supset \phi n'$ in L'. By choosing suitable definitions, Con (L) becomes a theorem of L'. Con (L) and Con $(L\#)$ are theorems of L''. It follows that Con (L') is also a theorem of L''. (Received May 22, 1950.)

STATISTICS AND PROBABILITY

442*t*. Murray Rosenblatt: *On distributions of certain Wiener functionals.*

Let $x_1(t)$, $x_2(t)$ be elements of two independent Wiener spaces. $V(x, y)$ is assumed to be non-negative and sufficiently regular except on a rectifiable curve of discontinuity C. The Wiener functional $\int_0^t V(x(\tau), y(\tau))d\tau$ is investigated. The principal result is the following: If $\sigma(\alpha; t)$ is the distribution of $\int_0^t V(x(\tau), y(\tau))d\tau$, then $\int\int_0^\infty \exp(-u\alpha - \delta t)d_\alpha\sigma(\alpha; t) = \int\int_{-\infty}^{\infty}\psi(x, y)dxdy$ where $\psi(x, y)$ is the fundamental solution of $2^{-1}\nabla^2\psi - (s+uV(x, y))\psi = 0$, $(x, y) \neq (0, 0)$, $\lim_{\epsilon\to 0}\int_0^{2\pi}\partial\psi(\epsilon\cos\theta, \epsilon\sin\theta)/\partial t d\theta = -2$. The methods used are similar to those of M. Kac (*On the distributions of certain Wiener functionals*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 1–13). A small discussion indicates that the results of the paper can be immediately extended to the product space of n Wiener spaces in like manner. (Received April 25, 1950.)

443*t*. Abraham Wald and Jacob Wolfowitz: *Characterization of the minimal complete class of decision functions when the number of distributions and decisions is finite.*

Let m be the number of cumulative distribution functions $\{f_i\}$ (of any kind whatever) and ξ the generic designation of an a priori probability distribution on $\{f_i\}$. A Bayes solution with respect to the sequence $\xi_1, \cdots, \xi_k$ is defined inductively as follows: When $h=1$ it is a Bayes solution with respect to ξ_1. For $h>1$ it is a Bayes solution with respect to ξ_h among all decision functions which are Bayes solutions with respect to the sequence $\xi_1, \cdots, \xi_{h-1}$. Let ξ_{ij} be the probability of f_j according to ξ_i. The authors prove the following: In order that a decision function be admissible it is necessary and sufficient that it be a Bayes solution with respect to a sequence of h $(\leqq m)$ a priori distribution functions $(\xi_1, \cdots, \xi_h)$, such that the matrix $\{\xi_{ij}\}$, $i=1, \cdots, h; j=1, \cdots, m$, has the following properties: (a) for any j there exists an i such that $\xi_{ij}>0$, (b) The matrix $\{\xi_{ij}\}$, $i=1, \cdots, (h-1); j=1, \cdots, m$, does not possess property (a). (Received April 24, 1950.)

TOPOLOGY

444*t*. S. K. B. Stein: *Convex maps.*

The following theorem is proved: Let M be a regular map in the plane with the two additional properties that each country is simply connected and that the union of two touching countries is simply connected. Then it is possible to deform this map

M into a map in which all the countries are convex still preserving all incidence relations. The proof is inductive and tells how to construct the convex map. (Received May 3, 1950.)

445t. A. D. Wallace: *Cohomology groups of collections.*

Fix a space X and an abelian group G. Let $\mathfrak{X} \supset \mathfrak{A} \supset \mathfrak{B}$ be three collections of subsets of X. We define cohomology groups $H^p(\mathfrak{X}, \mathfrak{A}), \cdots$ with G as coefficient group. Each of these groups depends on X as well as the displayed arguments. The standard algebraic theorems hold as well as an exact sequence theorem for $\cdots \rightarrow H^p(\mathfrak{X}, \mathfrak{A}) \rightarrow H^p(\mathfrak{X}, \mathfrak{B}) \rightarrow H^p(\mathfrak{A}, \mathfrak{B}) \rightarrow \cdots$. It is possible also to define groups $H^p(X, \mathfrak{A}), \cdots$. For these there are comparable results with an exact sequence $\cdots \rightarrow H^p(X, \mathfrak{A}) \rightarrow H^p(X) \rightarrow H^p(\mathfrak{A}) \rightarrow \cdots$, the last group depending on X as well as on the displayed argument. This last case is interesting when the collection A is $\{f^{-1}(y) \mid y \in Y\}$, where $f: X \rightarrow Y$ is a map. (Received March 20, 1950.)

R. M. Winger

VIRGIL SNYDER

1869–1950

Professor Virgil Snyder of Cornell University, President of the American Mathematical Society in 1927 and 1928, died on January 4, 1950, at the age of eighty-one. It is appropriate, therefore, to record at this time some account of his services to the Society and of his contributions to mathematics over his active career of forty-three years. Since a biography of Professor Snyder appears in Volume One of the Semicentennial Publications of the Society,[1] the following account is devoted more particularly to his activity in research.

Professor Snyder began his work at a time when geometers were exploring the superstructures of their subject, particularly in space and hyperspace. By adding the radius of a sphere to its coefficients, Lie had defined a sphere by six homogeneous coordinates subject to a non-singular quadratic relation. This situation also occurs with the Plücker line-coordinates so that the parallel between line geometry in three-space and Lie's "Kugelgeometrie" was apparent. Snyder's doctoral dissertation (Göttingen, 1895) was concerned with linear complexes of spheres. Of twenty-one papers he published in the next ten years, twelve were concerned with the metric side of this parallel and dealt with annular, tubular, and developable surfaces, their asymptotic lines, and lines of curvature, or with the development of collateral algebra. In [3] he gives a metric classification of the Dupin cyclide, the envelope of a quadratic system of spheres. The other nine have to do with algebraic ruled surfaces or scrolls, largely from the line geometry, or projective, point of view. After a preliminary check on a classification by Schwartz of quintic scrolls he undertook in three papers [**18, 19, 20**] in vol. 25 of the American Journal the classification of sextic scrolls according to the multiplicity of directrix curves. This classification yielded 68, 32, 11, 5, 2 types of genus (genus of a plane section) 0, 1, 2, 3, 4, respectively. Two years later in vol. 27, two papers [**22, 24**] amplified and corrected this classification with reference to earlier works of A. Wiman. This initial preoccupation with line geometry colored much of Snyder's thinking in later years, particularly in connection with a variety of birational and Cremona transformations which he introduced.

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[1] Pp. 218–223. To this biography there is appended a bibliography with eighty-five numbered entries. For brevity we use the brackets [ ] to indicate the number of a specific item in this bibliography and add thereto [**30A**], *On birational transformations of curves of high genus*, AJM, v. 30, 1908, pp. 10–18.

In five papers published in 1907–1908 Snyder turned his attention to the birational geometry of plane curves. In [29] he constructs a curve C_{2n} which has a complete but composite $g^2_{2(n-1)}$, analogous to the hyperelliptic curve, and shows that it is the projection of a space curve on a quadric surface. In [30A] he proves that curves of sufficiently high genus can be transformed into curves of the same order only by collineations, and in [32] the range of possible transforms is explored. In [31] the groups of birational transformations of curves of genus six are obtained. Finally in [33] there is given a proof of the existence of curves of order n with any prescribed number of double points up to the maximum number $(n-1)(n-2)/2$.

The cubic primal in S_4 with nine nodes, and the one with a double line, had been studied synthetically by Segre and Castelnuovo. In [34] and [35], respectively, Snyder treats these cases analytically, thus obtaining more precise results particularly with respect to the surface in S_3 which yields the apparent contour of the primal with respect to a point. It is indeed somewhat characteristic of his work in general that, while he would use synthetic methods for a first approximation, he would not seem to be entirely happy until he had obtained an analytic formulation as a basis for a more comprehensive treatment of individual cases.

In 1910 Snyder opened a long series of articles on surfaces invariant under infinite discontinuous groups of birational transformations. In [36, 39, 40] these groups are generated by involutorial elements such as the projection of a quartic surface into itself from a node, or as the interchange of the two contacts of a properly chosen system of bitangents. The surfaces in general are those which can be obtained as focal surfaces of line congruences, a number of which he had discussed in earlier papers from other points of view.

In the period 1914–1923, Snyder, in collaboration with his colleague F. R. Sharpe, also an accomplished geometer, continued the above series along with other topics in eight memoirs published in the Transactions of the Society. In [53] and [55] the birational groups of quartic surfaces with exceptional curves, either rational curves, or sextics of genus three, or sextics of genus two, are obtained, the birational transformations in the last case being excerpts from Cremona transformations. In these papers a notable increase in elegance and sophistication accrues from the use of a Severi basis for the curves on the surfaces. In [59] the mapping of one three-space on another by means of a web of quadrics is discussed and birational transformations of the jacobian of the web and its related symmetroid are given, again in terms of a Severi basis. These memoirs furnish interesting examples of the effectiveness of the basis theory for a study of the geometry on a surface. In [58] there is given an exhaustive discussion of (2, 2) point correspondences between two planes which yields two general types with respectively five and six subtypes. Another interesting paper [61] in this group relates to the construction of (n, n') correspondences between two algebraic curves. In the two memoirs [60], the very comprehensive class of Cremona involutions in space which arise from the pairs of a (2, 1) correspondence with another space is completely treated up to the cases where the web of surfaces which defines the correspondence has the order six. In [63] the inverse problem of determining the (2, 1) correspondence when the involution is given is considered in some special cases.

Following a determination [64] of the five types of monoidal Cremona involutions in space corresponding to the five types of ternary Cremona involutions, Snyder, in an address [65] to the Society, presented his reflections on the state (then quite unsatisfactory) of the theory of such involutions in space as contrasted with that of the ternary involutions. A number of his later papers, not reported here, deal with particular involutions. Some of these papers were done in collaboration with graduate students and he inspired a number of others which were published independently.

Two problems seemed to intrigue Snyder in his later years, these relating respectively to the rationality of the pairs of a space Cremona involution and to the rationality of the cubic primal in S_4. His address [70, cf. also 62, 74] as retiring president is devoted to a survey of the literature which bears on these problems. Here [cf. also 79] he frequently indicates the place of his own contributions in the complex situation which he portrays.

One of Snyder's major contributions was made as chairman of the committee of the National Research Council which prepared their Bulletin No. 63, entitled *Selected topics in algebraic geometry*. Of this digest of journal articles up to the date of publication he personally wrote about one quarter of the text and he took on with energy and enthusiasm the entire responsibility for editing and publishing the volume. The supplementary Bulletin, No. 96, covering the next six years, was almost entirely his own work.

Mr. and Mrs. Snyder traveled a great deal both on vacations and on leaves of absence for study and research. On these occasions he attended foreign mathematical meetings and made frequent reports to the Society concerning them. His acquaintance with foreign scholars and with national attitudes made him an especially useful member of the committee which represented the Society at the

Toronto Congress where international tensions resulting from the first world war were still acute [cf. History, pp. 19–20].

Professor Snyder retired from active teaching in 1938. He was fortunate in that he lived to see the Society which he had served so faithfully and so well become one of the leading organizations of its kind.

A. B. Coble

BOOK REVIEW

Divergent series. By G. H. Hardy. Oxford University Press, 1949. 16+396 pp. $8.00.

Hardy died on December 1, 1947; during his lifetime the theory of divergent series and its applications developed into an important branch of modern analysis. It is only natural that the book bears all the marks of his own research work, but it is also a comprehensive presentation of this vastly expanded subject. The final galley proofs were read and completed by some of his younger collaborators, in particular L. S. Bosanquet.

The book consists of thirteen chapters and five additional sections. Attached to each chapter are notes and references. Chapters I and II contain a lively historical survey, particularly on Euler's, Fourier's, and Heaviside's contributions. Also some principles are discussed on which to base methods for the summation of divergent series and integrals.

Chapter III discusses general theorems concerning linear transformations of sequences and functions, with some particular examples. Let us explain a few fundamental concepts. The general linear transformation of a sequence s_n is either another sequence

$$t_m = \sum_{n=0}^{\infty} c_{m,n}s_n, \qquad m = 0, 1, 2, \cdots,$$

or a function $t(x) = \sum c_n(x)s_n$, where x is a continuous parameter. A linear transform of a function $s(y)$ is $t(x) = \int_0^\infty c(x, y)s(y)dy$; $(c_{m,n})$ is the matrix of the transform t_m, $c(x, y)$ is the kernel of the transform $t(x)$.

In Chapter IV special methods are discussed (Nörlund, Euler, Abel, and others). Chapter V is concerned with Hölder, Cesàro, and Riesz means. Hölder and Cesàro means of first order, denoted by $(H, 1)$ and $(C, 1)$, are the same: $h_n' = (1/(n+1)) \sum_0^n s_\nu$; by iteration: $h_n^{(2)} = (1/(n+1)) \sum_0^n h_\nu'$, and $h_n^{(r)} = (1/(n+1)) \sum_0^n h_\nu^{(r-1)}$. The Cesàro means are defined by taking $s_n' = \sum_0^n s_\nu$, $s_n^{(r)} = \sum_0^n s_\nu^{(r-1)}$, $c_n^{(r)} = s_n^{(r)}/A_n^{(r)}$, where $A_n^{(r)}$ is $s_n^{(r)}$ when all $s_n = 1$.

The next two chapters are devoted to Tauberian theorems for Cesàro and for Abel means. Tauber proved in 1897 that Abel summability and $\lim na_n = 0$ imply convergence of $\sum a_n$; this elementary result was the starting point of a long chain of investigations to establish inverse theorems of summability dealing with the question:

Under what conditions for the terms a_n does summability of a series $\sum a_n$ imply its convergence? A significant result of Hardy and Landau is that if $\sum a_n$ is summable (C, k) for some k and if $na_n > -M$ (M constant for all n) then $\sum a_n$ is convergent. Many generalizations concerning Cesàro and Riesz's summability are discussed. The methods employed led to the concept of slowly oscillating sequences and functions and to a connection with Fourier transforms and closure theorems. Chapter VII concludes with the "high indices" Theorem of Hardy and Littlewood.

Chapter VIII is concerned with the methods of Euler and Borel and their generalizations, with application to analytic continuation and to summation of certain asymptotic series. Chapter IX analyzes Tauberian Problems for Borel and Euler summability. The principal Tauberian Theorem is that Borel summability and $a_n n^{1/2} = O(1)$ imply convergence of $\sum a_n$.

The main subject of Chapter X is to discuss the convergence and summability properties of the Cauchy product of two series. Chapter XI discusses Hausdorff means and related transforms. In Chapter XII, N. Wiener's theory of general Tauberian problems is presented; here the tool is the theory of Fourier transforms. It includes applications to the theory of primes, and to special summability methods, such as Borel summability and Riemann summability.

Chapter XIII discusses the Euler-Maclaurin sum formula. The first of the appendices discusses the evaluation of certain definite integrals by means of divergent series; appendices II and III give applications to Fourier series. In particular appendix II analyzes the Fourier kernel of certain methods of summation such as Cesàro's method and de la Vallée Poussin's method. In appendix III, Riemann and Abel summability are applied to Fourier series. Appendix IV deals with Lambert and Ingham summability and their role in the analytic theory of numbers. Appendix V discusses theorems concerning a method related to Abel summability.

The large amount of material evidently did not permit more detail concerning applications to trigonometric series, or a discussion of the subject of orthogonal series and of multiple series. It seems worthwhile to write another book on these subjects.

OTTO SZÁSZ

ABSTRACTS OF PAPERS

The abstracts below are abstracts of papers presented by title at the October Meeting in New York and the November Meeting in Evanston. Abstracts of papers presented in person at these meetings will be included in the reports of the meetings which will be published in the January issue of this BULLETIN.

Abstracts are numbered serially throughout this volume.

ALGEBRA AND THEORY OF NUMBERS

446t. R. H. Bruck and Erwin Kleinfeld: *New identities for alternative rings*. Preliminary report.

Let R be an alternative ring with (skew-symmetric) commutator $(x, y) = xy - yx$ and associator $(x, y, z) = xy \cdot z - x \cdot yz$. Known identities for the associator are mainly based upon (*) $(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$. Instead, define f by $(wx, y, z) = (x, y, z)w + x(w, y, z) + f(w, x, y, z)$, and use (*) to prove f skew-symmetric. Skew-symmetry of f, and one of several identities such as (i) $f(w, x, y, z) = ((w, x), y, z) + ((y, z), w, x)$, are equivalent to (*). Other functions are introduced in the same spirit; these and f are powerful tools in the study of alternative rings. Among the significant byproducts are: (ii) $((x, y, z)^2, y, z) = 0$; (iii) $((w, x)^2, y, z) = (w, x)f(w, x, y, z) + f(w, x, y, z)(w, x)$. (Received July 21, 1950.)

447t. D. O. Ellis: *An algebraic characterization of lattices among semilattices*.

A semilattice is an associative, commutative, idempotent groupoid. The set-theoretic product of the sets of units for each of two elements of a semilattice is a sub-semilattice. Cosets of this sub-semilattice by its own elements are considered. The original semilattice is said to have property M if at least one such coset consists of a single element for each two elements of the original semilattice. The principle result is: *Let L be a semilattice under $a+b$. It is possible to define a second operation, ab, in L so that L forms a lattice under $a+b$ and ab if and only if L has Property* M. *Moreover, the introduction of this second operation may be accomplished in exactly one way; namely, by construction from Property* M. (Received June 2, 1950.)

448t. D. O. Ellis and J. W. Gaddum: *On solutions of systems of linear equations in a Boolean algebra*.

Let B be a Boolean algebra with meet, join, and complement denoted by ab, $a+b$, and a', respectively. Consider the system of equations (1) $\sum_{j=1}^{n} a_{ij}x_j = k_i$; $i = 1, 2, \cdots, m$, where the a_{ij} and k_i are constants. The cardinals m and n are finite, $\leq \aleph_0$, or arbitrary according as B is arbitrary, σ-complete, or complete. Define $d_{ij} = \prod_{\lambda=1,\lambda\neq i}^{m}(a'_{\lambda j} + k_\lambda)$; $j = 1, 2, \cdots, n$. The principle results are: 1. If $k_i = 0$; $i = 1, 2, \cdots, m$ in (1), the complete solution is the n-parameter form $x_j = a_j d_{ij} a'_{ij}$; $j = 1, 2, \cdots, n$. 2. A necessary and sufficient condition that (1) have a solution is that $k_i \sum_{j=1}^{n} a_{ij} d_{ij} = k_i$; $i = 1, 2, \cdots, m$. 3. The solutions of (1) are those values of the n-parameter form $x_j = a_j d_{ij}(a'_{ij} + k_i)$; $j = 1, 2, \cdots, n$, for which the parameter values satisfy $k_i \sum_{j=1}^{n} a_{ij} d_{ij} a_j = k_i$; $i = 1, 2, \cdots, m$. Some of the results have been obtained for the finite case by Löwenheim (Math. Ann. vol. 79 (1919) pp. 223–236). (Received June 2, 1950.)

449*t*. Oscar Goldman: *On the theory of algebraic surfaces.*

Algebraic foundations for a theory of algebraic surfaces without singular points are developed. The considerations are essentially limited to that part of the subject which has direct bearing on the so-called Riemann-Roch Theorem. For the purpose of providing an algebraic proof of this theorem, the theory of double differentials is developed in some detail, enabling one to avoid the necessity of considering the adjoint surfaces. A proof of the birational invariance of the entire theory is included. (Received July 6, 1950.)

450*t*. B. W. Neumann: *Embedding nonassociative rings in division rings.*

Rings are here understood nonassociative, that is, with associativity of multiplication omitted from the usual ring postulates. Necessary and sufficient condition for a ring to be embeddable in a division ring (no matter whether one-sided or two-sided division be required) is that every nonzero element generate additively a cycle of the same order; this order, if finite, is a prime. The same condition is equivalent with embeddability in an algebra (of possibly infinite rank) over a field. Necessary and sufficient condition for a ring to be embeddable in a ring with *unique* division (on one side or on both sides) is that it possess no zero divisors. A one-sided or two-sided neutral element can be adjoined, and an obviously necessary condition is also sufficient to ensure that such adjunction does not disturb the uniqueness of division. Analogous results hold when multiplication is commutative ($xy = yx$ for all x, y), anticommutative ($xy = -yx$), or idempotent ($xx = x$). The method consists in the adjunction, as freely as possible, of single left quotients, and so forth, together with the usual transfinite building-up process. The author believes the results may be known already, but has been unable to find any references. (Received July 17, 1950.)

ANALYSIS

451*t*. J. G. Herriot: *The polarization of a lens.*

Consider a conductor placed in an electrostatic field uniform at infinity and having a given direction there. The intensity of the disturbance produced by the conductor may be measured by a quantity called the polarization in the given direction. (See M. Schiffer and G. Szegö, Trans. Amer. Math. Soc. vol. 67 (1949) pp. 130–205.) The mean polarization P_m is the average of the polarizations in any three mutually orthogonal directions. Let V denote volume and C electrostatic capacity. For the bowl (limiting case of lens) the author showed earlier (see Bull. Amer. Math. Soc. Abstract 54-11-470) that $P_m \geqq (8\pi/3)C^3$. It is now shown that $P_m \geqq 4\pi C^3$ for the bowl. The inequality $P_m + V \geqq 4\pi C^3$ previously proven for a lens with dielectric angle $\pi/2$ and for two tangent spheres is now established for the symmetric lens. (Received July 20, 1950.)

TOPOLOGY

452*t*. A. D. Wallace: *An isomorphism theorem.*

For the Alexander-Kolmogoroff cohomology groups (arbitrary coefficient group) the following is shown to hold: Let $\iota:(X, A)\rightarrow(Y, B)$ be the map which takes A onto the point B and maps $X-A$ topologically onto $Y-B$. Then $\iota^*:H^p(Y, B)\rightarrow H^p(X, A)$ is an isomorphism onto if either (i) X is fully normal and A is closed or (ii) X is locally compact Hausdorff, A is closed, and $F(A)$ is compact. This result (which seems to have been first stated by H. Cartan) seems not to be known in this generality except for the Čech groups based on finite open coverings. It is a consequence of suitable extension and reduction theorems valid under conditions analogous to (i) or (ii). In the proofs of these (under (ii)) use is made of a result of J. W. Keesee which follows from corresponding results using hypotheses of type (i). It is easily seen (and probably known) that other excision theorems are equivalent to this one. (Received July 10, 1950.)

NOTES

The school of Mathematics of the Institute for Advanced Study will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year 1951–1952. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, N. J., and are returnable by January 1, 1951.

Indiana University has established a Graduate Institute for Applied Mathematics. Professors T. Y. Thomas, Eberhard Hopf, and David Gilbarg have been transferred from the Mathematics Department of the University to the New Institute, and Dr. C. A. Truesdell of the Naval Research Laboratory will also serve as a visiting professor for the year 1950–51.

Professor John von Neumann of the Institute for Advanced Study, president elect of the American Mathematical Society, has been awarded the honorary degrees of Doctor of Science by Harvard University and the University of Pennsylvania.

Professor Lamberto Cesari of the University of Bologna is on leave of absence and has been appointed to a visiting professorship at Purdue University.

Assistant Professor F. T. Adler of the University of Wisconsin has been appointed to an associate professorship of physics at Carnegie Institute of Technology.

Associate Professor Harriet W. Allen of Connecticut College has been appointed to a professorship of physics at Western College, Oxford, Ohio.

Dr. T. M. Apostol of Massachusetts Institute of Technology has been appointed to an assistant professorship at California Institute of Technology.

Dr. J. E. Barney of the University of Kansas has accepted a position as chemist with the Whiting Research Laboratory of the Standard Oil Company.

Dr. P. T. Bateman of the Institute for Advanced Study has been appointed to an assistant professorship at the University of Illinois.

Mr. M. R. Bates of Union College has accepted a position as mathematician with the Bell Aircraft Corporation, Buffalo, New York.

Assistant Professor P. W. Carruth of Swarthmore College is on leave and has been appointed to a visting professorship at Reed College.

Dr. F. E. Clark of Tulane University of Louisiana has been appointed to an assistant professorship at Rutgers University.

Associate Professor John DeCicco of Illinois Institute of Technology has been appointed to a professorship at DePaul University.

Associate Professor C. H. Denbow of the U. S. Naval Academy has been appointed to a professorship at Ohio University.

Dr. N. J. Divinsky of the University of Chicago has been appointed to an assistant professorship at Ripon College.

Miss Genevieve H. Dixon of Cornell Aeronautical Laboratories, Inc. has accepted a position as senior computer with the Bell Aircraft Corporation, Buffalo, New York.

Dr. D. G. Duncan of the University of Michigan has been appointed to an assistant professorship at the University of Arizona.

Assistant Professor J. C. Eaves of the University of Alabama has been appointed to an associate professorship at the Alabama Polytechnic Institute.

Dr. D. O. Ellis of the University of Missouri has been appointed to an assistant professorship at the University of Florida.

Mr. E. S. Elyash of Cornell University has been appointed to an assistant professorship at the University of Pittsburgh.

Mr. W. B. Evans of the University of Illinois has been appointed to an assistant professorship at Georgia Institute of Technology.

Mr. F. D. Faulkner of the University of Michigan has been appointed to an assistant professorship at the U. S. Naval Postgraduate School, Annapolis, Maryland.

Professor William Feller of Cornell University has been appointed to a professorship at Princeton University.

Mr. J. V. Finch of the University of Wisconsin has been appointed to an assistant professorship at Beloit College.

Dr. David Gale of Princeton University has been appointed to an assistant professorship at Brown University.

Assistant Professor P. R. Garabedian of the University of California has been appointed to an acting assistant professorship at Stanford University.

Dr. Emil Grosswald of the University of Pennsylvania has been appointed lecturer at the University of Saskatchewan.

Dr. Ernest Ikenberry of Louisiana State University has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Assistant Professor Robert Kahal of Polytechnic Institute of Brooklyn has been appointed to an associate professorship of electrical engineering at Washington University.

Dr. L. H. Kanter of the University of Wisconsin has been appointed to an associate professorship at Mississippi State College.

Mr. E. L. Kaplan of Princeton University has accepted a position as mathematician with the Bell Telephone Laboratories, Murray Hill, New Jersey.

Dr. R. B. Leipnik of the Institute for Advanced Study has been appointed to an assistant professorship at the University of Washington.

Dr. J. A. Lewis of Brown University has accepted a position with the Corning Glass Works, Corning, New York.

Professor Eugene Lukacs of Our Lady of Cincinnati College is on leave of absence and has accepted a position as statistician with the National Bureau of Standards.

Dr. Nathaniel Macon of the University of North Carolina has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Dr. W. S. Massey of Princeton University has been appointed to an assistant professorship at Brown University.

Mr. J. W. Mettler of Lehigh University has accepted a position with the Educational Testing Service, Princeton, New Jersey.

Professor W. R. Murray of Franklin and Marshall College has been appointed to an associate professorship at Wagner College.

Dr. Margaret Owchar of the University of Minnesota has been appointed to an associate professorship at Southwest Missouri State College.

Dr. A. J. Owens of the University of Florida has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Dr. L. E. Payne of Iowa State College of Agriculture and Mechanic Arts has been appointed to an assistant professorship at the University of Arizona.

Mr. M. M. Resnikoff of Pennsylvania State College has accepted a position as aeronautical research scientist with the National Advisory Committee for Aeronautics, Moffett Field, California.

Dr. R. P. Rich has accepted a position as mathematician with the Applied Physics Laboratory, Johns Hopkins University.

Professor R. A. Rosenbaum of Reed College is on leave of absence and has been appointed to a visiting professorship at Swarthmore College.

Dr. W. A. Rutledge of the University of Tennessee has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Dr. A. E. Scheidegger of the University of Toronto has been appointed as lecturer in physics at Queen's University, Kingston, Ontario, Canada.

Mr. B. L. Schwartz of Brown University has been appointed to an assistant professorship at Duquesne University.

Mr. Bernard Sherman of the University of Vermont has been appointed to an assistant professorship at the University of Southern California.

Dr. Seymour Sherman of the Institute for Advanced Study has accepted a position as mathematician with the Lockheed Aircraft Corporation, Burbank, California.

Dr. Harold Shniad of the University of Southern California has been appointed to an assistant professorship at the University of Arkansas.

Associate Professor W. H. Spragens of Florida State University has been appointed to an associate professorship at the University of Mississippi.

Mr. J. J. Taylor of the Kellex Corporation, New York City, has accepted a position with Westinghouse Electrical Corporation, Pittsburgh, Pennsylvania.

Dr. M. C. Waddell of Johns Hopkins University has been appointed to an assistant professorship at Western Reserve University.

Dr. S. S. Walters of the University of California, Los Angeles, has accepted a position as mathematician with the Rand Corporation, Santa Monica.

Professor Emeritus F. B. Wiley of Denison University has been appointed to a professorship at Ashland College.

Dr. J. E. Wilkins of the American Optical Company has accepted a position as mathematician with Nuclear Development Associates, Inc., New York City.

Mr. Roger Williams of the McGraw-Hill Publishing Company is now president of Roger Williams, Inc., a firm of chemical engineering consultants.

Dr. Henry Wolf of Brown University has been appointed to an assistant professorship at Hofstra College.

The following promotions have been announced:

E. S. Kennedy, American University of Beirut, to an associate professorship.

Walter Nef, University of Bern, to a professorship.

R. D. Anderson, University of Pennsylvania, to an assistant professorship.

T. W. Anderson, Columbia University, to an associate professorship.

H. M. Bacon, Stanford University, to a professorship.

Z. W. Birnbaum, University of Washington, to a professorship.

Elizabeth Carlson, University of Minnesota, to an associate professorship.

R. M. Cohn, Rutgers University, to an assistant professorship.

R. H. Cole, University of Western Ontario, to a professorship.

D. A. Darling, University of Michigan, to an assistant professorship.

R. F. Dressler, New York University, to an assistant professorship.

G. M. Ewing, University of Missouri, to a professorship.

C. H. Fischer, University of Michigan, to a professorship.

Evelyn Frank, University of Illinois, to an associate professorship.

H. S. Grant, Rutgers University, to a professorship.

Mary C. Graustein, Tufts College, to an associate professorship.

J. B. Greeley, Utica College, Syracuse University, to an associate professorship.

Edwin Hewitt, University of Washington, to an associate professorship.

H. H. Hinman, City College, New York City, to an assistant professorship.

R. V. Hogg, State University of Iowa, to an assistant professorship.

M. Gweneth Humphreys, Randolph-Macon Woman's College, to a professorship.

P. B. Johnson, Occidental College, to an associate professorship.

Shizuo Kakutani, Yale University, to an associate professorship.

H. R. Kingston, University of Western Ontario, to principal of University College and Dean of the College of Arts and Science.

E. R. Kolchin, Columbia University, to an associate professorship.

H. R. Leifer, Veterans Administration, Pittsburgh, Pennsylvania, to Chief of Registration and Research Section.

R. C. Lyndon, Princeton University, to an assistant professorship.

W. C. McDaniel, Southern Illinois University, to a professorship.

G. R. Magee, University of Western Ontario, to a professorship.

L. E. Malvern, Carnegie Institute of Technology, to an assistant professorship.

Morris Morduchow, Polytechnic Institute of Brooklyn, to an assistant professorship.

Abba V. Newton, Vassar College, to an associate professorship.

Ivan Niven, University of Oregon, to a professorship.

E. P. Northrop, University of Chicago, to a professorship.

E. R. Ott, Rutgers University, to a professorship.

C. R. Phelps, Rutgers University, to an associate professorship.

A. L. Putnam, University of Chicago, to an associate professorship.

Russell Remage, University of Delaware, to an assistant professorship.

Herbert Robbins, University of North Carolina, to a professorship.

M. S. Robertson, Rutgers University, to a professorship.

The following appointment to instructorships are announced: Alabama Polytechnic Institute: Mr. H. W. Burnette, Mr. Verne Dietrich; Brown University: Dr. J. E. L. Peck; Case Institute of Technology: Dr. R. A. Clark; Columbia University: Mr. Fritz Steinhardt; University of Connecticut: Dr. J. K. Goldhaber, Mr. E. S. Wolk; City College, New York City: Dr. Gerald Freilich; University of Illinois: Mr. Martin Davis; Illinois Institute of Technology: Dr. Gerald Berman; Lehigh University: Dr. Samuel Goldberg; University of Maine: Mr. G. F. Simmons; Massachusetts Institute of Technology: Dr. Walter Rudin; New York University: Dr. I. I. Kolodner, Dr. K. S. Miller; Ohio State University: Dr. D. J. Lewis; University of Oklahoma: Mr. A. E. Labarre; Oregon State College: Mr. J. C. Gysbers; Princeton University: Mr. D. W. Blackett, Dr. K. T. Chen; University of Rochester: Mr. D. C. Barton; University of Tulsa: Dr. R. N. Goss.

Professor W. J. Bruns, retired, of Syracuse University died on July 18, 1950, at the age of seventy-four years.

Professor R. I. Pepper of Winthrop College died on July 2, 1950, at the age of fifty-eight years.

Professor H. L. Smith of Louisiana State University died on June 13, 1950. He had been a member of the Society for thirty years.

NEW PUBLICATIONS

Beth, E. W. *Les fondements logiques des mathématiques*. Paris, Gauthier-Villars, 1950. 222 pp. 1400 fr.

Blaschke, W. *Einführung in die Differential geometrie*. Berlin, Springer, 1950. 8+146 pp. 16 DM.

Čebotarev, N. G. *Sobranie Sočineniĭ*. Moscow, Akademii Nauk SSSR, 1949. Vol. 1, 616 pp., vol. 2, 588 pp.

Chandrasekhar, S. *Radiative transfer*. Oxford University Press, 1950. 14+393 pp.

Colloque de géometrie algebrique tenue à Liege les 19, 20, *et* 21 *décembre* 1949. Paris, Masson, 1950. 195 pp. 1400 fr.

Courant, R. *Dirichlet's principle, conformal mapping and minimum surfaces*. New York, Interscience, 1950. $4.50.

Deming, W. E. *Some theory of sampling*. New York, Wiley, 1950. 18+602 pp. $9.00.

Duschek, A., and Hochrainer, A. *Grundzüge der Tensorrechnung in analytischer Darstellung*. Vol. 2, *Tensoranalysis*. Vienna, Springer, 1950. 8+338 pp. $6.00.

Feller, W. *An introduction to probability theory and its application*. New York, Wiley, 1950. 12+419 pp. $6.00.

Golomb, M., and Shanks, M. *Elements of ordinary differential equations*. New York, McGraw-Hill, 1950. 10+356 pp. $3.50.

Hochrainer, A. See Duschek, A.

Hornich, H. *Lehrbuch der Funktionentheorie*. Vienna, Springer, 1950. 8+216 pp. $5.20.

Jaeger, J. C. *An introduction the Laplace transformation*. New York, Wiley, 1949. 132 pp. $1.50.

Kershner, R. B., and Wilcox, L. R. *The anatomy of mathematics*. New York, Ronald, 1950. 11+416 pp. $6.00.

Kočin, N. E. *Sobranie sočineniĭ*. Moscow, Akademii Nauk SSSR., 1949. Vol. 1, 344 pp., vol. 2, 420 pp.

Kuerti, G. See Sommerfeld, A.

Lindelöf, E., and Ullrich, E. *Einfuhrung in die höhere Analysis*. Leipzig, Teubner, 1950. 9+526 pp. $3.55.

Mushelišvili, N. I. *Singular integral equations. Boundary problems of function theory and their applications to mathematical physics*. Trans. by J. R. M. Radok and W. G. Woolnough. Melbourne, Department of Supply and Development, Aeronautical Research Laboratories, 1949. 11+404 pp.

Nagell, T. *Elementär talteori*. Uppsala, Almqvist and Wiksells, 1950. 271 pp. 21 kronor.

Natanson, I. P. *Konstruktivnaya teoriya funkciĭ*. Moscow, 1949. 688 pp.

Piazzolla Beloch, M. *Teoria diametrale delle curve algebriche piane*. University of Ferrara, 1949. 95 pp.

Quine, W. *Methods of logic*. New York, Holt, 1950. 21+264 pp. $2.90.

Radok, J. R. M. See Mushelišvili, N. I.

Schilling, O. F. G. *The theory of valuations*. (Mathematical Surveys, no. 4.) New York, American Mathematical Society, 1950. 8+253 pp. $6.00.

Schwerdtfeger, H. See Tschebotarow, N.

Shanks, M. See Golomb, M.

Sommerfeld, A. *Lectures on theoretical physics*. Vol. 2. *Mechanics of deformable bodies*. Trans. by G. Kuerti. New York, Academic Press, 1950. 11+396 pp. $6.60.

Symposium on stochastic processes. London, Royal Statistical Society, 1949. 133 pp. 12s, 6d.

Tables of the binomial probability distribution. Washington, National Bureau of Standards, 1950. 10+387 pp. $2.50.

Tietze, H. *Gelöste und ungelöste mathematische Probleme aus alter und neuer Zeit*. Munich, Biederstein, 1949. Vol. 1, 20+256 pp., vol. 2, 4+305 pp.

Topologie algébrique. (Colloques Internationaux du Centre National de la Recherche Scientifique, no. 12.) Paris, Centre de la Recherche Scientifique, 1949. 6+115 pp. 600 fr.

Tschebotarow, N. *Grundzuge der Galois'schen Theorie*. Trans. and ed. by H. Schwerdtfeger. Groningen, Noordhoff, 1950. $5.25.

Ullrich, E. See Lindelof, E.

Vlasov, V. Z. *Obščaya teoriya oboloček i eë priloženiya v tehnike*. Moscow, 1949. 784 pp.

van der Waerden, B. L. *Ontwakende Wetenschap*. (Historische Bibliotheek voor de Exacte Wetenschappen, vol. 7.) Groningen, Noordhoff, 1950. 332 pp.

Walsh, J. L. *The location of critical points of analytic and harmonic functions*. (American Mathematical Society Colloquium Publications, vol. 34.) New York, American Mathematical Society, 1950. 8+384 pp. $6.00.

Wiener, N. *The human use of human beings*. Boston, Houghton Mifflin, 1950. 241 pp. $3.00.

Wilcox, L. R. See Kershner, R. B.

Wilkes, M. V. *Oscillations of the earth's atmosphere*. Cambridge University Press, 1949. 10+76 pp. $2.50.

Woolnough, W. G. See Mushelišvili, N. I.

Zygmund, A. *Trigonometric interpolation*. University of Chicago, 1950. 4+99 pp.

DUALITY FOR GROUPS

SAUNDERS MACLANE[1]

I. THE PHENOMENON OF DUALITY

1. Abelian groups. Certain dualities arise in those theorems of group theory which deal, not with the elements of groups, but with subgroups and homomorphisms. For example, a free abelian group F may be characterized in terms of the following diagram of homomorphisms:

$$\begin{tikzcd} F \arrow[r, "\beta"] \arrow[dr, "\alpha"'] & B \arrow[d, "\rho"] \\ & A \end{tikzcd} \tag{1.1}$$

THEOREM 1.1. *The abelian group F is free if and only if, whenever ρ: $B \rightarrow A$ is a homomorphism of an abelian group B onto an abelian group A and α: $F \rightarrow A$ a homomorphism of F into A, there exists a homomorphism β: $F \rightarrow B$ with*

$$\rho\beta = \alpha. \tag{1.2}$$

If F is known to be free, with generators g_i, β may be constructed by setting $\beta g_i = b_i$, with b_i so chosen that $\rho b_i = \alpha g_i$. Conversely, let F have the cited property and represent F as a quotient group F_0/R_0, where F_0 is a free abelian group. Choose $A = F$ and $B = F_0$ in (1.1), let α be the identity, and ρ the given homomorphism of F_0 onto F with kernel R_0. Then, by (1.2), $\alpha = \rho\beta$ is an isomorphism, hence β has kernel 0 and thus is an isomorphism of F into F_0. Therefore F is isomorphic to a subgroup of a free group F_0, so is itself free.

The analogous theorem is true for free nonabelian groups, when A and B are interpreted as arbitrary (not necessarily abelian) groups; the proof uses the Schreier theorem [14][2] that a subgroup of a free group is free.

An abelian group D is said to be *infinitely divisible* if for each $d \in D$ and each integer m there exists in D an element x such that $mx = d$. Such groups may be characterized by a similar diagram

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[1] Essential portions of this paper were developed while the author held a fellowship from the John Simon Guggenheim Memorial Foundation.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

(1.1′)
$$\begin{tikzcd} D & B \arrow[l, "\beta"'] \\ & A \arrow[u, "\lambda"'] \arrow[ul, "\alpha"] \end{tikzcd}$$

THEOREM 1.1′. *The abelian group D is infinitely divisble if and only if, whenever λ: $A \rightarrow B$ is an isomorphism of an abelian group A into an abelian group B and α: $A \rightarrow D$ a homomorphism of A into D, there exists a homomorphism β: $B \rightarrow D$ with*

$$\beta\lambda = \alpha. \tag{1.2′}$$

PROOF. Let D be infinitely divisible. Since λ is an isomorphism of A into B, the construction of β is essentially that of extending a homomorphism α of A into D to a larger group containing A; this construction, using suitable transfinite methods, is well known.

Conversely, let D be an abelian group with the property cited in the theorem, and let d be an element in D and m any integer. The cyclic subgroup Z generated by d can be embedded in a cyclic group B generated by an element b in such fashion that $mb=d$. In (1.1′) take $A=Z$, α the identity, and λ the identity injection of Z into B. There is then a β as in (1.2′); if $\beta b=x$, then $mx=\beta(mb)=\beta\lambda(d)=\alpha(d)=d$. Hence the equation $mx=d$ has a solution in D, and D is infinitely divisible.

In this pair of "dual" theorems the hypotheses differ only in the direction of the arrows in the diagrams (1.1) and (1.1′) and in the replacement of ρ, a "homomorphism onto," by λ an "isomorphism into"; the conclusions differ only in the direction of the arrows and in the inversion of the order of factors in the products (1.2) and (1.2′). In this sense free abelian groups[3] are dual to infinitely divisible abelian groups.

There are other "dual" properties of free and infinitely divisible abelian groups. Any subgroup of a free group is free; any quotient group of an infinitely divisible group is infinitely divisible. Any abelian group is isomorphic to a quotient group of a free abelian group; any abelian group is isomorphic to a subgroup of an infinitely divisible group (that is, can be embedded in an infinitely divisible group). If a free abelian group F is a factor group of an abelian group, it is a direct factor; if an infinitely divisible group D is a subgroup of an abelian group, it is a direct factor.

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[3] Call the dual (in this sense) of a free (nonabelian) group a fascist group. R. Baer has shown me a proof of the elegant theorem: every fascist group consists only of the identity element.

The last property has an application to the characterization of group extensions. Given abelian groups G and A, an (abelian) extension of G by A is an abelian group E with G as subgroup and A as corresponding factor group $A = E/G$. More exactly, an extension is a diagram

$$G \xrightarrow{\lambda} E \xrightarrow{\rho} A \tag{1.3}$$

where λ is an isomorphism of G into E, ρ a homomorphism of E onto A, and the kernel of ρ is the image of λ. The set of all extensions of G by A, with a suitable equivalence relation and a suitable composition, constitutes a group Ext (A, G). This group has an alternate expression ([7] Theorem 10.1) in terms of any representation of A as a quotient group of a free group.

THEOREM 1.2. *If A is isomorphic to F/R, where F is a free abelian group with subgroup R, then*

$$\operatorname{Ext}(A, G) \cong \operatorname{Hom}(R, G)/\operatorname{Hom}(F, G) \mid R, \tag{1.4}$$

where Hom (R, G) *denotes the group of homomorphisms of the group R into G, and* Hom $(F, G)|R$ *the subgroup of those homomorphisms of R into G which can be extended to homomorphisms of F into G.*

A similar and relatively simple argument will prove a dual theorem. The dual of an extension of G by A is described by a diagram like (1.3) with the arrows reversed and the terms "isomorphism into" and "homomorphism onto" interchanged; thus the dual of an extension of G by A is an extension of A by G. The dual theorem now reads

THEOREM 1.2′. *If A is isomorphic to a subgroup S of an infinitely divisible group D, then*

$$\operatorname{Ext}(G, A) \cong \operatorname{Hom}(G, D/S)/\operatorname{Hom}_S(G, D), \tag{1.4′}$$

where Hom $(G, D/S)$ *denotes the group of homomorphisms of G into the factor group D/S, and* Hom_S (G, D) *the subgroup of those homomorphisms of G into D/S which can be obtained by a homomorphism of G into D, followed by the canonical projection of D onto D/S.*

It should be noted that the subgroups Hom$(F, G)|R$ and Hom_S (G, D) which appear in (1.4) and (1.4′) can be described in strictly dual fashion. Thus in (1.4) let κ: $R \rightarrow F$ denote the identity injection of R into F. Then a homomorphism γ of R into G lies in the subgroup Hom $(F, G)|R$ if and only if there exists a β,

$$\beta: F \rightarrow G \quad \text{with} \quad \beta\kappa = \gamma.$$

Dually, let π: $D\rightarrow D/S$ be the canonical projection of the infinitely divisible group D onto the factor group D/S. Then a homomorphism γ of G into D/S lies in the subgroup Hom_S (G, D) if and only if there exists a β

$$\beta: G \rightarrow D \quad \text{with} \quad \pi\beta = \gamma.$$

2. A table of dualities. The duality under discussion is a process which assigns a dual statement to each of certain statements about groups and homomorphisms. Each homomorphism α is understood to be a homomorphism of a specified group G into a specified group H; we write α: $G\rightarrow H$ and call G the domain and H the range of α. Note that the range may be larger than the image $\alpha(G)$.

If S is a subgroup of G (notation $S\subset G$), then the *injection* $\kappa = [G\supset S]$ of S into G is that homomorphism of S into G with $\kappa(s)=s$ for every $s\in S$. If N is a normal subgroup of G, then $Q=G/N$ is a quotient group of G (notation $Q\leqq G$), and the *projection* $\pi=[Q\leqq G]$ of G onto Q is that homomorphism of G onto Q for which $\pi(g)$ is the coset gN for every $g\in G$. The systematic use of these injection and projection homomorphisms is at the heart of our formulation of the duality phenomena.

We consider any statement $\mathfrak{S}$ about groups which does not make reference to the elements of the groups involved, but only to homomorphisms with these groups as domains and ranges, to the products of homomorphisms, to subgroups and quotient groups, injection and projection. The statement dual to $\mathfrak{S}$ is then the statement obtained by carrying out the following interchanges in $\mathfrak{S}$.

$\alpha: G\rightarrow H$	α: $H\rightarrow G$
domain $\alpha=G$	range $\alpha=G$
α is an isomorphism into	α is a homomorphism onto
product $\beta\alpha$	product $\alpha\beta$
S is a subgroup of G	Q is a quotient group of G
the injection $[G\supset S]$: $S\rightarrow G$	the projection $[Q\leqq G]$: $G\rightarrow Q$.

A difficulty appears at once. The relation of inclusion for subgroups is transitive, in that $T\subset S$ and $S\subset G$ imply $T\subset G$. However, if $R\leqq Q$ and $Q\leqq G$, then R cannot be a quotient group of G, consisting of cosets of G, because R consists of cosets of cosets of G. The second isomorphism theorem shows only that R is isomorphic to a quotient group of G. By suitably redefining the notion of a quotient group, this difficulty will be removed (cf. §12 below).

To apply this duality to nonabelian groups we consider the dual to the statement "S is a normal subgroup of G." The usual definition of normality refers to elements of the groups, hence cannot be dualized directly. Let us say that a homomorphism β: $G\rightarrow H$ is a zero homomorphism (briefly, $\beta=0$) if β maps all of G into the identity element of H. The statement "$\beta=0$" turns out to be self dual (§15). If S and S' are subgroups of G, $[G\supset S]$ and $[G\supset S']$ the corresponding injections into G, we say that S *dominates* S' in G if and only if, for every homomorphism α: $G\rightarrow H$,

$$\alpha[G\supset S]=0 \quad \text{implies} \quad \alpha[G\supset S']=0.$$

It then appears that a subgroup S is normal in G if and only if S contains every subgroup S' of G which it dominates. This characterization can be dualized to define a *conormal* quotient group of G; it then appears that *every* quotient group is conormal in this sense!

This illustrates the fact that the dual of a true statement about groups need not be true. Another more familiar example is the following. Every subgroup S of a quotient group G/N of G is a quotient group of a subgroup of G; indeed $S=M/N$, for a suitable subgroup M, with $G\supset M\supset N$. However, a quotient group S/N of a subgroup S of G need not be a subgroup of a quotient group of G, because N, though normal in the subgroup S, need not be normal in G. Another example is the Kurosch-Birkhoff-Jordan-Hölder theorem for transfinite ascending sequences [5, p. 89]; the dual is not true, even for abelian groups.

It is nevertheless true that the duals of a large class of true statements about groups are true, and it is our objective to delimit this class of statements.

3. **Free products and direct products.** Let $A\times B$ be the direct (or cartesian) product of the groups A and B, defined as the group of pairs (a, b) for $a\in A$, $b\in B$. Let α and β denote the natural homomorphisms $\alpha(a, b)=a$, $\beta(a, b)=b$ of the direct product onto its respective factors. The direct product may then be described conceptually in terms of α and β and the diagram

$$\begin{tikzcd} A & A\times B \arrow[l, "\alpha"'] \arrow[r, "\beta"] & B \\ & C \arrow[ul, "\alpha'"] \arrow[u, "\gamma"'] \arrow[ur, "\beta'"'] & \end{tikzcd} \tag{3.1}$$

as follows. Given any group C, and any homomorphisms α' and β' of C into A and B respectively, there exists one and only one homomorphism γ: $C\rightarrow A\times B$ with

$$\alpha\gamma=\alpha', \qquad \beta\gamma=\beta'. \tag{3.2}$$

Specifically, $\gamma(c) = (\alpha' c, \beta' c)$ is such a homomorphism, and is clearly the only such. This property of the diagram (3.1) determines the direct product $A \times B$ and its mappings α and β up to an isomorphism; hence it may serve as a definition of the direct product.

Let $A * B$ be the free product of the groups A and B, defined in the usual fashion (cf. [18, p. 45; 1; 21]) as the set of all words $a_1b_1a_2b_2 \cdots a_nb_n$ for $a_i \in A$, $b_i \in B$ with multiplication defined by juxtaposition of words, equality by the process of removing any b_i (or any a_i) equal to 1, and multiplying the two adjacent a's, together with the inverse and iterations of this process. With this free product we associate the isomorphism α of the first factor A into $A * B$, obtained by mapping each a in A into the word "a," and the analogous isomorphism $\beta: B \rightarrow A * B$. The free product may then be described conceptually by the diagram

(3.1′)
$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, "\alpha'"'] & A * B \arrow[d, "\gamma"] & B \arrow[l, "\beta"'] \arrow[dl, "\beta'"] \\
& C &
\end{tikzcd}$$

as follows. Given any group C, and any homomorphisms α' and β' of A and B, respectively, into C, there exists one and only one homomorphism $\gamma: A * B \rightarrow C$ such that

(3.2′)
$$\gamma\alpha = \alpha', \qquad \gamma\beta = \beta'.$$

Specifically, $\gamma(a_1b_1 \cdots a_nb_n) = (\alpha' a_1)(\beta' b_1) \cdots (\alpha' a_n)(\beta' b_n)$ is such a homomorphism, and is the only one. Again, this description determines the free product, together with its mappings α and β, up to an isomorphism.

The diagram (3.1) is dual to (3.1′), while the product relations (3.2) are dual to (3.2′). The mappings α and β of (3.1) are homomorphisms onto; in (3.1′) they are isomorphisms into. If the intersection of the kernels of α' and β' in (3.1) is the unit group, then γ in (3.1) is an isomorphism into. If the union of the images of α' and β' in (3.1′) is the group C, then γ in (3.1′) is a homomorphism onto. These two statements can be so reformulated as to be strictly dual in our sense (the notion of "kernel" must be replaced by a suitable notion of "coimage," as described in §16 below.)

The proof of the existence of the direct product is not dual to the proof of the existence of the free product, for both proofs involve reference to the elements of the groups concerned. However, the proof that the direct product is unique up to an isomorphism can be phrased so as to be exactly dual to the proof of the uniqueness of the free product up to an isomorphism. Similarly, the proofs of the associative and commutative laws for direct products, formulated in terms of diagrams like (3.1), are dual to the proofs of the corresponding laws for the free product, as will be indicated again in §18 below.

4. **Composition series and chief series.** A *composition series* for the (finite) group G is a sequence of subgroups

$$G = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{k-1} \supset M_k = (1), \tag{4.1}$$

such that each subgroup M_i is a maximal proper normal subgroup of the group M_{i-1}, for $i=1, \cdots, k$. A *chief series* for the finite group G is a sequence of normal subgroups of G,

$$G = N_m \supset N_{m-1} \supset N_{m-2} \supset \cdots \supset N_1 \supset N_0 = (1), \tag{4.2}$$

such that each N_i is maximal among the proper subgroups of N_{i+1} normal in G. This description is not a dual of the preceding description, but consider instead the quotient groups $Q_i = G/N_i$. Then the chief series becomes a sequence of quotient groups of G,

$$G = Q_0 \geqq Q_1 \geqq Q_2 \geqq \cdots \geqq Q_{m-1} \geqq Q_m = (1'), \tag{4.1$'$}$$

with each Q_i a maximal proper quotient group of Q_{i-1}, for $i=1, \cdots, m$. Conversely, any such sequence of quotient groups of G determines uniquely the corresponding chief series (4.2). The description of (4.1′) is dual to that of (4.1), because the maximal proper quotient groups may be described equivalently as maximal proper *conormal* quotient groups, with our previous definition of conormality. The last term of the series (4.1′) is the unit quotient group $(1') = G/G$ of G, which can be described as that quotient group of G which is a quotient of every quotient group of G. This is dual to a description of the unit subgroup (1) of G.

We next dualize the formula G/M. The quotient group $Q = G/M$, for M normal in G, can be characterized in terms of injections and projections by the properties

(i) $[G/M \leqq G][G \supset M] = 0$;

(ii) If $[R \leqq G][G \supset M] = 0$, then $R \leqq G/M$.

Dually, each conormal quotient group Q of G determines a corresponding normal subgroup of G, denoted by $G \div Q$ and characterized by the dual properties

(i′) $[Q \leqq G][G \supset (G \div Q)] = 0$;

(ii′) If $[Q \leqq G][G \supset S] = 0$, then $S \subset (G \div Q)$,

which state in effect that $G \div Q$ is the kernel of the projection of G on Q. Hence, in ordinary terms,

$$(4.3) \qquad G \div (G/M) = M \qquad G/(G \div Q) = Q,$$

for M a normal subgroup, Q a conormal quotient group of G.

The composition series (4.1) determines a set of quotient groups M_i/M_{i+1}, for $i=0, \cdots, k-1$. The chief series in the dual form (4.1′) determines a set of groups $Q_i \div Q_{i+1}$. Since $Q_i = G/N_i$, $Q_{i+1} = G/N_{i+1}$, the symbol $Q_i \div Q_{i+1}$ designates the kernel of the projection of Q_i onto Q_{i+1}, and this kernel is exactly the group N_{i+1}/N_i. Hence the set of quotient groups of the composition series (4.1) corresponds dually to the set of quotients N_{i+1}/N_i of the chief series (4.2). The Jordan-Hölder theorem asserting the uniqueness of the set of these quotient groups for a finite group G thus becomes the dual of the Jordan-Hölder theorem for the chief series.

5. **Ascending and descending central series.** It has long been recognized that the ascending and the descending central series of a group G (cf. [12; 22, chap. IV]) are dual concepts. We may show that they can be described in strictly dual fashion, in our sense, relative to the group of inner automorphisms of G. For any element a in G we denote by $\phi = \phi_a$ the corresponding inner automorphism of G, with $\phi(g) = aga^{-1}$ for all $g \in G$, and by $I(G)$ the group of all these inner automorphisms. If S is a subgroup of G, then $\phi(S)$ denotes the image of S under ϕ.

The ascending central series $G \supset \cdots \supset Z_2 \supset Z_1 \supset Z_0$ for a group G consists of Z_0, the identity subgroup, Z_1, the center of G, and Z_n, defined inductively as that normal subgroup of G such that Z_n/Z_{n-1} is the center of G/Z_{n-1}. In particular, the center Z_1 consists of all elements of G left fixed by every inner automorphism of G. Thus if S is a normal subgroup of G such that

$$(5.1) \qquad \phi(S) = S, \text{ and } \phi \text{ induces the identity automorphism on } S,$$

for every $\phi \in I(G)$, then $S \subset Z_1$. Since Z_1 itself has the property (5.1), it may be characterized as *the* maximal normal subgroup S of G with this property. This description suggests the following readily proved characterization of the groups $M_n = Z_n$ of the ascending central series.

THEOREM 5.1. *If $I(G)$ is the group of inner automorphisms of G, then there exists a unique sequence $G \supset \cdots \supset M_n \supset \cdots \supset M_1 \supset M_0$ of normal subgroups of G such that M_0 is the identity subgroup and, for each $n > 0$, M_n is the maximal subgroup N of G with the properties*

(i) *$N \supset M_{n-1}$, N normal in G;*

(ii) *For each ϕ in $I(G)$, $\phi(N) = N$, and ϕ induces the identity automorphism on N/M_{n-1}.*

To formulate the dual of (ii), we must use our duality table to interpret the "image" $\phi(N)$ and the "induced" automorphism. Let α be any automorphism of G. If S is any subgroup of G and $[G \supset S]$ the corresponding injection of S into G, then the product $\alpha[G \supset S]$ is an isomorphism of S onto some subgroup T of G, hence may be written uniquely in the form

$$\alpha[G \supset S] = [G \supset T]\alpha_S, \tag{5.2}$$

where α_S is an isomorphism. In this decomposition we may define the group T to be the *image* of S under α, and the isomorphism α_S: $S \to T$ to be the isomorphism *induced* by α on S, in agreement with the usual meaning of an "induced" isomorphism. Dually, let Q be a quotient group of G, and $[Q \leqq G]$ the corresponding projection. The product $[Q \leqq G]\alpha$ is then a homomorphism of G onto Q, hence may be written uniquely in the form

$$[Q \leqq G]\alpha = \alpha_Q[R \leqq G], \tag{5.2'}$$

where R is a quotient group of G, and α_Q an isomorphism of R onto Q. We define R to be the *coimage* of Q under α (in symbols, $R = \phi'(Q)$), and α_Q: $R \to Q$ to be the isomorphism *induced* by α on the coimage. In these terms, condition (ii) of Theorem 5.1 becomes the requirement that for each ϕ there exists an isomorphism ϕ_N: $N \to N$ such that

$$\phi[G \supset N] = [G \supset N]\phi_N, \ [N/M_{n-1} \leqq N]\phi_N = [N/M_{n-1} \leqq N]. \tag{5.3}$$

The dual of Theorem 5.1 may now be formulated as follows.

THEOREM 5.1′. *If $I(G)$ is the group of inner automorphisms of G, then there exists a unique sequence of (conormal) quotient groups $G \geqq \cdots \geqq Q_n \geqq \cdots \geqq Q_1 \geqq Q_0$ of G such that $Q_0 = G/G$ is the identity quotient group and, for each $n > 0$, Q_n is the maximal quotient group R of G with the properties*:

(i′) *$R \geqq Q_{n-1}$, R conormal in G;*

(ii′) *For each $\phi \in I(G)$, $\phi'(R) = R$, and ϕ induces the identity automorphism on $R \div Q_{n-1}$.*

Here again the condition (ii′) means that for each ϕ there exists an isomorphism ϕ_R: $R \to R$ such that

$$\begin{aligned} [R \leqq G]\phi &= \phi_R[R \leqq G], \\ \phi_R[R \supset (R \div Q_{n-1})] &= [R \supset (R \div Q_{n-1})]. \end{aligned} \tag{5.3'}$$

By setting $Q_n = G/L_n$, this theorem may be translated into a theorem about a descending chain of normal subgroups L_n of G. One readily shows that the theorem is valid with L_n the groups of the usual descending central series; that is, with $L_1 = [G, G]$, the subgroup of G generated by all commutators in G, and in general with $L_n = [G, L_{n-1}]$, the subgroup generated by all commutators of elements in G with elements in L_{n-1}.

We have thus shown that in this formulation the descending central series of G is indeed the dual of the ascending central series. We remark in passing that the theorems above are still valid if the group $I(G)$ of inner automorphisms of G is replaced by any group $B(G)$ of automorphisms of G.

It may be noted that our formulation of duality in terms of homomorphisms does not suffice to subsume all known "duality" phenomena. In particular, it does not appear to explain the duality between "verbal" and "marginal" subgroups [13], which is, however, an extension of the above duality between ascending and descending central series.

6. **Functional and axiomatic duality.** For a topological space the duality between homology and cohomology groups with locally compact abelian coefficient groups can be formulated in terms of character groups. Another formulation is suggested by the axiomatic homology theory of Eilenberg and Steenrod [9; 10]. In this formulation, the axioms for a homology theory refer not to elements of the (relative) homology groups, but only to certain homomorphisms; the dual statements are exactly the axioms for a cohomology theory. For example, any continuous mapping $\xi: X \rightarrow Y$ of one space into a second induces a mapping in the *same* direction on the homology groups, and in the *reverse* direction on the cohomology groups of these spaces. One of our chief objectives is that of providing a background in which the proofs for axiomatic homology theory become exactly dual to those for cohomology theory.[4]

Duality phenomena also appear in the case of vector spaces. Each finite-dimensional vector space V over a field F has a dual or conjugate space V^*, consisting of all linear functionals on V to F, and to each linear transformation T: $V \rightarrow W$ there is a conjugate transformation T^*: $W^* \rightarrow V^*$. The passage from a set of transformations to their conjugates inverts the direction of all transformations, interchanges the order of factors in a product of transformations, and replaces a transformation onto by a one-to-one transformation into, hence provides an explicit realization of the type of duality we have discussed. Much the same remarks apply to locally compact abelian groups, under the formation of character groups.

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[4] This consideration was suggested to the author by his study of the manuscript of [10].

In these instances there is a process assigning to each object a dual object and to each transformation a dual transformation, so that a "functional" duality is present. Similarly, the duality of (plane) projective geometry may be formulated in two ways: *functional*, by assigning to each figure its polar reciprocal with respect to a fixed conic; *axiomatic*, by observing that the axioms for plane projective geometry are invariant under the interchange of "point" with "line."

Even for discrete abelian groups or for discrete (infinite-dimensional) vector spaces, a functional duality does not exist. We aim to provide an axiomatic duality covering such cases.

II. BICATEGORIES

7. Categories. The notion of an abstract group arises by consideration of the formal properties of one-to-one transformations of a set onto itself. Similarly, the notion of a category [8] is obtained from the formal properties of the class of all transformations $\xi: X \rightarrow Y$ of any one set into another, or of continuous transformations of one topological space into another, or of homomorphisms of one group into another, and so on. Each transformation ξ is associated with a definite domain X and a definite range Y; the product or composite $\eta\xi$ of two such transformations $\xi: X \rightarrow Y$ and $\eta: Y' \rightarrow Z$ is to be defined only when $Y = Y'$ (range ξ = domain η). With these conventions, one has the following formal properties.[5]

DEFINITION. A *category* $\mathcal{C}$ is a class of elements $\alpha, \beta, \gamma, \cdots$, called "mappings" for certain pairs of which a product $\alpha\beta \in \mathcal{C}$ is defined, subject to the axioms C-0 to C-4.

C-0. (Equality axiom). If $\alpha = \alpha'$, $\beta = \beta'$,and the product $\alpha\beta$ is defined, so is the product $\alpha'\beta'$, and $\alpha\beta = \alpha'\beta'$.

C-1. If the products $\alpha\beta$ and $(\alpha\beta)\gamma$ are defined, so is $\beta\gamma$.

C-1′. If the products $\beta\gamma$ and $\alpha(\beta\gamma)$ are defined, so is $\alpha\beta$.

C-2. (Associative law). If the products $\alpha\beta$ and $\beta\gamma$ are defined, so are the products $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$, and these products are equal.

A mapping I of $\mathcal{C}$ is called an *identity* of $\mathcal{C}$ if (i) II is defined; (ii) $I\alpha = \alpha$ whenever $I\alpha$ is defined; (iii) $\beta I = \beta$ whenever βI is defined.

C-3. (Existence of domain I and range I'). For each $\alpha \in \mathcal{C}$ there are identities I and I' in $\mathcal{C}$ such that αI and $I'\alpha$ are defined.

C-4. For given identities I and I' the class of all mappings α of $\mathcal{C}$ such that both αI and $I'\alpha$ are defined is a set (cf. §8 below).

The axioms C-1, C-1′, and C-2 together state that the associative law $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds, with all terms defined, whenever both products on the left, or both products on the right, or both products

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[5] These axioms (with C-4 omitted) are equivalent to those given in [8].

$\alpha\beta$ and $\beta\gamma$ are known to be defined.

It is convenient to introduce a class of "objects" $A, B, \cdots$ in one-to-one correspondence $A \leftrightarrow I_A$ with the identity mappings I of $\mathcal{C}$. Since the identities I, I' described in C-3 are unique, the object $A = D(\alpha)$ such that αI_A is defined is called the ***domain*** of α, and the object B such that $I_B\alpha$ is defined is the *range* $B = R(\alpha)$; we then write $\alpha: A \rightarrow B$. These objects have the following properties.

(i) $\alpha = \beta$ implies $D(\alpha) = D(\beta)$, $R(\alpha) = R(\beta)$;

(ii) $A = B$ if and only if $I_A = I_B$;

(iii) The product $\alpha\beta$ is defined exactly when $D(\alpha) = R(\beta)$, and then $D(\alpha\beta) = D(\beta)$, $R(\alpha\beta) = R(\alpha)$;

(iv) $D(I_A) = R(I_A) = A$;

(v) $\alpha I_{D(\alpha)} = \alpha = I_{R(\alpha)}\alpha$;

(vi) If $D(\alpha) = R(\beta)$ and $D(\beta) = R(\gamma)$, then $\alpha(\beta\gamma) = (\alpha\beta)\gamma$;

(vii) For given objects A and B, the class of all α with $R(\alpha) = B$, $D(\alpha) = A$ is a set.

Conversely, these seven properties could serve as a definition of a "category with objects"; they imply the original axioms (C-0)–(C-4) for the mappings of such a category.

A mapping θ is an *equivalence* in $\mathcal{C}$ if there are mappings ϕ and ψ in $\mathcal{C}$ such that $\theta\phi$ and $\psi\theta$ are defined and are identities. Then $\phi = \psi$ is the unique inverse θ^{-1} of θ. It is itself an equivalence, and

$$R(\theta^{-1}) = D(\theta), \qquad D(\theta^{-1}) = R(\theta), \qquad (\theta^{-1})^{-1} = \theta.$$

8. **Foundations.** We shall use the category $\mathcal{G}$ of all groups, in which the objects are *all* groups, and the mappings are *all* homomorphisms of one group into another—and similarly the category of all topological spaces and continuous mappings, and so on. The apparently illegitimate totalities of "all" groups and "all" homomorphisms can be justified by using the standard von Neumann-Bernays-Gödel axiomatics for set theory [2; 11] in which both the notions of "class" and "set" appear, the sets being more restricted than the classes. We then understand a group to be a *set* G, with multiplication defined by a suitable *set* of triples; a homomorphism is likewise described by *sets*. With the cited axioms for set theory, one can then correctly speak of the *class* of all groups and of the *class* of all homomorphisms of one group into another. For this reason, we have described a category as a *class* and have inserted axiom C-4.

9. **Bicategories.** The primitive concepts of a category are not sufficient to formulate all the duality phenomena, and in particular do not provide for "subgroups versus quotient groups," or "homomorphisms onto versus isomorphisms into." To extend our formula-

tion, we axiomatize the terms "injection homomorphism of a subgroup into a larger group" and "projection homomorphism of a group onto a quotient group." We can then define homomorphisms onto and isomorphisms into as "supermaps" and "submaps," respectively.

DEFINITION. A *bicategory*[6] $\mathcal{C}$ is a category with two given subclasses of mappings, the classes of "injections" (κ) and "projections" (π) subject to the axioms BC-0 to BC-6 below.[7]

BC-0. A mapping equal to an injection (projection) is itself an injection (projection).

BC-1. Every identity of $\mathcal{C}$ is both an injection and a projection.

BC-2. If the product of two injections (projections) is defined, it is an injection (projection).

BC-3. (Canonical decomposition). Every mapping α of the bicategory can be represented uniquely as a product $\alpha=\kappa\theta\pi$, in which κ is an injection, θ an equivalence, and π a projection.

Any mapping of the form $\lambda=\kappa\theta$ (that is, any mapping with π equal to an identity in the canonical decomposition) is called a ***submap***; any mapping of the form $\rho=\theta\pi$ is called a ***supermap***.

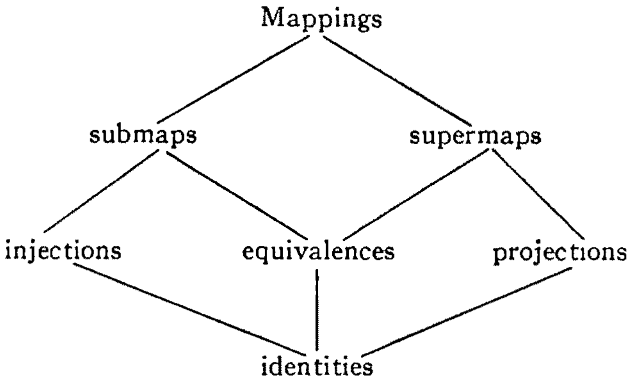
BC-4. If the product of two submaps (supermaps) is defined, it is a submap (supermap).

Any product $\kappa_1\pi_1 \cdots \kappa_n\pi_n$ of injections κ_i and projections π_i is called an *idemmap*.

BC-5. If two idemmaps have the same range and the same domain, they are equal.

BC-6. For each object A, the class of all injections with range A is a set, and the class of all projections with domain A is a set.

The inclusion relations between the various classes of mappings can be represented by the following Hasse diagram.



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[6] The term "bicategory" was suggested by Professor Grace Rose.

[7] In the preliminary announcement [16], axiom BC-6 did not appear, and axiom BC-5 was present only in weaker form.

The axioms show that the intersection of any two such classes is exactly the intersection which may be read off this diagram. Each class of mappings is closed under multiplication (whenever the product is defined). It is convenient to reserve the letters λ, μ, ν for submaps, ρ, σ, τ for supermaps.

10. **The duality principle.** The concept of the "dual" of a statement about homomorphisms may now be defined precisely. In a category, the only primitive statements are statements of the forms

$$\alpha = \beta, \qquad \alpha\beta = \gamma; \tag{10.1}$$

We interpret the latter to mean "the product $\alpha\beta$ is defined and is equal to γ." All other statements can be expressed in terms of these primitive statements; in particular, we understand the statement "$\alpha\beta$ is defined" to be interpreted as "there exists a γ such that $\alpha\beta=\gamma$." A *first order statement* S in a finite number of letters (which designate mappings of the category in question) is any statement formed from a number of primitive statements of the types (10.1), combined by the standard logical connectives (including quantifiers "for all α" and "$\exists\alpha$"). The *dual* of S is the statement obtained from S by the following typographical process: replace each primitive statement $\alpha\beta=\gamma$ by the statement $\beta\alpha=\gamma$, leaving the other primitive statements, all the letters, and all the logical connectives unchanged. The dual is thus obtained by "inverting all products." A similar process will apply to statements which are not of the first order, in that they involve variables for sets of mappings, sets of sets of mappings, and so on.

The dual of any axiom for a category is also an axiom; in particular, C-1′ is (except for change in notation) the dual of C-1, and the other axioms are self-dual. A simple metamathematical argument thus proves the

DUALITY PRINCIPLE. If any statement about a category is deducible from the axioms for a category, the dual statement is likewise deducible.

In a bicategory, the only added primitive statements are

(10.2) α is an injection, α is a projection.

The dual of a statement about mappings in a bicategory is now obtained as before, with the added interchange of the terms "injection" and "projection." An inspection of the axioms shows that the duality principle holds for bicategories also.

A statement of group theory may often be formulated in (bi-) *categorical form*; that is, as a statement about homomorphisms, injections, projections, identities, and their products. When so formulated, it has a definite dual, but note that there may be several such formulations which lead to essentially different duals. For example, "Q is a quotient group of G" (that is, there is a projection with domain G and range Q) is equivalent to "Q is a conormal quotient group of G." The duals—"M is a subgroup of G" and "M is a normal subgroup of G"—are not equivalent.

11. **Partial order in a bicategory.** The axioms (especially axiom BC-5) suffice to introduce a relation of partial order (under "inclusion") in the objects of a bicategory. We define a mapping β to be *left cancellable* in a category if $\beta\alpha_1=\beta\alpha_2$ always implies $\alpha_1=\alpha_2$, and *left invertible* if β has a left inverse γ, with $\gamma\beta=I_{D(\beta)}$. One may readily prove, in succession, the following results.

LEMMA 11.1. *Two injections κ_1 and κ_2 such that $\kappa_1\kappa_2$ is an identity are themselves identities.*

LEMMA 11.2. *Every right factor of a submapping is a submapping.*

LEMMA 11.3. *If $\alpha\beta$ is an identity, α is a supermap and β a submap.*

LEMMA 11.4. *Every left invertible mapping is a submap, and every submap is left cancellable.*

THEOREM 11.5. *The class of objects in a bicategory is partially ordered by either of the relations*

(11.1) *$S\subset B$ if and only if there is an injection κ: $S\rightarrow B$;*
(11.1′) *$Q\leqq A$ if and only if there is a projection π: $A\rightarrow Q$.*

If $S\subset B$, we call S a *subobject* of B, while if $Q\leqq A$, Q is a *quotient-object* of A, the terms corresponding to those in group theory. By axiom BC-5 the mappings κ and π which appear in the dual definitions (11.1) and (11.1′) are unique; it is more suggestive to denote them as

$$(11.2)\qquad \kappa = [B \supset S]: S \rightarrow B; \qquad \pi = [Q \leqq A]: A \rightarrow Q.$$

Thus $[B\supset S]$ is a mapping, defined precisely when $S\subset B$ and is then an injection; every injection has this form. The notation is so chosen that

$$(11.3)\quad [B \supset S][S \supset T] = [B \supset T], \ \ [R \leqq Q][Q \leqq A] = [R \leqq A],$$

by BC-5, whenever the terms on the left are defined.

In examining prospective examples of bicategories, it is easier to formulate the axioms directly in terms of these constructions on the objects.

DEFINITION. A *bicategory with objects* is a category with objects, in the sense of §7, in which there are mappings $[B\supset S]$ and $[Q\leqq A]$ assigned to certain pairs of objects B, S or Q, A, subject to conditions (i)–(vi) below and their duals. Here "$S\subset B$" means "the mapping $[B\supset S]$ is defined," and dually, while an equation involving the mappings $[B\supset S]$, $\cdots$ is understood to include the assertion that $[B\supset S]$, $\cdots$ is defined. An idemmap is any product $\alpha_1\alpha_2\cdots\alpha_m$, where each α_i has one of the forms $[B\supset S]$ or $[Q\leqq A]$. The axioms are

(i) (Equality). $B=B'$, $S=S'$, and $S\subset B$ imply $[B\supset S]=[B'\supset S']$.

(ii) For all objects A, $[A\supset A]=I_A=[A\leqq A]$.

(iii) Every mapping α has a unique representation

$$\alpha = [B\supset S]\theta[Q\leqq A],\qquad \theta \text{ an equivalence.} \tag{11.4}$$

(iv) If θ: $A\rightarrow B$ is an equivalence and $T\subset A$, there is an object S and an equivalence θ' such that

$$\theta[A\supset T] = [B\supset S]\theta'. \tag{11.5}$$

(v) If two idemmaps have the same range and the same domain, they are equal.

(vi) For each object A, the class of all $Q\leqq A$ is a set.

Note that axiom (v) includes the statement (11.3).

Every bicategory determines a bicategory with objects; conversely, the mappings of a bicategory with objects form a bicategory, if the injections are the mappings $[B\supset S]$ and the projection the mappings $[Q\leqq A]$.

In the canonical decomposition (11.4) call S the *image* of α, Q the *coimage* of α; in symbols,

$$\operatorname{Im}(\alpha) = S\subset R(\alpha),\qquad \operatorname{Coim}(\alpha) = Q\leqq D(\alpha). \tag{11.6}$$

By (vi), each object A determines the *set* $\mathfrak{S}(A)$ of all subobjects T, with $T\subset A$, and the set $\mathfrak{Q}(A)$ of all quotients $R\leqq A$. Given α: $A\rightarrow B$, each $T\subset A$ has an "image" $\alpha_s T\subset B$, and dually, as defined by

$$\alpha_s T = \operatorname{Im}(\alpha[A\supset T])\qquad \alpha_q Q = \operatorname{Coim}([Q\leqq B]\alpha). \tag{11.7}$$

Then α_s is an (inclusion) order preserving transformation of the set $\mathfrak{S}(A)$ into the set $\mathfrak{S}(B)$, and, if $\alpha\beta$ is defined, $(\alpha\beta)_s=\alpha_s\beta_s$. One readily proves the following theorem.

THEOREM 11.6. *If α is an identity map, α_s is the identity. If θ is an equivalence θ: $A\rightarrow B$, θ_s is a one-to-one transformation of $\mathfrak{S}(A)$ onto $\mathfrak{S}(B)$, with inverse $(\theta^{-1})_s$. If κ is an injection κ: $S\rightarrow B$, κ_s is the identity transformation of $\mathfrak{S}(S)$ into $\mathfrak{S}(B)$. If λ: $A\rightarrow B$ is a submapping, λ_s*

is a one-to-one transformation of $\mathfrak{S}(A)$ *onto the subset* $\mathfrak{S}(\lambda_* A)$ *of* $\mathfrak{S}(B)$.

A corresponding result for projections or supermappings does not hold in general. Note that the dual of Theorem 11.6 will assert, for example, that if $\rho: A \rightarrow B$ is a supermapping, then ρ_q is a one-to-one transformation $\mathfrak{Q}(B)$ onto a subset $\mathfrak{Q}(\rho_q B)$ of $\mathfrak{Q}(A)$—the logical phrases (one-to-one onto, and so on) are *not* changed by dualization.

12. **Equality and examples of bicategories.** We have already observed in §2 that groups and homomorphisms, with the natural interpretation of an injection as the identity mapping applied to a subgroup and of a projection as a canonical homomorphism of G on G/N, do not satisfy the axioms for a bicategory, because BC-2′, on products of projections, fails. We avoid this difficulty by abandoning the current coset fashion for the classical congruence idea, and regarding G/N as a group whose elements are simply the elements of G, with a new equality:—$g_1 \equiv g_2 \pmod{N}$ if and only if $g_1 g_2^{-1} \in N$. A quotient group of a quotient group is then a quotient group.

More explicitly, we regard a group G as the mathematical system $[M_G, =_G, \times_G]$ consisting of a set M_G, a relation $=_G$, reflexive, symmetric, and transitive for the elements of M_G, and a binary operation $\times_G$ defined for all pairs of elements of M_G. To all the usual axioms and definitions we then append the appropriate equality axioms; for the group axioms they are

$$\begin{gathered} a =_G b \text{ implies } a \in M_G, \quad b \in M_G, \\ a =_G b \text{ and } g =_G h \text{ imply } a \times_G g =_G b \times_G h; \end{gathered}$$

for a homomorphism $\alpha: G \rightarrow H$ they are

$$\begin{aligned} & a =_G b \text{ implies } \alpha a =_H \alpha b, \\ & \alpha(a \times_G b) =_H (\alpha a) \times_H (\alpha b), \qquad \text{for } a, b \in M_G; \end{aligned}$$

for a subgroup $S \subset G$ they are

$$\begin{aligned} & a =_S b \text{ if and only if } a =_G b, \qquad \text{for all } a \in M_S, \\ & a \times_S b =_G a \times_G b, \qquad \text{for all } a, b \in M_S. \end{aligned}$$

Finally, a quotient group Q of G is a group such that

$$\begin{aligned} & h \in M_Q \text{ implies } h \in M_G, \\ & a =_G b \text{ implies } a =_Q b, \qquad \text{for } a, b \in M_G, \\ & a \times_G b =_Q a \times_Q b \qquad \text{whenever } a, b \in M_Q. \end{aligned}$$

It follows that M_G and M_Q have the same elements.

With similar definitions for the equality of groups and homomorphisms, and with an injection (projection) defined as a homomorphism α of a subgroup S into G (of G upon a quotient group) determined by the identity function j, it readily follows that the class of all groups and homomorphisms constitutes a bicategory with objects. Similarly, we have the bicategory of all abelian groups, of all finite groups, of all rings, and so on.

The bicategory $\mathfrak{S}$ of nonvoid sets will thus be interpreted as the bicategory of sets where each set S carries with it an equivalence relation $=_S$, and with the appropriate transformations as mappings. This bicategory has several special properties. The axiom of choice is equivalent to the theorem that for every mapping ξ of $\mathfrak{S}$ there is a mapping ξ^* of $\mathfrak{S}$ with $\xi\xi^*\xi=\xi$. The left cancellable mappings are identical with the left invertible mappings and hence with the submappings (Lemma 11.4). The dual statement is also true. However, the dual of a true statement about the bicategory of sets need not always be true. For example, there exists a subclass $\mathfrak{S}^*$ of mappings which contains exactly one mapping with any given domain and any given range, and such that any right multiple of a mapping of $\mathfrak{S}^*$ is also in $\mathfrak{S}^*$. The dual assertion is false.

The bicategory of topological spaces has as objects all topological spaces and as mappings all continuous transformations of one such space into another. The definitions are again the standard ones, *plus* equality axioms; in particular Q is a quotient space of X if the set M_Q is a quotient set of M_X, and if the identity transformation of M_X onto M_Q induces a continuous transformation of X onto Q. Thus, if X is decomposed into disjoint subsets, the usual decomposition space Q may be interpreted as a quotient space of X—but these are not the only quotient spaces of X, since the same set M_Q and the same equality may form a quotient space with *fewer* open sets. The axioms for a bicategory may be verified, the essential feature being the unique factorization axiom BC-3, for a continuous transformation $\xi: X\rightarrow Y$. If we define the image $\xi(X)\subset Y$ with the usual relative topology, the set quotient of the transformation ξ is then a set $Q\leq X$ in one-to-one correspondence with the image space $\xi(X)$; using this correspondence we impose a topology on Q which makes Q a quotient space of X. This gives the factorization of ξ, the uniqueness following readily. The essential feature of this argument is the fact that every continuous image $\xi(X)$ of a topological space in another such is itself a topological space. We may thus speak of the bicategories of all T_0, T_1, T_2, or all compact Hausdorff spaces.

13. Universal algebra. These examples indicate that most types of algebraic, topological, or other mathematical systems, together with the appropriate type of transformations, yield bicategories. The bicategory language appears to be the appropriate vehicle for many of the theorems of universal algebra (cf. [3; 4; 6; 15; 19; 20])—often giving simpler formulations, because the axiomatic formulation avoids the inevitably cumbersome explicit description of the general form of any algebraic or mathematical system. This is especially the case when universal algebra is extended to include those algebraic systems, which occur so frequently, in which several groups, homomorphisms, functions, and so forth, together constitute a single algebraic system. Using the notions of covariance and contravariance [8] one can in fact give a general definition of mathematical systems and prove, under general hypotheses, that the class of all systems of a given type is the class of objects of a bicategory.

The crucial point of this development is the definition of the "homomorphisms" appropriate to the type of system at hand. For example, a topological space may be regarded as a set X together with a suitable, selected subset $\mathcal{U}_X$ (open sets) of 2^X. A homomorphism (continuous transformation) ξ: $X \to Y$ must then carry $\mathcal{U}_Y$ into $\mathcal{U}_X$; in other words 2^X must be regarded as a contravariant functor of X (and not, as in some other cases, as a covariant functor). The general definition can then be extended to include algebraic systems defined by functors contravariant in one argument and covariant in another —a good illustrative example being the algebraic homotopy types considered in [17].

Leaving this development aside, we shall next show merely that the axioms for a bicategory can be extended so as to include also all the phenomena of universal algebra treated by lattice-theoretic means (the new axioms being valid in all standard examples).

14. **Lattice ordered categories.** A lattice ordered category (LC) is a bicategory $\mathcal{C}$ satisfying two[8] additional axioms (and their duals).

LC-1. For each object A, every nonvoid subset of $\mathfrak{S}(A)$ has a least upper bound (l.u.b.) in the partially ordered set $\mathfrak{S}(A)$.

Here $\mathfrak{S}(A)$ denotes the set of all subobjects $S, T, \cdots$ of A with the partial order $S \subset T$. The l.u.b. of a collection $\{S_i\}$ of subobjects will be denoted by $\sum_A S_i$, that of two subobjects by $S_1 \cup_A S_2$.

LC-2. If $\alpha: A \to B$ and $\{S_i\}$ is a nonvoid subset of $\mathfrak{S}(A)$,

$$\alpha_*\left(\sum_A S_i\right) \subset \sum_B \alpha_* S_i. \tag{14.1}$$

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[8] It might be desirable to add an axiom LC-3 (and its dual) requiring that $\alpha_1, \alpha_2: A \to B$, $A = \sum_A S_i$, and $\alpha_1[A \supset S_i] = \alpha_2[A \supset S_i]$ for all i imply that $\alpha_1 = \alpha_2$.

Since α_s preserves order, we deduce that equality actually holds in (14.1). A nonvoid set of objects S_i with upper and lower bounds $C \subset S_i \subset A$ has, by the usual proof, a g.l.b. $\prod S_i$ which is independent of the choice of C and A. If $\alpha: A \to B$, then $\alpha_s(\prod S_i) \subset \prod \alpha_s S_i$, with equality when α is a submapping (the latter by Theorem 11.6). The set of all S with $C \subset S \subset A$ is a complete lattice [5]. The l.u.b. $\sum_A S_i$ may depend on the choice of the "universe" A, but

$$(14.2) \qquad \sum_A S_i = \sum_B S_i, \qquad \text{if } S_i \subset A \subset B, \text{ all } i.$$

If we use only LC-1, the *inverse image* of any subobject $T \subset B$ under a mapping $\alpha: A \to B$ is defined as

$$(14.3) \qquad \alpha_s^* T = \sum_A S, \qquad \text{over all } S \subset A \text{ with } \alpha_s S \subset T.$$

In this notation LC-2 is equivalent to the requirement that

$$(14.4) \qquad \alpha_s \alpha_s^* T \subset T, \qquad \text{for all } \alpha: A \to B,\ T \subset B.$$

Also, $S = \alpha_s^* T$ is characterized as a subobject of A by the properties (i) $\alpha_s S \subset T$; (ii) $\alpha_s S' \subset T$ for $S' \subset A$ implies $S' \subset S$. Furthermore, α_s^* is an order preserving transformation of $\mathcal{S}(B)$ into $\mathcal{S}(A)$, with $(\alpha\beta)_s^* = \beta_s^* \alpha_s^*$ whenever $\alpha\beta$ is defined. One has also the following theorem.

THEOREM 14.1. *If $\alpha: A \to B$ is an identity, so is α_s^*; if α is an equivalence, α_s^* is a one-to-one transformation of $\mathcal{S}(B)$ onto $\mathcal{S}(A)$, with α_s as inverse, while if α is a submapping, α_s^* maps $\mathcal{S}(B)$ onto $\mathcal{S}(A)$. If $\kappa = [A \supset S]$ is an injection, and $T \subset A$, then $\kappa_s^* T = T \cap S$.*

The dual statements hold; in particular any two quotients $Q_1 \leqq A$ and $Q_2 \leqq A$ have an l.u.b. $Q_1 \vee Q_2$ and a g.l.b. $Q_1 \wedge Q_2$, the latter whenever they have some lower bound. For $\alpha: A \to B$ the inverse coimage α_q^* yields a transformation of $\mathfrak{Q}(A)$ into $\mathfrak{Q}(B)$.

The notion of a lattice ordered category is a simultaneous generalization of the notions of group and lattice; specifically, every group is a lattice ordered category, in which every mapping is an equivalence and there is only one identity, and every lattice is a lattice ordered category in which every mapping is an injection (or dually).

15. **Zeros and zero mappings.** A category $\mathcal{C}$ has a *zero* if it satisfies the following axiom.

Z. There is an object Z such that for every object A of $\mathcal{C}$ there exists exactly one mapping $\zeta: A \to Z$ and exactly one mapping $\eta: Z \to A$. We call Z a *zero* object.

In the category of groups (abelian groups, rings) any group with only the identity element (ring with only the zero) is a zero in this sense. The category of topological space does not satisfy axiom Z; however that of topological spaces with a distinguished base point (as used in homotopy theory) does. The investigations [15] of Jónsson and Tarski have already indicated that the presence (or absence) of a zero is of critical importance in many investigations of universal algebra.[9]

Given axiom Z, it follows at once that any two zero objects are equivalent. A mapping α is said to be a *zero mapping* if it can be factored as $\alpha=\eta\zeta$, where $R(\zeta)=D(\eta)=Z$. This definition is independent of the choice of the zero Z. Given any two objects A and B in the category, there is exactly one zero mapping α: $A\rightarrow B$; we denote this mapping as 0_{BA}, so that $0_{CB}0_{BA}=0_{CA}$. In the category of all groups 0_{BA} is the homomorphism mapping all elements of A into the identity element of B. If $\alpha=0$ (that is, if α is a zero mapping), then $\alpha\beta=0$ and $\gamma\alpha=0$ whenever these products are defined.

Under the axioms BCZ (bicategory with a zero) we define, for each object A,

$$(15.1)\qquad 1_A = \mathrm{Im}\ (0_{AZ}),\qquad 1'_A = \mathrm{Coim}\ (0_{ZA}).$$

For groups, 1_A is the identity subgroup of A, and $1'_A$ the quotient group A/A. In general (BCZ) $1_A\subset A$; for $S\subset A$, $1_A\subset S\subset A$; and 1_A is a zero object. The mapping 0_{ZA} is a supermap, while 0_{AZ} is a submap, and any zero mapping 0_{BA} has

$$(15.2)\qquad 0_{BA} = [B\supset 1_B]0_{1_B1'_A}[1'_A \leqq A]$$

as its canonical decomposition.

Lemma 15.1 (BCZ). *If λ is a submap and ρ a supermap, then*

$$(15.3)\qquad \lambda\beta = 0 \textit{ implies } \beta = 0;\qquad \alpha\rho = 0 \textit{ implies } \alpha = 0.$$

Proof. Since λ is a submap, $\lambda=\kappa\theta$ with κ an injection, θ an equivalence. Let $\theta\beta$ have the canonical decomposition $\theta\beta=\kappa_1\theta_1\pi_1$. Then $\lambda\beta=\kappa\theta\beta$ has the canonical decomposition $(\kappa\kappa_1)\theta_1\pi_1$. This must agree with a canonical decomposition of the form (15.2), whence $\theta_1=0$ and $\beta=\theta^{-1}(\theta\beta)=\theta^{-1}\kappa_1\theta_1\pi_1=0$.

Finally, under the axioms LCZ (lattice ordered category with Z) we observe that each $\mathfrak{S}(A)$ and each $\mathfrak{Q}(A)$ is a complete lattice (with unit 1_A resp. $1'_A$) and that

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[9] The Jónsson-Tarski zeros give a zero in the sense of this axiom, but the author has been unable to prove their direct decomposition theorems in categorical form.

(15.4) $\alpha\colon A \to B$ implies $\alpha_s 1_A = 1_B$, $\alpha_q 1'_B = 1'_A$.

The axioms LCZ hold in the bicategories of all groups, of all rings, of all vector spaces over a fixed division ring, and so on.

16. **Normality and kernels.** A categorical definition of normal subobject can be formulated conveniently under the axioms LCZ. For subobjects $S \subset A$ and quotient objects $Q \leqq A$ we define an orthogonality relation

(16.1) $S \perp_A Q$ if and only if $[Q \leqq A][A \supset S] = 0$;

using the axiom LC-1 we then give dual definitions:

(16.2) $A/S =$ l.u.b. in $\mathfrak{Q}(A)$ of all $Q \leqq A$ with $S \perp_A Q$,

(16.3) $A \div Q =$ l.u.b. in $\mathfrak{S}(A)$ of all $S \subset A$ with $S \perp_A Q$.

For groups, A/S will then be the usual factor group A/N, where $N \supset S$ is the least normal subgroup of A containing S. In general, one then proves, using LC-2 in (16.5), that

(16.4) $A/1_A = A$, $A/A = 1'_A$,

(16.5) $S \perp_A (A/S)$, $(A \div Q) \perp_A Q$,

(16.6) $S_1 \subset S_2$ implies $A/S_2 \leqq A/S_1$,

(16.7) $S \subset A \div (A/S)$, $A/S = A/(A \div (A/S))$;

in other words $S \to A/S$, $Q \to A \div Q$ provides a Galois connection [5, p. 56] between $\mathfrak{S}(A)$ and $\mathfrak{Q}(A)$.

Finally, we define

(16.8) S is *normal* in A if and only if $S = A \div (A/S)$,

and dually. Alternative definitions are given by the following theorem.

THEOREM 16.1. *A subobject $S \subset A$ is normal in A if and only if any one of the following conditions hold.*

(i) *There exists $Q \leqq A$ such that $S = A \div Q$;*

(ii) *There exists a Q conormal in A such that $S = A \div Q$;*

(iii) *If, for all Q, $S \perp_A Q$ implies $S' \perp_A Q$, then $S' \subset S$;*

(iv) *If, for all α: $A \to B$, $\alpha[A \supset S] = 0$ implies $\alpha[A \supset S'] = 0$, then $S' \subset S$;*

(v) *There exists a $Q \leqq A$ such that $S \perp_A Q$ and $S' \subset S$ whenever $S' \perp_A Q$.*

The proof of (iv) makes essential use of Lemma 15.1. Also, it may readily be proved that 1_A and A are normal in A, that $N \subset T \subset A$ with N normal in A implies N normal in T, and that the intersection

πN_i of objects normal in A is normal in A. (Apparently the similar assertion for union, though true for groups, is not a consequence of the axiom LCZ.)

For any mapping $\alpha: A \rightarrow B$ the kernel $K(\alpha)$ may be defined as

$$K(\alpha) = A \div (\text{Coim } \alpha) = \alpha_s^* 1_B. \tag{16.9}$$

Then a subobject S is normal in A if and only if it is the kernel of some α; when so, it is in fact the kernel of $[A/S \leqq A]$. Furthermore $K(\alpha\beta) \supset K(\beta)$, and, for a canonical decomposition, $K(\kappa\theta\pi) = K(\pi)$.

A further development giving the first and second isomorphism theorems, and so on, can be made by introducing additional carefully chosen dual axioms. This will be done below only in the more symmetrical abelian case. It is also possible to give a definition of normality in lattice ordered categories without a zero, by using the criterion (iv) of Theorem 16.1, with the zero object suitably replaced by objects acting like spaces or sets with only one element. This definition applies in any category, agrees with the above definition when a zero is present, and in the category of T_1 spaces has the amusing property that the normal subobjects of a topological space are exactly the closed subsets of that space! (Because of the failure of separation axioms in a decomposition space, this is *not* true in the category of T_2 spaces.) However, in the category of all compact Hausdorff spaces (with mappings all continuous transformations) every subspace (being closed) is normal, but there are nonconormal quotient spaces. This reverses the group-theoretic phenomenon that every quotient group is conormal.

III. ABELIAN CATEGORIES

17. The group of integers. Our objective in this chapter is that of providing a self-dual set of axioms for abelian groups and their homomorphisms sufficient to prove all categorical theorems which refer to a finite number of such groups—and hence adequate to explain the apparent perfect duality present for such theorems on abelian groups. We also obtain a representation theorem for certain abstract categories, using the following purely categorical characterization of the additive group of integers.

DEFINITION. An object J in a category $\mathcal{C}$ is *integral* in $\mathcal{C}$ if

(i) For two distinct mappings $\alpha_1, \alpha_2: G \rightarrow H$ of $\mathcal{C}$ with the same domain G and the same range H there exists a mapping $\beta: J \rightarrow G$ such that $\alpha_1\beta \neq \alpha_2\beta$.

(ii) If J' is another object of $\mathcal{C}$ with the property (i), there exists in $\mathcal{C}$ a mapping $\sigma: J' \rightarrow J$ with a right inverse in $\mathcal{C}$.

(iii) If $\alpha_1\alpha_2=I_J$ for two mappings α_1, α_2: $J\rightarrow J$, then α_1 and α_2 are equivalences.

One then readily proves the following theorems.

THEOREM 17.1. *The additive group of integers is integral* (*satisfies* (i), (ii), *and* (iii)) *in the category of all abelian groups.*

THEOREM 17.2. *In any category, any two objects with the properties* (i), (ii), *and* (iii) *are equivalent.*

PROOF. Given J, J' both integral, there exist by (ii) mappings σ: $J'\rightarrow J$ and τ: $J\rightarrow J'$, each with right inverse. Then $\tau\sigma$: $J'\rightarrow J'$, and $\sigma\tau$: $J\rightarrow J$ have right inverses, hence by (iii) are equivalences. Therefore $(\tau\sigma)^{-1}\tau\sigma=I_{J'}$, and σ has a left inverse; therefore σ is an equivalence σ: $J'\rightarrow J$, as asserted.

A somewhat more difficult proof gives the following theorem.

THEOREM 17.3. *The additive group L of rational numbers modulo* 1 *is cointegral* (*satisfies the duals of* (i), (ii), *and* (iii)) *in the category of all abelian groups.*

Hence the group L can also be characterized up to isomorphism. We note by the way that the space consisting of one point only is integral in the category of all topological spaces, but we do not know of (and doubt the existence of) a "cointegral" object in this category. However, in the category of all vector spaces over a fixed division ring D, the vector space D is both integral and cointegral.

18. **Abelian categories.** For abelian groups and for vector spaces the cartesian product (of a finite number of factors) can be regarded simultaneously as a free and a direct product, in the sense of §3 above. We thus introduce corresponding axioms on a category.

DEFINITION. In a category with zero, a (simultaneous) free-and-direct product of objects A and B is a diagram

$$A \leftrightarrows A \times B \rightleftarrows B \tag{18.1}$$

consisting of an object $A\times B$ and four mappings Γ^1, Γ^2, Λ^1, Λ^2,

$$\begin{array}{ll} \overset{1}{\Gamma}_{A\times B}: A \times B \rightarrow A, & \overset{2}{\Gamma}_{A\times B}: A \times B \rightarrow B, \\ \overset{1}{\Lambda}_{A\times B}: A \rightarrow A \times B, & \overset{2}{\Lambda}_{A\times B}: B \rightarrow A \times B, \end{array} \tag{18.2}$$

with the following three properties (omitting the subscript $A\times B$ to simplify notation):

(i) $\Gamma^1\Lambda^1=I_A$, $\Gamma^1\Lambda^2=0_{AB}$, $\Gamma^2\Lambda^1=0_{BA}$, $\Gamma^2\Lambda^2=I_B$;

(ii) For any pair of mappings α_1: $C\rightarrow A$, α_2: $C\rightarrow B$, there exists a unique γ: $C\rightarrow A\times B$ with $\Gamma^1\gamma=\alpha_1$, $\Gamma^2\gamma=\alpha_2$;

(iii) For any pair of mappings β_1: $A\rightarrow D$, β_2: $B\rightarrow D$, there exists a unique δ: $A\times B\rightarrow D$ with $\delta\Lambda^1=\beta_1$, $\delta\Lambda^2=\beta_2$.

Here property (ii) asserts that we have a direct product and (iii) that we have a free product, both in the sense of §3.

Definition. An abelian category (AC) is a category $\mathcal{C}$ satisfying the axiom Z (existence of zero) and the axioms

AC-1. There exists an integral and a cointegral object in $\mathcal{C}$.

AC-2. There exists in $\mathcal{C}$ a free-and-direct product for any two objects of $\mathcal{C}$.

Theorem 18.1. *Any two free-and-direct products of two objects A and B have isomorphic diagrams* (18.1).

If $A*B$ is the second product, the isomorphism of the diagrams means that there is an equivalence mapping θ: $A\times B\rightarrow A*B$ with

$$\theta\overset{i}{\Lambda}_{A\times B}=\overset{i}{\Lambda}_{A*B},\qquad \overset{i}{\Gamma}_{A*B}\theta=\overset{i}{\Gamma}_{A\times B},\qquad i=1,2. \tag{18.3}$$

The proof is straightforward. This result and a similar theorem for triple products will prove the following theorem.

Theorem 18.2. *The operation of forming free-and-direct products is commutative and associative* (*in the sense of diagram isomorphism*).

This operation also has the zero object Z as identity, as follows:

Theorem 18.3. *The diagram*

$$\begin{tikzcd} G \arrow[r, shift right] & G \arrow[l, shift right] \arrow[r, shift left] & Z \arrow[l, shift left] \end{tikzcd}$$

with identity mappings on the left, zero mappings on the right, represents G as the free-and-direct product $G\times Z$.

Two homomorphisms α_1, α_2 for groups have a cartesian product defined by

$$(\alpha_1\times\alpha_2)(a, b)=(\alpha_1 a, \alpha_2 b).$$

This product may be introduced on the basis of our axioms.

Theorem 18.4. *Given mappings α_1: $A\rightarrow A'$, α_2: $B\rightarrow B'$, there exists one and only one mapping $\alpha_1\times\alpha_2$: $A\times B\rightarrow A'\times B'$ with*

$$\overset{i}{\Gamma}_{A'\times B'}(\alpha_1\times\alpha_2)=\alpha_i\overset{i}{\Gamma}_{A\times B},\quad (\alpha_1\times\alpha_2)\overset{i}{\Lambda}_{A\times B}=\overset{i}{\Lambda}_{A'\times B'}\alpha_i,\quad i=1,2. \tag{18.4}$$

The mapping $\alpha_1\times\alpha_2$ is determined by either one of these two equations.

The proof (here and elsewhere) may be visualized most readily by drawing the appropriate diagrams. By (ii), there exists a unique mapping $\gamma = \alpha_1 \times \alpha_2$ with the first property of (18.4). Then by (i)

$$\begin{aligned} \Gamma'^{i}(\alpha_1 \times \alpha_2)\Lambda^{j} = \alpha_i \Gamma^{i} \Lambda^{j} &= \alpha_i && (\text{for } i = j) \\ &= 0 && (\text{for } i \neq j). \end{aligned}$$

But on the other hand, by (i),

$$\begin{aligned} \Gamma'^{i}\Lambda'^{j}\alpha_j &= \alpha_i && (\text{for } i = j) \\ &= 0 && (\text{for } i \neq j). \end{aligned}$$

Hence, by the uniqueness assertion of (ii) for $A' \times B'$, the second equation of (18.4) holds.

One shows also that $0 \times 0 = 0$ and $I_A \times I_B = I_{A \times B}$.

For the cartesian square $A \times A$ of an abelian group there is a *diagonal* homomorphism

$$\nabla_A : A \to A \times A \qquad \text{with } \nabla_A(a) = (a, a)$$

and a *codiagonal* homomorphism

$$\Delta_A : A \times A \to A \qquad \text{with } \Delta_A(a, b) = a + b.$$

Under our axioms, these mappings can be characterized dually as the unique mappings such that

$$\nabla_A : A \to A \times A, \qquad \Gamma^{i}_{A \times A} \nabla_A = I_A, \qquad i = 1, 2; \tag{18.5}$$

$$\Delta_A : A \times A \to A, \qquad \Delta_A \Lambda^{i}_{A \times A} = I_A, \qquad i = 1, 2. \tag{18.5$'$}$$

The mappings Γ^i, Λ^i, γ, and δ appearing in the definition of the free-and-direct product may then be expressed as

$$\Gamma^1 = \Delta_A(I_A \times 0_{AB}), \qquad \Gamma^2 = \Delta_B(0_{BA} \times I_B), \tag{18.6}$$

$$\Lambda^1 = (I_A \times 0_{BA})\nabla_A, \qquad \Lambda^2 = (0_{AB} \times I_B)\nabla_B, \tag{18.6$'$}$$

$$\gamma = (\alpha_1 \times \alpha_2)\nabla_C, \qquad \delta = \Delta_D(\beta_1 \times \beta_2). \tag{18.7}$$

THEOREM 18.5. *The axiom* AC-2 *asserting the existence of free-and-direct products may be replaced by the assumption that there exists to each pair of objects an object $A \times B$, to each pair of mappings $\alpha_1 : A \to A'$, $\alpha_2 : B \to B'$ a mapping $\alpha_1 \times \alpha_2 : A \times B \to A' \times B'$, and to each object A two mappings $\Delta_A : A \times A \to A$, $\nabla_A : A \to A \times A$ such that*

(i) $(\alpha_1 \times \alpha_2)(\beta_1 \times \beta_2) = (\alpha_1\beta_1) \times (\alpha_2\beta_2)$ *whenever $\alpha_1\beta_1$ and $\alpha_2\beta_2$ are defined*;

(ii) *If $\alpha : A \to B$, then*

$$\Delta_B(\alpha \times 0_{BA})\nabla_A = \alpha = \Delta_B(0_{BA} \times \alpha)\nabla_A;$$

(iii) *Every* $\gamma: C \to A \times B$ *has the form*

$$\gamma = (\alpha \times \beta)\nabla_C \qquad \textit{for some } \alpha: C \to A,\ \beta: C \to B;$$

(iii′) *Every* $\delta: A \times B \to C$ *has the form*

$$\delta = \Delta_C(\alpha \times \beta) \qquad \textit{for some } \alpha: A \to C,\ \beta: B \to C.$$

Proof. Given AC-2, (i) is readily derived, while (ii) results from the equations

$$\begin{aligned}
\Delta_B(\alpha \times 0_{BA})\nabla_A &= \Delta_B(\alpha \times 0_{BZ})(I_A \times 0_{ZA})\nabla_A \\
&= \left\{\Delta_B(\alpha \times 0_{BZ})\Delta^{1}_{A\times Z}\right\}\left\{\Gamma^{1}_{A\times Z}(I_A \times 0_{ZA})\nabla_A\right\} \\
&= \left\{\Delta_B\Delta^{1}_{B\times B}\alpha\right\}\left\{I_A\Gamma^{1}_{A\times A}\nabla_A\right\} \\
&= \left\{I_B\alpha\right\}\left\{I_AI_A\right\} = \alpha.
\end{aligned}$$

To prove (iii) we set $\alpha = \Gamma^1_{A\times B}\gamma$, $\beta = \Gamma^2_{A\times B}\gamma$ and apply the appropriate definitions; (iii′) is the dual.

Conversely, given the conditions (i) to (iii′), we use the definitions (18.6), (18.6′), and (18.7) to construct the free-and-direct product diagram on $A \times B$ and to prove its properties.

The "existential" character of condition (iii) in this theorem, corresponding to the existential character of (ii) in (18.2), can be avoided, by using the readily proved identities

$$(18.8) \qquad (\alpha \times \alpha)\nabla_A = \nabla_B\alpha, \qquad \Delta_B(\alpha \times \alpha) = \alpha\Delta_A, \qquad \text{for } \alpha: A \to B,$$

$$(18.9) \qquad [\Delta_A(I_A \times 0_{AB})] \times [\Delta_B(0_{BA} \times I_B)]\nabla_{A\times B} = I_{A\times B}.$$

Theorem 18.6. *Theorem* 18.5 *remains valid if conditions* (iii) *and* (iii′) *are replaced by* (18.8), (18.9), *and the dual of* (18.9).

19. **Addition of mappings.** Two homomorphisms α, β: $A \to B$ for abelian groups A and B have a sum $\alpha+\beta: A \to B$ defined by $(\alpha+\beta)(a) = \alpha(a)+\beta(a)$. For groups one readily verifies that this sum can be expressed in terms of diagonal mappings, and so on, as

$$(19.1) \qquad \alpha + \beta = \Delta_B(\alpha \times \beta)\nabla_A.$$

We adopt the self-dual expression (19.1) as the definition of the sum of two coterminal mappings in any abelian category. By suitable use of the Uniqueness Theorem 18.1, one shows that the sum is independent of the choice of the free-and-direct products $B \times B$ and $A \times A$ entering its definition. For $0 = 0_{BA}$ the equation

$$(19.2) \qquad 0 + \beta = \beta = \beta + 0$$

follows from Theorem 18.3. The commutative and associative laws

$$(19.3) \qquad \alpha + \beta = \beta + \alpha, \qquad (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

similarly follow from commutative and associative properties of the product $A \times B$, using a lengthy but straightforward manipulation of diagrams. One may also establish the distributive laws

$$(19.4) \qquad (\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma, \qquad \gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta,$$

each valid whenever the left side is defined. A similar but lengthy proof shows that the mappings Γ^i, Λ^i entering the definitions (18.2) of the product satisfy the equation

$$(19.5) \qquad \Lambda^1\Gamma^1 + \Lambda^2\Gamma^2 = I_{A\times B}.$$

This gives an alternative characterization of the direct-and-free product as follows.

THEOREM 19.1. *A diagram* (18.1) *in an abelian category, with mappings satisfying* (18.1), (18.2), *condition* (i) *below* (18.2), *and* (19.5) *is isomorphic to the free-and-direct product diagram for A and B.*

In an abelian category which is also a bicategory we have $\Gamma^i\Lambda^i$ = identity by (i) of (18.2), hence by Lemma 11.4, the mappings Γ^i appearing in a free-and-direct product are supermaps, the mappings Λ^i are submaps (exactly as in the usual group-theoretic case). Similarly, by (18.5), Δ_A is a supermap, and ∇_A a submap.

20. **The representation theorem.** We understand an abelian semigroup G to be a set closed under a commutative and associative operation of addition and containing a zero for this operation (we do not assume a cancellation law). The class of all abelian semigroups and all homomorphisms of one such into another (with or without the precautions of §12 as to equality) is then a category. Our representation theorem now is

THEOREM 20.1. *Any abelian category $\mathcal{C}$ is isomorphic to a category of semigroups.*

PROOF. Let J be the essentially unique integral object of $\mathcal{C}$. To each object A of $\mathcal{C}$ assign the semigroup G_A consisting of all mappings $\xi: J \rightarrow A$ in $\mathcal{C}$, with addition defined as in §19. Then G_A is a semigroup with 0_{AJ} as zero. To each mapping $\alpha: A \rightarrow B$ construct the transformation $M_\alpha: G_A \rightarrow G_B$ defined by $M_\alpha\xi = \alpha\xi: J \rightarrow B$, for each ξ in G_A. By the distributive law (19.4), M_α is a semigroup homomorphism. By the first property of an integral object, $\alpha_1 \neq \alpha_2: A \rightarrow B$ implies $M_{\alpha_1} \neq M_{\alpha_2}$. By the associative law in a category, $M_{\alpha\beta} = M_\alpha M_\beta$ whenever $\alpha\beta$ is defined. Hence the correspondence $A \rightarrow G_A$, $\alpha \rightarrow M_\alpha$ provides the desired isomorphic representation of the category $\mathcal{C}$.

This theorem shows that our axioms for an abelian category include all the "purely formal" properties of homomorphisms for abelian groups, except for properties dependent upon the existence of the inverse.

21. **Abelian bicategories.** By an abelian bicategory (ABC) we understand a lattice ordered bicategory (§14) which is also an abelian category (§18) and which is subject to the following additional axioms[10] ABC-1 to ABC-5.

ABC-1. For each object A there exists a map $V_A\colon A\to A$ such that $V_A+I_A=0_{AA}$.

It follows readily that V_A is unique, and that $V_A V_A=I_A$. Furthermore, if $\alpha\colon A\to B$, then $\alpha V_A+\alpha=\alpha(V_A+I_A)=0$ and $V_B\alpha+\alpha=0$, hence $V_B\alpha=\alpha V_A$. We may thus define the (additive) inverse of α as $(-\alpha)=\alpha V_A=V_B\alpha$; it has the properties

$$\alpha+(-\alpha)=0,\qquad (-\alpha)\beta=-(\alpha\beta)=\alpha(-\beta).$$

Moreover, it may be shown, using (18.8), that the kernel of Δ_A is the image of $(I_A\times V_A)\nabla_A\colon A\to A\times A$, as in the case of groups.

ABC-2. If κ_1, κ_2 are injections, so is $\kappa_1\times\kappa_2$, and dually.

This gives the canonical decomposition of any $\alpha_1\times\alpha_2$; it is not used further below.

ABC-3. Every subobject is normal (in the sense of §16) and dually.

ABC-4. If $A\supset S\supset T$, then $S/T\subset A/T$, and dually.

ABC-5. If $A\supset T$, $A/T\supset M$, then $M=S/T$ for some S with $A\supset S\supset T$, and dually.

ABC-3 and the dual of ABC-5 fail in the category of all groups[10] but hold for abelian groups. Using them, one can prove that any projection $\pi\colon A\to Q$ with kernel $K(\pi)=K$ has $Q=A/K$, and hence that the canonical decomposition of any mapping $\alpha\colon A\to B$ has the customary form

$$\alpha=[B\supset \operatorname{Im}\alpha]\theta[A/K(\alpha)\leqq A]. \tag{21.1}$$

Furthermore, if $T\subset B$ and $S\subset A$,

$$\alpha_q(B/T)=A/\alpha_s^*T,\qquad \alpha_q^*(A/S)=B/\alpha_s S \tag{21.2}$$

and dually. If $A\supset S\supset T$ with $A\neq S$, then $A/T\neq S/T$. If $\kappa=[A\supset S]$ is an injection, the induced transformations κ_s, and so forth, are given for any $T \subset S$ and any $U \subset A$ by the formulas

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[10] Axiom ABC-3 and its dual are valid in the category of all groups under the hypothesis that $\pi=[A/T\leqq A]$ is a "conormal" mapping—that is, that π_q carries conormal quotients of A/T into conormal quotients of A. By this and analogous devices the proofs of the first and second isomorphism theorems for all groups can be based on "categorical" axioms.

$$(21.3)\qquad \begin{aligned} [A \supset S]_s T &= T, & [A \supset S]_s^* U &= U \cap S, \\ [A \supset S]_q (A/U) &= S/U \cap S, & [A \supset S]_q^* (S/T) &= A/T. \end{aligned}$$

If $\{S_i\}$ is a nonvoid collection of subobjects of A, then

$$(21.4)\qquad \sum_A (A/S_i) = A/\prod S_i, \qquad \prod (A/S_i) = A/\sum_A S_i.$$

The second isomorphism theorem holds in the form

$$(21.5)\qquad A \supset S \supset T \quad \text{implies} \quad A/S = (A/T)/(S/T);$$

the proof using the fact that

$$[A/T \supset S/T][S/T \leqq S] = [A/T \leqq A][A \supset S],$$

by ABC-4 and BC-5. (This is a typical indication of the force of BC-5.)

The first isomorphism theorem also holds in the form that

$$(21.6)\qquad A \supset S,\ A \supset N \quad \text{imply} \quad (S \cup N)/N \cong S/S \cap N,$$

where the isomorphism in question is the equivalence factor of the mapping $[A/N \leqq A][A \supset S]$. If $\pi = [A/M \leqq A]$ is a projection, the induced transformations π_s, and so on, are given by formulas analogous to (21.3), for $M \subset T \subset A$, $L \subset A$, as

$$(21.7)\qquad \begin{aligned} &[A/M \leqq A]_q (A/T) = A/T, && [A/M \leqq A]_q^* (A/L) = A/M \cup L, \\ &[A/M \leqq A]_s L = (M \cup L)/M, && [A/M \leqq A]_s^* (T/M) = T. \end{aligned}$$

In none of these results is ABC-5 used, except in the last case to justify the representation of any subobject of A/M in the form T/M.

In axiom ABC-5, one may show that the S whose existence is asserted actually has the form $\pi_s^* M$, where $\pi = [A/T \leqq A]$, and thence that $\pi_s^* \pi_s S = S$.

In an ABC-category the class of submappings is entirely determined by the category structure (without intervention of the injections) according to the following readily proved result.

THEOREM 21.1 (ABC). *A mapping α is a submapping if and only if it is left cancellable.*

Furthermore, if ρ is a supermapping $\rho\colon A \to B$, the transformation ρ_s^* is a right inverse of $\rho_s\colon \mathfrak{S}(A) \to \mathfrak{S}(B)$, and if λ is a submapping $\lambda\colon A \to B$, λ_s^* is a left inverse of the transformation $\lambda_s\colon \mathfrak{S}(A) \to \mathfrak{S}(B)$. Note that these statements are not *dual* to each other—though the dual statements about λ_q, λ_q^*, ρ_q, ρ_q^* are, of course, also demonstrable. Another pair of such pseudo-dual statements is indicated by the following theorem.

THEOREM 21.2 (ABC). *A mapping α is a supermapping if and only if either α_s is a transformation of $\mathfrak{S}(A)$ onto $\mathfrak{S}(B)$ or α_s^* is a one-to-one transformation of $\mathfrak{S}(B)$ into $\mathfrak{S}(A)$.*

One may also establish the "extension equivalence" theorem. As in (1.3) an extension of G by Q is a diagram

$$G \xrightarrow{\lambda} E \xrightarrow{\rho} Q, \tag{21.8}$$

such that λ is a submapping, ρ a supermapping, and image (λ) $=$kernel (ρ). If

$$G \xrightarrow{\lambda'} E' \xrightarrow{\rho'} Q \tag{21.9}$$

is a second such extension (with the same G and Q), a "homomorphism" of (21.8) into (21.9) is given by any mapping γ: $E \rightarrow E'$ which satisfies the usual commutation relations

$$\rho'\gamma = \rho, \qquad \lambda' = \gamma\lambda. \tag{21.10}$$

THEOREM 21.3 (ABC extension equivalence theorem). *Any homomorphism γ of one extension* (21.8) *of G by Q into another is an equivalence.*

The usual (group-theoretic) proof proceeds by calculating the image and the kernel of γ; the same calculations may now be reproduced in categorical language, eventually leaving only the equivalence factor θ in the decomposition (21.1) for γ. This theorem is also of interest because the Eilenberg-Steenrod "5 Lemma" on two exact sequences of length 5 can be reduced to this theorem, also by categorical arguments.

We mention also that various alternative characterizations of free-and-direct products (such as Theorem 19.1) as well as the Eilenberg-Steenrod "hexagon Lemma" can be derived from the axioms of an ABC-category; these derivations making heavy use of the addition of homomorphisms introduced in §19.

These deductions indicate that most of the standard properties[11] of homomorphisms between abelian groups (or, of linear transformations between vector spaces) can be deduced from our self-dual system of axioms for an abelian bicategory. We thus provide a precise axiomatic proof for the duality phenomena in abelian groups.

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[11] Facts such as the existence of infinite direct and infinite free products (no longer identical) or the existence, within the category, of the group Hom (A, B) of all homomorphisms of A into B are of course not deducible from our axioms.

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BOOK REVIEWS

Enciclopedia delle matematiche elementari e complementi con estensione alle principali teorie analitiche, geometriche e fisiche. Loro applicazione e notizie storico-bibliografiche. Vol. 3. Ed. by Luigi Berzolari. Milan, Hoepli. Part 1, 1947, xii+966 pp. 2600 lire; part 2, 1950, xx+1040 pp. 3800 lire.

The first Encyclopedia of Elementary Mathematics was the popular three-volume work of H. M. Weber (1842–1913) and J. Wellstein (1869–1919), *Encyklopädie der Elementar-Mathematik*, first published in 1903–1907; it went through several editions.

Already in a 1909 session of the Society Mathesis there were advocates for the preparation and publication of an Italian Encyclopedia of Mathematics, but it was nearly a score of years later before the project actually took tangible form. One may, however, regard as a sort of preliminary presentation of Encyclopedia topics the remarkably successful work edited by F. Enriques (1871–1946), *Questioni riguardanti le matematiche elementari*, in two large volumes, 1912–1914. The volumes, in which numerous specialists wrote on the various topics, were appreciably enlarged in the third edition, a dozen years later.

But finally, in 1930, under the editorship of Luigi Berzolari (1863–1949), Giulio Vivanti (1859–1949), and Duilio Gigli (1878–1933), appeared the first part of the first volume of *Enciclopedia delle matematiche elementari*; the second part was published in 1932, both parts (each with its own author index) making a volume of nearly 1100 pages. This volume, devoted to arithmetic, computation, algebra, and analysis, by 9 authors, was the only one of the *Enciclopedia* previously reviewed in this Bulletin (vol. 38 (1932) pp. 156–157). The general editorial plan of this volume was also carried out in the second. Only the more fundamental theorems were proved, but there is a constant wealth of bibliographic references where further information may be gleaned.

The second volume, in two parts, 1937–1938, of over 1200 pages and by 16 authors, was devoted to topics in geometry. Gigli had then died and it was the last one in which Vivanti appeared as joint editor.

Of the third volume, 1947–1950, Berzolari was the sole editor, but he died before the final printing of the great second part of the volume was published. In this part is a brief biographical sketch of the editor and the frontispiece is his portrait. The volume is dedicated to varied applications of mathematics, pure and applied, to their history, and to didactic questions. The bibliographic references are not quite as full as in the earlier volumes, especially in connection with some of the monographs, but they are still valuable.

The contents of part 1 of this volume are as follows: Giovanni Giorgi, *Systems of units and measures*, pp. 1–37; Paolo Straneo, *General theory of dimensions in physics—its characteristic applications*, pp. 39–97; G. Giorgi, *Methods of vector calculus in space—critical and comparative remarks*, pp. 99–124; G. Giorgi, *Calculus of matrices*, pp. 125–166; Attilio Palatini, *Rational mechanics*, pp. 167–249; Ermenegildo Daniele, *Graphical calculus and graphical statics*, pp. 251–293; Mariano Pierucci, *Classical physics*, pp. 295–506; Luigi Gabba, *Cosmography and a sketch of celestial mechanics*, pp. 507–538; Gino Giotti, *Geometrical optics*, pp. 539–585; Rocco Serini, *Crystallography and crystallographic physics*, pp. 587–703; Gino Cassinis and Luigi Solaini, *Geodesy and topography*, pp. 705–774; Attilio Talatini, *Theory of relativity*, pp. 775–819; Paolo Straneo, *Matter, radiation, and quantum physics*, pp. 821–950; *List of authors cited*, pp. 951–962; *Errata*, pp. 963–964.

The contents of part 2 are as follows: Ugo Casina, *Numerical approximations*, pp. 1–191; Filippo Sibirani, *Calculus of probability*, pp. 193–244; Corrado Gini, *Statistical methods, the measure of collective phenomena*, pp. 245–321; Tommaso Boggio, *Financial mathematics*, pp. 323–409; T. Boggio and Fernando Giaccardi, *Actuarial mathematics*, pp. 411–480; Michele Cipolla (1880–1947), *Mathematical recreations*, pp. 481–538; Ettore Bortolotti (1866–1947), *History of elementary mathematics*, pp. 539–750; Francesco Severi and Fabio Conforto, *Character and directions of modern mathematics*, pp. 751–813; Mario Gliozzi, *History of physical thought*, pp. 815–883; Luigi Brusotti, *Didactic questions*, pp. 885–973; Giovanni Giorgi, *Appendix. On the foundations of geometry*, pp. 975–1014; *List of authors cited*, pp. 1015–1038. Giorgi's appendix is simply a second edition of three conferences held at the University of Rome in 1912 and published in *Bollettino della' Mathesis* for 1912. At various points Bortolotti's history will be highly suggestive for future writers, especially in directions where he had carried on personal research.

To the completion of the *Enciclopedia* 49 authors—all of them distinguished specialists—contributed during the 20 years of its publication, and it is indeed most worthy of a place on a shelf of every university library. Almost every mathematician would be sure to find something of interest in this admirably edited work.

R. C. ARCHIBALD

Nonlinear vibrations in mechanical and electrical systems. By J. J. Stoker. New York, Interscience, 1950. 20+273 pp. $5.00.

Professor Stoker's book is an important landmark in the first decade of the studies of nonlinear problems in the United States and should be recommended to those who are interested in a systematic study of this subject. The presentation renders it particularly attractive to physicists and engineers owing to a considerable number of examples illustrating the various mathematical concepts such as singularities, limit cycles, and so on, which thus acquire an almost intuitive aspect. On the other hand, once this intuitive approach has been sufficiently prepared, the author proceeds with the necessary rigor to prove the existence theorems for these concepts introduced more or less on physical grounds.

As the author states in the introduction, it was necessary to restrict the subject to a somewhat limited number of selected topics in order to keep within the limits of the book. In fact, the Russian literature alone between 1929 and 1937 covers over 2000 pages of the various publications to say nothing of a considerable amount of the earlier material accumulated by Poincaré, Liapounoff, Bendixson, Birkhoff, and other forerunners in these studies. Nevertheless, Stoker's treatment of these selected topics is sufficiently broad to enable the reader to acquire the grasp of a much wider situation.

Perhaps the best way of abstracting the book is to proceed chapter by chapter as he does himself in his introduction. Chapter I is a short summary of the theory of linear vibrations for a system with one degree of freedom; it serves both as a reference and contrast for the nonlinear theory. Chapter II introduces the reader into the simplest case of nonlinear problems associated with the behavior of nonlinear conservative systems. The phase plane representation of integral curves is introduced in this chapter and is illustrated by a number of examples. Chapter III is also of a rather elementary character and deals with the differential equations of a more general type, namely, those which, in addition to the nonlinear restoring force, possess also a nonlinear damping. In this chapter the author gives an outline of the Liénard graphical method and closes the theoretical part by a summary of the theory and classification of simple singularities of the differential equations in real domain. The end of the chapter contains a number of physical examples illustrating the concept of singular points as various types of equilibria of dynamical systems. The last example concerning the behavior of the synchronous motor covers nearly 14 pages of the text and is treated in great detail.

Chapter IV is of a less elementary character and deals with the so-called Duffing equation with forcing term. This subject is of interest mostly in mechanical problems and is treated by two different methods, the iteration and the perturbation methods. §10 of this chapter touches a far more advanced point which has formed the subject of many discussions in the past, namely, the difficulty of the so-called "small divisors" which was encountered by Poincaré, Gylden, Lindstedt, and other mathematicians of that time in connection with astronomical problems. A more detailed treatment of this subject is given in Appendix II.

Chapter V is perhaps the most interesting and important of the whole book and is split into two parts. Part A deals with the self-excited systems. In this chapter the author introduces the fundamental concept of limit cycles on physical grounds, but a fuller mathematical treatment of this subject is given in Appendices III and VI. The second half of Part A is devoted to the so-called relaxation oscillations and contains a considerable amount of original work done by Stoker who has contributed much to recent studies of this difficult subject. Particularly interesting are asymptotic expansions appearing as approximations of higher orders in this case.

Part B deals with the forced oscillations in self-sustained systems. After a brief review of the earlier work of van der Pol, the author gives a very interesting and detailed presentation of the Andronow-Witt method, appearing as a further extension of the van der Pol theory. The method consists of investigating the distribution of singularities in a special parameter plane having "detuning" as one of the variables. Following this line of attack, the author treats the problem of stability of harmonic as well as combination oscillations in seven sections occupying nearly thirty pages of the text and extending considerably the original method of Andronow and Witt on the basis of "contact curves" of Poincaré. This is perhaps the most interesting part of the book. The only point to be regretted is that the connection between this theory and the well known phenomenon of synchronization (or "entrainment") is hardly mentioned at all. For a mathematical reader, this may not be so important, but for a physicist or an engineer more familiar with the phenomenon itself there is a risk of overlooking this important connection.

The last chapter is in some respects an anti-climax to the really remarkable Chapter V. The beginning of Chapter VI is devoted to a somewhat long review of the theory of linear differential equations with periodic coefficients (the Floquet theory), covering nearly 24 pages of the text. The reason for doing this, as Stoker explains, is that "the problem of the infinitesimal stability of the periodic solutions of nonlinear systems always leads to a Hill equation." Following this thought, the author reduces the problem of stability to the analysis of the local behavior of the variational equations (of the Hill type) in the various parts of the response curve and applies this procedure to the Duffing equation. It turns out, however, that, on this basis, the free oscillations are unstable. Having ascertained this seemingly paradoxical result, the author ascribes it to the fact that the criterion of the "infinitesimal stability" (that is, the stability in the sense of the variational equations) ought to be replaced by that of the orbital stability. In fact, after a somewhat delicate argument, in §7, the author proves this point. In spite of this, the reader, particularly the beginner, must inevitably feel somewhat confused as to when to use one criterion and when to use the other. This question does not seem to find a definite answer in the text, probably because the author, as he said in his introduction, had to curtail considerably the theory of stability owing to lack of space. It seems, however, sufficiently simple to show that if the differential equations are referred to the "amplitude-phase" plane (a, ϕ) instead of the usual $(x, \dot{x})$ phase plane (namely, $da/dt=f_1(a, \phi)$, $d\phi/dt=f_2(a, \phi)$) the singular point $f_1(a_0, \phi_0)=f_2(a_0, \phi_0)=0$ in this case represents the stationary periodic motion (if $a_0\neq 0$) and the variational equations ("the infinitesimal stability") give precisely the orbital stability in such a case, without any necessity of applying the theory of characteristic exponents of Poincaré. Reduction to this form is always possible if the differential equations do not contain time explicitly.

N. MINORSKY

Differential algebra. By Joseph Fels Ritt. (American Mathematical Society Colloquium Publications, vol. 33.) New York, American Mathematical Society, 1950. 8+181 pp. $4.40.

It was a gigantic task that J. F. Ritt undertook twenty years ago: to give the classical theory of nonlinear differential equations a rigorous algebraic foundation. Emmy Noether and her school had done the same thing for the theory of algebraic equations and algebraic varieties, but differential equations are much more difficult than algebraic equations. Luckily, Ritt has gathered around himself a whole school of able collaborators: Raudenbusch, Strodt, Kolchin, Howard Levi, Gourin, R. M. Cohn.

The present book is not just a revised and enlarged edition of the author's *Differential equations from the algebraic standpoint* (Colloquium Publications, vol. 14). It is written from a much higher point of view and based upon new principles. The first edition was a tentative, this a classic.

One important new feature is that the whole theory is now based upon the Ritt-Raudenbusch *basis theorem.*

Let $\mathcal{F}$ be a field in which an operation of differentiation is performable, for example a field of functions of one variable x closed with respect to differentiation. Let y_i $(i=1, \cdots, n)$ be symbols of unknown functions, and y_{ij} symbols of their derivatives (for example $y_{i0}=y_i$ and $y_{i1}=y_i{}')$. Polynomials in the indeterminates y_{ij} are called d.p. (differential polynomials). They form a ring in which a differentiation is defined.

Let Σ be any infinite set of d.p. A finite subset Λ is said to form a *basis* of Σ if for every d.p. A in Σ, a power A^p is a sum of multiples of elements of Λ and their derivatives. The basis theorem now states: *Every set Σ has a finite basis.*

The basis theorem was implicit in Ritt's work, but it was Raudenbush who brought it to the present complete form. By placing this theorem right at the beginning, in Chapter I, the great line of thought is made much clearer than it was in the original treatment.

Starting from the basis theorem, the theory of differential ideals is developed up to the representation of "perfect ideals" as intersections of prime ideals.

If for the indeterminates $y_1, \cdots, y_n$ field elements $\eta_1, \cdots, \eta_n$ are substituted, it may happen that all d.p. of a set Σ become zero. In this case, the set $\{\eta_1, \cdots, \eta_n\}$ is called a *zero* or *solution* of Σ. The totality of all zeros is the *manifold* of Σ.

Chapter II deals with manifolds and their decomposition into irreducible manifolds. Just as in algebraic geometry, the *dimension* of an irreducible manifold can be defined as the maximum number of parametric indeterminates among the y_i.

For one single differential equation $F=0$ a distinction between *singular* and *nonsingular* solutions is introduced. In the manifold of F, there is always one irreducible component, called the *general solution* which does not consist of singular solutions only. Perhaps *nonsingular component* would be a better expression.

The discussion of the solutions of one single differential equation is continued in Chapter III. The "low power theorem" gives the necessary and sufficient condition that the nonsingular component of one differential equation be at the same time a component of another differential equation. This theorem solves a problem first treated by Laplace and Poisson by heuristic methods.

In Chapter IV, systems of algebraic equations are treated by a new algorithmic method. The results of this chapter are used in Chapter V to give an algorithmic treatment of various questions connected with finite systems of differential equations. For the case of a field $\mathcal{J}$ consisting of analytic functions, a very useful approximation theorem is proved.

Chapter V deals with constructive methods and tests.

In Chapter VI, the case of a field of analytic functions is treated by analytic methods. For this case, another proof of the low power theorem is given.

Chapter VII deals with intersections of algebraic differential manifolds, especially with their dimensions. A result of Jacobi proves true in some special cases, but false in general.

Chapters VIII and IX deal with partial differential equations. In Chapter VIII, a very important existence theorem, due to Riquier, is proved. In Chapter IX this theorem is used to extend some of the main results of the preceding chapters to partial differential polynomials.

B. L. VAN DER WAERDEN

Transcendental numbers. By Carl Ludwig Siegel. (Annals of Mathematics Studies, no. 16.) Princeton University Press, 1949. 8+102 pp. $2.00.

As the author states in a short preface, this book is based on lectures given at Princeton in 1946. In Chapter I, *The exponential function,* proofs are given of the irrationality of e and π, and then a general method is introduced.

Let $\rho_1, \cdots, \rho_m$ be complex numbers, $n_1, \cdots, n_m$, non-negative integers, and let

$$N + 1 = \sum_{k=1}^{m} (n_k + 1).$$

It is shown that polynomials $P_1(x), \cdots, P_m(x)$ of degrees $n_1, \cdots, n_m$, respectively, may be determined uniquely (up to a constant factor) such that the function

$$R(x) = \sum_{k=1}^{m} P_k(x)e^{\rho_k x}$$

vanishes at $x=0$ of order N. Such a function is called an approximation form. An explicit formula for $R(x)$ as a multiple integral provides an upper bound for $|R(1)|$ and shows that $R(1)>0$ when $\rho_1, \cdots, \rho_m$ are real.

The stage is then set for the proof that e^a is transcendental for real algebraic $a \neq 0$. If e^a were algebraic, there would be an algebraic integer ξ, not zero, but whose norm is less than one. This is clearly a contradiction. A similar argument, based on m approximation forms, leads to the Lindemann-Weierstrass theorem which states that, if $b_1, \cdots, b_r$ are different algebraic numbers, then $e^{b_1}, \cdots, e^{b_r}$, are not related by a homogeneous linear equation with algebraic coefficients not all zero.

The exponential function $y = e^x$ satisfies the differential equation $y' = y$ and has the addition theorem $e^x e^t = e^{x+t}$. The remaining chapters are concerned with functions that are either solutions of linear differential equations or possess addition theorems.

Chapter II, *Solutions of linear differential equations*, deals with what are called E-functions. A function $f(x)$ is an E-function if

$$f(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!},$$

where the coefficients c_n belong to the same algebraic number field of finite degree over the rational number field, and where the maximum of the absolute values of all conjugates of c_n and the least common (rational) denominator of $c_1, \cdots, c_n$, regarded as functions of n, are each $O(n^{n\epsilon})$ for any $\epsilon > 0$. (The exponential function is clearly an E-function.)

Let $E_1, \cdots, E_m$ be E-functions that are solutions of a system of m homogeneous linear differential equations of the first order,

$$y_k' = \sum_{j=1}^{m} Q_{kj}(x) y_j \qquad (k = 1, \cdots, m),$$

whose coefficients Q_{kj} are rational functions of x with algebraic numerical coefficients. If the products $E_1^{\nu_1} \cdots E_m^{\nu_m}$, $\nu_1 + \cdots + \nu_m \leqq \nu$, form what is called a normal system for all positive integral ν, and if α is any algebraic number different from zero and the poles of Q_{kj}, it is proved that $E_1(\alpha), \cdots, E_m(\alpha)$ are not related by an algebraic equation with algebraic coefficients. This result includes the Lindemann-Weierstrass theorem.

A particular E-function is

$$K_\lambda(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!(\lambda + 1) \cdots (\lambda + n) 2^{2n}}, \qquad \lambda \neq -1, -2, \cdots,$$

which is related to the Bessel function $J_\lambda(x)$ by the equation

$$J_\lambda(x) = \frac{1}{\Gamma(\lambda+1)}\frac{x^\lambda}{2^\lambda}K_\lambda(x).$$

It is shown that the normality condition is satisfied for all rational $\lambda \neq \pm 1/2, \pm 3/2, \cdots$. With these restrictions, it follows that $K_\lambda(\alpha)$ is transcendental for any algebraic $\alpha \neq 0$. Hence all the zeros of $K_\lambda(x)$ and all the zeros not equal to 0 of $J_\lambda(x)$ are transcendental.

The function

$$w = f_\lambda(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!(\lambda+1)\cdots(\lambda+n)}$$

satisfies the differential equation $xw''+(\lambda+1)\omega'=w$, which leads to the continued fraction

$$\frac{w}{w'} = \lambda + 1 + \frac{x}{\lambda + 2 + x/(\lambda+3+\cdots)}.$$

Since $f_\lambda(-x^2/4)=K_\lambda(x)$, the previous results show that the continued fraction is transcendental for all algebraic $x\neq 0$ and all rational λ. In particular,

$$1+\cfrac{1}{2+\cfrac{1}{3+\cdots}}$$

is transcendental.

Chapter III, *The transcendency of a^b for irrational algebraic b and algebraic $a\neq 0$*, is a short one giving Schneider's proof and a simplified form of Gelfond's. Each proof is similar in principle to the proof of the transcendence of e^a, but is naturally much deeper.

Chapter IV, *Elliptic functions*, deals with some results due to Schneider. Let $\wp(z)$ and $\zeta(z)$ denote the usual Weierstrass functions, where $\zeta'(z) = -\wp(z)$. It is shown that, if λ, μ, g_2, g_3, $\wp(z_0)$ are algebraic and λ, μ are not both zero, then $\lambda z_0+\mu\zeta(z_0)$ is transcendental. As a special case it follows, for example, that the arc length of the lemniscate $(x^2+y^2)^2=2a^2(x^2-y^2)$ is transcendental for algebraic a between points with algebraic coordinates.

It can be seen from the foregoing paragraphs that the book deals authoritatively with the problems of transcendental numbers. Unfortunately, it is not an easy book to follow. The subject matter is admittedly difficult, but, in the opinion of the reviewer, much could be done to present the material in a way that is clear and well organized.

The Editors of the series and Princeton University Press must share the blame for the poor form and style of the book. Many of the lines in the text are badly set up, some symbols are all too obviously handwritten, and the formulas are generally difficult to read.

The organization of the book leaves much to be desired, and, in the opinion of the reviewer, the fault lies in the method of presenting a proof. The author seldom takes the trouble to let the reader know what he is doing and why. This is clearly illustrated in the first chapter where a proof is given of the transcendency of e^a. The reader may find the proofs involved, but not until the end of the chapter, if he gets that far, does he discover that the author has a perfectly good reason for presenting the proofs in the way he does.

Pólya, writing in the American Mathematical Monthly, December 1949, page 684, describes the situation exactly in the following words. "A mathematical lecture should be, first of all, correct and unambiguous. Still, we know from painful experience that a perfectly unambiguous and correct exposition can be far from satisfactory and may appear uninspiring, tiresome or disappointing, even if the subject-matter presented is interesting in itself."

R. D. JAMES

Fourier transforms. By S. Bochner and K. Chandrasekharan. (Annals of Mathematics Studies, no. 19.) Princeton University Press, 1949. 10+219 pp. $3.50.

This is a very readable introduction to the craft of the authors, and as such fills a very real need. The subject matter serves to a large extent as a springboard for the presentation of interesting techniques, viewpoints, and concepts, and is treated with great deftness and remarkable continuity. There is a good deal of explanatory and motivating material, and altogether the book is a very appropriate one for study by an apprentice to the guild of semi-classical analysts.

The book is apparently not intended for reference use, nor is its subject matter and development the sort that are best adapted to the needs of a mathematician with a merely general interest in the subject, or to the needs of a theoretical physicist. The topics treated are interesting but the basis of their selection seems to have been esthetic and subjective, rather than a function of their relative significance within the general framework of mathematics. Among the important topics not treated are Fourier transforms in the complex domain, Fourier-Stieltjes transforms, and generalized harmonic analysis; from a purely informational standpoint the book is hardly a well-rounded survey of Fourier analysis. The proofs and approaches employed in the book are technically elementary, but often somewhat intricate and delicate, and demand either a certain sophistication or special attentiveness of the reader. The very considerable smoothness, coherency, and local clarity of the book seem to be achieved by a unity of mathematical viewpoint and technique, as well as by niceness of style, rather than by logical organization.

Fourier transforms of functions in L_1 and L_2 on the real line and in euclidean n-space are treated along familiar lines, but with much intrinsically interesting illustrative material. In particular, transforms of derivatives, and Gauss and Abel summability, are presented in some detail. Special but valuable topics more briefly treated include transforms of radial functions, closure of translations of functions in L_2, and bounded transformations on L_2 commuting with translations. All these developments take up the first four chapters of the book, and there are two additional chapters, concerning more specialized matters, along lines reflecting some of the particular interests of the authors. The fifth chapter gives an integro-differential representation for the most general unitary operator on L_2, from which a study of the Watson transform evolves. The sixth chapter treats aspects of Tauberian theorems from a combination of viewpoints due originally to Karamata and Wiener. These last two chapters both have a certain roundness and charm almost, which make relatively complex subjects seem quite approachable. This feature may be related to the circumstance that each of these chapters is apparently based on a paper by Bochner.

There is also a short treatment of Banach spaces, along with a discussion of the L_p spaces for general p (the Fourier transforms of functions in those spaces are not considered). None of the deeper theorems about Banach spaces is covered, and on the whole the "abstract" viewpoint is avoided. The outlook of the book is indicated by the following quotation (pp. 211–212): " . . . theorem 7 . . . has the disadvantage (*if disadvantage it is*) of being peculiar to the euclidean setup . . . " (reviewer's italics). The book contains essentially no orientation of the results with respect to the theories of harmonic analysis on topological groups, Banach algebras, or operators on Hilbert space. (One of the very few comments on the operator-theoretic background of a result is unsound,—the operators described on page 215 as having simple spectrum can have nearly arbitrary spectral multiplicities.) It opens with a definition of Fourier transform on the line, and no explanation is even given of why this particular transform, rather than some other, is the object of such intensive study. The fact that there exists a general theory of transforms on locally compact abelian groups, which establishes such basic results as the uniqueness theorem for L_1 transforms, the closure theorems for translations of functions in L_1 and L_2, and the generalized Plancherel theorem, in forms which apply to Fourier integrals and series on n-dimensional euclidean space, as well as to almost periodic functions, expansions into Walsh-Rademacher functions, functions on finite groups, and so forth, is devoid of any significant recognition.

Thus the book is intensive rather than extensive in scope, and strength of logical organization has been somewhat sacrificed for elegance of technique and mathematical style. It is a really valuable addition to the literature, and substantial numbers of students should find it both pleasant and instructive.

I. E. SEGAL

ERRATA, VOLUME 56

Electromagnetic theory. (Review by L. Brillouin.)

p. 375, line 15. For "Rutherford" read "Retherford."

Coxeter, H. S. M. *The real projective plane.* (Review by P. DuVal.)

p. 377, lines 2–4. The statement "but curiously it is never shown, though constantly taken for granted in the sequel, that there is always one such collineation" is in error. It has been pointed out to the author of the review that it is proved on p. 53 of the book that a collineation always exists in which one given quadrangle corresponds to another.

NOTES

A Colloquium under the auspices of the Edinburgh Mathematical Society will be held in St. Andrews July 18–28, 1951. Detailed information may be obtained from Dr. D. E. Rutherford, United College, St. Andrews, Scotland.

The following have received appointments to National Research Fellowships in Mathematics for the academic year 1950–1951: Dr. Alex Heller, Columbia University; Dr. R. V. Kadison, University of Chicago; and Dr. M. A. Rosenlicht, University of Chicago.

Dr. N. H. Kuiper of the Institute of Technology, Delft, Netherlands, has been appointed to a professorship at the School of Agriculture, Wageningen, Netherlands.

Dr. Cengiz Uluçay has been appointed to an assistant professorship at Ankara University, Ankara, Turkey.

Professor A. C. Zaanen of the University of Indonesia has been appointed to a professorship at the Institute of Technology, Delft, Netherlands.

The following members of the Society are currently members of the Institute for Advanced Study: Emilio Baiada, Dorothy L. Bernstein, Lipman Bers, Raoul Bott, Werner Fenchel, J. S. Frame, Bent Fuglede, William Gustin, Alex Heller, A. J. Hoffman, S. T. Hu, R. V. Kadison, J. B. Kelly, Kunihiko Kodaira, Benjamin Lepson, G. A. Miller, L. J. Mordell, J. D. Newburgh, Ilse L. Novak, D. S. Park, C. L. Pekeris, R. S. Phillips, C. R. Putnam, K. G. Ramanathan, H. E. Rauch, Giovanni Ricci, Robert Schatten, M. H. Stone, Leo Zippin.

Miss Florence R. Anderson of the University of Southern California has accepted a position as research laboratory analyst with Northrop Aircraft, Inc.

Mr. Jonas Beraru of the North American Aviation Corporation has accepted a position as mathematician with the Rand Corporation, Santa Monica, California.

Dr. F. E. Bothwell of the University of Chicago has been appointed to an associate professorship at Northwestern University.

Assistant Professor J. L. Botsford of San Jose State College has been appointed to an associate professorship at the University of Idaho.

Mr. C. F. Briggs of the University of Michigan has been appointed to an acting assistant professorship at Emory University.

Dr. Eleazer Bromberg of the Reeves Instrument Corporation has accepted a position as head of the Mechanics Branch, Office of Naval Research, Washington, D. C.

Dr. F. W. Brown of North American Aviation, Inc. has accepted a position as associate director for research and development, U. S. Naval Ordnance Test Station, Inyokern, California.

Dr. F. H. Brownell of Princeton University has been appointed to an assistant professorship at the University of Washington.

Dr. K. A. Bush has been appointed to an associate professorship at Champlain College.

Dr. M. N. Chase has accepted a position as research engineer with the Battelle Memorial Institute of Columbus, Ohio.

Dr. Y. W. Chen of the Institute for Advanced Study has been appointed to an associate professorship at the University of Oklahoma.

Dr. George Copp of the University of Texas has been appointed to an assistant professorship at North Texas State College.

Dr. Avron Douglis of the California Institute of Technology has been appointed to an assistant professorship at New York University.

Dr. Nat Edmonson of Johns Hopkins University has accepted a position as mathematician with Fairchild Engine and Airplane Corporation, Oak Ridge, Tennessee.

Mr. W. C. Foreman of the University of Kansas has been appointed to a professorship at Baker University.

Dr. Abraham Franck of the University of Minnesota has been appointed to an assistant professorship at Kansas State College.

Dr. J. R. Garrett of Duke University has been appointed to an assistant professorship at the Georgia Institute of Technology.

Dr. V. D. Gokhale of Rutgers University has been appointed to a professorship at Atlanta University.

Assistant Professor R. D. Gordon of the University of Buffalo has been appointed to a professorship at Washington College.

Dr. Violet G. Hachmeister of the University of Wisconsin has been appointed to an assistant professorship at Kansas State College.

Dr. Harish-Chandra of Harvard University has been appointed visiting lecturer at Columbia University.

Dr. V. C. Harris of Northwestern University has been appointed to an assistant professorship at San Diego State College.

Dr. S. L. Isaacson of Columbia University has been appointed to an assistant professorship of statistics at Iowa State College of Agriculture and Mechanic Arts.

Dr. S. L. Jamison of the University of California has been appointed to an assistant professorship at the Florida State University.

Mr. J. X. Jamrich of the University of Wisconsin has been appointed supervisor of student teaching in the School of Education at Northwestern University.

Dr. J. A. Jenkins of Harvard University has been appointed to an assistant professorship at Johns Hopkins University.

Associate Professor J. L. Kelley of the University of California is on leave of absence and has been appointed to a visiting associate professorship at Tulane University of Louisiana.

Mr. S. A. Kushneriuk has accepted a position as assistant research officer with the Atomic Energy Project, Chalk River, Ontario, Canada.

Assistant Professor J. S. Leech of the University of Chicago has been appointed to a professorship at Colorado College.

Assistant Professor Lee Lorch of Pennsylvania State College has been appointed to an associate professorship at Fisk University.

Assistant Professor Nathaniel Macon of Alabama Polytechnic Institute is on leave of absence at the University of Amsterdam, Amsterdam, Netherlands.

Mr. J. H. Manheim has accepted a teaching position with Wilson Junior College, Chicago, Illinois.

Dr. George Marsaglia of Ohio State University has been appointed to an assistant professorship at Montana State University.

Mr. P. T. Mielke of Purdue University has been appointed to an assistant professorship at Wabash College.

Dr. E. F. Moore has been appointed to an assistant professorship at the University of Illinois.

Dr. D. R. Morrison of the University of Wisconsin has been appointed to an assistant professorship at Tulane University of Louisiana.

Dr. M. O. Peach of the Carnegie Institute of Technology has been appointed to an associate professorship of engineering and mechanics at the University of Notre Dame.

Dr. W. A. Pierce of Harvard University has been appointed to an assistant professorship at Syracuse University.

Dr. R. W. Rempfer of the Farrand Optical Company has been appointed to a professorship at Antioch College.

Assistant Professor R. R. Reynolds of Oklahoma Agricultural and Mechanical College has accepted a position as mathematician at the Institute for Numerical Analysis of the National Bureau of Standards, Los Angeles, California.

Dr. Hugo Ribeiro of the University of California has been appointed to an associate professorship at the University of Nebraska.

Dr. L. G. Riggs of Northwestern University has been appointed to an assistant professorship at San Diego State College.

Professor J. B. Scarborough of the United States Naval Academy has retired with the title Emeritus.

Dr. J. P. Scholz of Wilson College has been appointed to an assistant professorship at Lebanon Valley College.

Dr. Esther Seiden of the University of California has been appointed to an assistant professorship at the University of Buffalo.

Mr. D. H. Shaftman has accepted a position as junior mathematician at the Argonne National Laboratory, Chicago, Illinois.

Mr. W. T. Sharp has been appointed to an assistant professorship at the University of Maryland.

Assistant Professor J. J. Sopka of Tufts College has accepted a position as mathematician at the Applied Physics Laboratory, Johns Hopkins University.

Dr. D. B. Sumner of the University of Reading has been appointed to an assistant professorship at Hamilton College of McMaster University.

Dr. L. W. Swanson of Coe College has accepted a position in the Applied Science Department of International Business Machines Corporation.

Assistant Professor G. L. Tiller of Utica College has been appointed to an associate professorship at Southwestern University.

Mr. P. M. Treuenfels has accepted a position as mathematician in the Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland.

Mr. G. E. Uhrich of the University of Washington has been appointed to an assistant professorship at Montana State College.

Dr. S. E. Urner of Los Angeles City College has been appointed to a professorship at the Los Angeles State College of Arts and Sciences.

Mr. J. R. Van Andel of the University of Pennsylvania has accepted the position of assistant research engineer with Burroughs Adding Machine Company.

Associate Professor R. W. Wagner of Oberlin College has been appointed to a professorship at the University of Massachusetts.

Mr. E. S. Williams has been appointed to a professorship at State Colored Normal, Industrial, Agricultural and Mechanical College of South Carolina.

Mrs. Marie N. Williams has accepted a position as analyst with the Quaker Oats Company, Chicago, Illinois.

Mr. J. W. Wray of the University of Illinois has been appointed to an assistant professorship at the University of Idaho.

Dr. C. S. Yih of the University of British Columbia has been appointed to an associate professorship at Colorado Agricultural and Mechanical College, Fort Collins, Colorado.

Mr. R. W. Young of the University of Florida has been appointed to a professorship at Henderson State Teachers College.

Dr. J. L. Zemmer of the University of Wisconsin has been appointed to an assistant professorship at the University of Missouri.

Dr. A. D. Ziebur of the University of Wisconsin has been appointed to an assistant professorship at Oklahoma Agricultural and Mechanical College.

The following promotions have been announced:

R. C. F. Bartels, University of Michigan, to an associate professorship.

W. E. Bleick, U. S. Naval Postgraduate School, to a professorship.

A. C. Burdette, University of California, Davis, California, to an associate professorship.

A. J. Cook, University of Alberta, to a professorship.

Helen W. Dodson, University of Michigan, to an associate professorship.

H. W. Ellis, Queen's University, to an assistant professorship.

F. C. Gentry, Arizona State College at Tempe, to a professorship.

David Gilbarg, Indiana University, to an associate professorship.

A. M. Gleason, Harvard University, to an assistant professorship.

W. J. Harrington, Pennsylvania State College, to an associate professorship.

Ernest Hawkins, U. S. Naval Academy, to a professorship.

E. A. Hedberg, University of South Carolina, to a professorship.

A. S. Hendler, Rensselaer Polytechnic Institute, to an assistant professorship.

G. P. Hochschild, University of Illinois, to an associate professorship.

P. E. Irick, Purdue University, to an assistant professorship.

W. J. Jaffe, Newark College of Engineering, to an associate professorship.

Walter Jennings, U. S. Naval Postgraduate School, Annapolis, Maryland, to an associate professorship.

D. M. Krabill, Bowling Green State University, to a professorship.

B. P. Kremer, John Carroll University, to an assistant professorship.

W. J. LeVeque, University of Michigan, to an assistant professorship.

R. E. Lowney, Montana State College, to an associate professorship.

V. O. McBrien, College of the Holy Cross, to an associate professorship.

Betty McKnight, Centenary College of Louisiana, to an assistant professorship.

W. R. Mann, University of North Carolina, to an assistant professorship.

R. C. Meacham, Carnegie Institute of Technology, to an assistant professorship.

N. S. Mendelsohn, University of Manitoba, to an associate professorship.

Ruth E. O'Donnell, Duquesne University, to an associate professorship.

C. D. Olds, San Jose State College, to a professorship.

Emily C. Pixley, University of Detroit, to an associate professorship.

L. L. Rauch, University of Michigan, to an associate professorship.

N. E. Steenrod, Princeton University, to a professorship.

E. G. Straus, University of California, Los Angeles, to an assistant professorship.

Walter Strodt, Columbia University, to an associate professorship.

G. B. Thomas, Massachusetts Institute of Technology, to an associate professorship.

C. C. Torrance, U. S. Naval Postgraduate School, Annapolis, Maryland, to a professorship.

V. J. Varineau, University of Wyoming, to an associate professorship.

Charles Walmsley, Dalhousie University, to a professorship.

J. F. Wardwell, Colgate University, to a professorship.

Betty R. Weber, University of South Carolina, to an adjunct professorship.

Martin Wright, University of Houston, to an associate professorship.

Max Wyman, University of Alberta, to an associate professorship.

L. A. Zadeh, Columbia University, to an assistant professorship.

The following appointments to instructorships are announced: University of California: Dr. R. M. Lakness, Dr. T. K. Pan; Catholic University of America: Mr. J. F. Hannan; Columbia University: Mr. R. H. Brown, Mr. G. N. Raney, Dr. R. L. Taylor; University of Connecticut: Dr. Samuel Bourne, Dr. Helene Reschovsky, Mr. David Wellinger; Cornell University: Dr. W. L. Murdock; Duke University, Mr. J. A. Silva; University of Florida; Dr. F. Virginia Rohde; Harvard University: Dr. R. G. Ayoub, Dr. D. M. Young; Illinois Institute of Technology: Dr. Allen Devinatz; Indiana University: Dr. D. H. Porter; Iowa State Teachers College: Miss Augusta L. Schurrer; Johns Hopkins University: Dr. A. B. Novikoff, Mr. J. A. Zilber; Kansas State Teachers College, Emporia: Mr. A. H. Albert; Lehigh University: Mr. Samuel Schecter; Louisiana State University: Mr. G. J. Trammell; Massachusetts Institute of Technology: Dr. I. M. Singer; University of Michigan: Dr. H. C. Davis, Dr. J. R. Lee, Mr. G. R. Livesay, Dr. J. E. McLaughlin, Mr. Daniel Resch; University of Notre Dame: Mr. J. A. Sullivan; Ohio State University: Mr. W. M. Myers; Pennsylvania State College: Mr. D. J. Dickinson; Princeton University: Mr. Eugenio Calabi, Dr. J. F. Nash, Dr. Theodor Teichmann; Purdue University: Mrs. Elizabeth H. Cuthill; University of Richmond: Miss Marion B. Jeffries; Southern Illinois University: Mr. Daniel Orloff; State College of Washington: Mr. K. L. Cooke; Sue Bennett College, Stanford, Kentucky: Miss Mary Pettus; Temple University: Mr. Albert Schild; University of Tennessee: Mr. E. E. Posey; Tennessee Polytechnic Institute: Mr. J. J. Hart; Texas State University for Negroes: Mr. J. E. Westberry; University of Texas: Mr. S. E. Dyer; Trinity College: Mr. R. C. Stewart, Mr. J. N. Williams; University of Vermont and State Agricultural College: Mr. W. T. Fishback; University of Washington: Dr. P. R. Peterson; Washington State College: Mr. R. R. Heacock; Wells College: Miss Marjorie B. Djorup; University of Wisconsin: Dr. L. E. Fuller; University of Wisconsin in Milwaukee: Mr. R. J. Weeks.

Notice has been received of the death of Assistant Professor L. C. Dawson of Colorado Agricultural and Mechanical College.

Professor J. T. McCormick of the College of the Holy Cross died in May, 1950. He had been a member of the Society for twenty-eight years.

Professor Emeritus S. W. Reaves of the University of Oklahoma died on August 2, 1950 at the age of seventy-five years. He had been a member of the Society for forty-two years.

Professor W. M. Smith of Lafayette College died on July 3, 1950 at the age of sixty-nine years. He had been a member of the Society for forty-two years.

Professor J. A. G. Shirk of Kansas State Teachers College died April 15, 1950 at the age of sixty-nine years. He had been a member of the Society for twenty-four years.

Mr. G. M. V. Tryon of Fenton, Michigan died on May 1, 1950. He had been a member of the Society for twenty-seven years.

ABSTRACTS OF PAPERS

The abstracts below are abstracts of papers presented by title at the October Meeting in New York and the November meeting in Evanston. Abstracts of papers presented in person at these meetings will be included in the reports of the meetings which will be published in the January issue of this BULLETIN.

Abstracts are numbered serially throughout this volume.

ALGEBRA AND THEORY OF NUMBERS

453*t*. R. H. Bruck and Erwin Kleinfeld: *The structure of alternative division rings.*

The following theorem is proved: (A) Every alternative division ring R of characteristic other than two is either (i) a field or skew-field or (ii) a Cayley-Dickson division algebra (of order eight over its centre). The essence of the proof is to show that if R, with centre F, is not associative, then every element of R satisfies a quadratic equation over F. This reduces (A) to a theorem of A. A. Albert (*Absolute-valued algebraic algebras*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 763–768). Call a projective plane alternative if Ruth Moufang's law of the complete quadrangle or Marshall Hall's Theorem L holds universally, but no complete quadrangle has collinear diagonal points. Then (A), together with published work of Moufang, Hall, R. D. Schafer, and others, implies the following theorems: (B) Every alternative plane characterizes a coordinate (alternative division) ring uniquely in the sense of isomorphism. (C) If an ordered plane is alternative it is Desarguesian. That (A) implies (B) was previously pointed out by M. F. Smiley, and (C) answers a question originally posed by Max Dehn to Ruth Moufang some years ago. (This abstract was originally submitted July 27, 1950 to the International Congress, too late for acceptance.) (Received September 15, 1950.)

454*t*. Trevor Evans: *On multiplicative systems defined by generators and relations.* I. *General theory.*

Using the characterization of a loop as a system with three binary operations, a study is made of the properties of loops defined in terms of generators and relations. A normal form theorem is obtained for loops defined by a certain type of relation and it is shown that every finitely related loop can be put in this form. The structure of subloops of loops defined by generators and relations is completely determined and results obtained analogous to those of Grace E. Bates, *The theory of free loops and nets and their generalizations*, Amer. J. Math. vol. 69 (1947) pp. 449–550. Corresponding results hold also for other nonassociative multiplicative systems such as quasigroups, groupoids, and groupoids with unique division on one side. The word problem is solved for loops and the cancellation and division problems for groupoids. (Received August 14, 1950.)

455*t*. Trevor Evans: *On multiplicative systems defined by generators and relations.* II. *Monogenic loops.*

The methods developed in a previous paper by the author *On the theory of multiplicative systems defined by generators and relations.* I. *General theory*, are used to study some properties of monogenic loops (loops generated by one element). The following results are obtained. The group of automorphisms of the free monogenic loop is the free cyclic group. A monogenic loop defined by a finite, nonzero number of relations has only a finite number of endomorphisms. This means that the isomorphism problem can be solved for monogenic loops. Any countable loop can be embedded in a monogenic loop. A monogenic loop is constructed which is isomorphic to a proper factor loop of itself. (Received August 14, 1950.)

456t. D. C. Murdoch: *Intersections of primary ideals in a noncommutative ring.*

Let R be a noncommutative ring in which the ascending chain condition holds for two-sided ideals. An ideal $\mathfrak{q}$ is right primary if when a is not in $\mathfrak{q}$, $aRb \subseteq \mathfrak{q}$ implies $b \in \mathfrak{r}(\mathfrak{q})$, where $\mathfrak{r}(\mathfrak{q})$ is the radical of $\mathfrak{q}$ in the sense of McCoy (Amer. J. Math. vol. 71 (1949) pp. 823–833). The radical of a right primary ideal is prime. If $\mathfrak{a}$ is the intersection of a finite number of right primary ideals then $\mathfrak{a}$ is right primary with radical $\mathfrak{p}$ if and only if all its right primary components have radical $\mathfrak{p}$. Hence any ideal $\mathfrak{a}$ which is the intersection of right primary ideals has a short representation in which the right primary components all have different radicals. In any two short representations of $\mathfrak{a}$ the number of right primary components is the same and the radicals of the components of one representation are equal in some order to the radicals of the components of the other. (Received September 12, 1950.)

457t. H. J. Ryser: *A combinatorial theorem with an application to latin rectangles.*

Let A be a matrix of r rows and n columns composed entirely of zeros and ones, where $1 \leqq r < n$. Let there be exactly k ones in each row, and let $N(i)$ denote the number of ones in the ith column of A. If for each i, $k-(n-r) \leqq N(i) \leqq k$, then $n-r$ rows of zeros and ones may be adjoined to A to obtain a square matrix with exactly k ones in each row and column. Let T be an array of r rows and s columns, formed from the integers $1, \cdots, n$ in such a way that the integers in each row and column are distinct. Let $N(i)$ denote the number of times that i occurs in T. The preceding theorem implies that a necessary and sufficient condition in order that T may be extended to an n by n latin square is that for each $i=1, \cdots, n$, $N(i) \geqq r+s-n$. The latter result generalizes M. Hall's existence theorem for latin squares (Bull. Amer. Math. Soc. vol. 51 (1945) pp. 387–388). (Received September 18, 1950.)

ANALYSIS

458t. P. R. Garabedian: *A partial differential equation arising in conformal mapping.*

Let $K(z, t)$ and $k(z, t)$ be the kernel functions in a plane region D which are associated, respectively, with the norms $\iint_D |f(z)|^2 \rho\, dx dy$ and $\iint_D u(x, y)^2 \rho\, dx dy$ of analytic functions $f(z)$ and real harmonic functions $u(x, y)$ in D, where ρ is a positive weight function. These kernels are given in terms of the Green's functions $G(z, t)$ and $g(z, t)$ of the partial differential equations $\partial/\partial \bar{z}(\rho^{-1}\partial/\partial z)G = 0$ and $\Delta \rho^{-1} \Delta g = 0$ by the formulas $K(z, t) = -2(\pi\rho(z)\rho(t))^{-1}(\partial^2 G(z, t)/\partial z \partial \bar{t})$ and $k(z, t) = -(8\pi\rho(z)\rho(t))^{-1}\Delta_z \Delta_t g(z, t)$. The Friedrichs eigenfunctions ψ_ν and eigenvalues μ_ν for $\mu = \left|\iint_D \psi^2 dx dy\right| / \iint_D |\psi|^2 dx dy = \text{maximum}$, $\partial\psi/\partial\bar{z} = 0$, are given in terms of the eigenfunctions U_ν of the partial differential equation $\partial^2 U/\partial \bar{z}^2 + \mu \partial^2 \overline{U}/\partial z \partial \bar{z} = 0$ by the formula $(1-\mu_\nu^2)\psi_\nu = \partial U_\nu/\partial \bar{z} + \mu_\nu \partial \overline{U}_\nu/\partial z$. Both problems have the same eigenvalues μ_ν. The eigenfunctions m_ν and eigenvalues λ_ν for the problem $\lambda = \iint_D |\partial m/\partial z|^2 dx dy / \iint_D |\partial m/\partial \bar{z}|^2 dx dy$

$=$minimum, $\partial^2 m/\partial \bar{z}^2=0$, are given by the formulas $\partial m_\nu/\partial \bar{z}=\psi_\nu$, $\partial m_\nu/\partial z=\partial U_\nu/\partial z$, $\lambda_\nu=1-\mu_\nu^2$. For $\rho=1$, the identity $\partial^2 g/\partial z\partial\bar{t}=G(z, t)-(\pi/2)\sum(1-\mu_\nu^2)^{-1}U_\nu(z)\overline{U_\nu(t)}$ is proved. Applications are made to the case where D is an ellipse. (Received September 20, 1950.)

459*t*. P. R. Garabedian: *Asymptotic identities among periods of integrals of the first kind.*

In a multiply-connected domain D of the z-plane bounded by curves $C_1, \cdots, C_{g+1}$, the harmonic measures $\omega_k(z)$ with boundary values 1 on C_k and 0 on the C_j with $j\neq k$, $k=1, \cdots, g$, have an interpretation as the real parts of the normal integrals of the first kind on a symmetric closed Riemann surface of genus g associated with D. Since the conformal type of D depends on $3g-3$ parameters, the periods $P_{ij}=\oint_{C_j}(\partial\omega_i/\partial n)ds$ of the integrals of the first kind satisfy $(g-2)(g-3)/2$ nontrivial identities. Asymptotic forms of these identities are obtained which correspond to degeneration of D when, first, all the C_j shrink to points a_j and, second, the C_j with $j\leqq g$ shrink to points a_j and C_{g+1} is the unit circle. In the first case, the asymptotic identities correspond to those which exist among the Euclidean distances between the points a_j; in the second case the identities originate from those which exist among the non-Euclidean distances amongst the a_j in the unit circle. In a further study, Poincaré's identity $(P_{12}P_{13}P_{24}P_{34})^{1/2}+(P_{13}P_{14}P_{23}P_{24})^{1/2}=(P_{12}P_{14}P_{23}P_{34})^{1/2}$ for $P_{ij}\rightarrow 0$, $i\neq j$, and P_{ii} bounded away from zero and infinity is derived. (Received September 20, 1950.)

460*t*. P. R. Garabedian: *The classes L_p and conformal mapping.*

Let D be a region of the z-plane bounded by disjoint smooth curves $C_1, \cdots, C_n$, and let t be a fixed point of D. We denote by $K(z)$ the analytic function in D with $K(t)=1$ and with $\alpha=\oint_C|K(z)|^p ds=$minimum, $p>1$; and we denote by $L(z)$ the function analytic in D except for a simple pole of residue $1/2\pi i$ at $z=t$ and with $\beta=\int_C|L(z)|^q ds=$minimum, $q>1$. For $p^{-1}+q^{-1}=1$ we obtain $L(z)K(z)dz=\alpha^{-1}|K(z)|^p ds=\beta^{-1}|L(z)|^q ds$ on C, and the minimum values α and β satisfy $\alpha^{1/p}\beta^{1/q}=1$. Proofs are based on Holder's inequality and on a consideration of the class L_p of all complex-valued functions $f(z)$ defined on C with $\oint_C|f|^p ds<\infty$ and with $\oint_C f\phi dz=0$ for every function $\phi(z)$ analytic in $D+C$. Further normalizations are discussed, and it is shown in one case where t is the point at infinity that all extremal functions with any value of $p\geqq 1$ are related and are given in terms of the Szego kernel function. Some of the results are extended to the case of values of p with $0<p<1$. (Received September 20, 1950.)

461*t*. J. G. Herriot: *An example of an analytic function with a singular line.*

The function $f(z)=\int_0^\infty(1-\tanh x\tanh zx)dx$, $0<z\leqq 1$, was first encountered in the study of the electrostatic capacity of a symmetric lens. It is easily seen that the above integral represents a function analytic in the half-plane $R(z)>0$. A series representation of $f(z)$ is obtained and used to continue $f(z)$ analytically throughout the rest of the complex plane with the exception of the negative real axis. The negative real axis is shown to be a line of singularities of $f(z)$. This is done by showing that $f(z)$ becomes infinite as z approaches any point of the form $-(2p+1)/(2q+1)$ where p and q are integers greater than one and $2p+1$ and $2q+1$ are relatively prime; z moves so that its real part is constant. (Received August 15, 1950.)

462*t*. Werner Leutert: *On the convergence of approximate solutions of the heat equation to the exact solution.*

It is shown that an approximate solution of the heat equation can be obtained from a three line difference equation by using only half of the particular solutions of the form $e^{i\beta x}e^{\alpha t}$. The approximate solution will converge to the exact solution for all positive values of the mesh ratio $r=\Delta t/(\Delta x)^2$ and it will be stable in the sense that small changes in the initial condition vanish as the time t is increased. von Neumann's test shows instability for all values of $r>0$. (Received July 31, 1950.)

463*t*. Bertram Yood: *On fixed points for semi-groups of linear operators.*

Let G be a semi-group of bounded linear operators on a normed linear space X, and G^* be the family of adjoints of elements of G. Sets of conditions are given on G which imply the existence of a nonzero fixed element for G^* (in X^*). In particular if X is the space of bounded functions on a set S, the results show, as a special case, the existence of a finitely-additive measure defined for all subsets of S invariant under a solvable group of 1-1 transformations of S onto S. This fact is due to von Neumann (Fund. Math. vol. 13 (1929)). (Received September 14, 1950.)

APPLIED MATHEMATICS

464*t*. C. N. Mooers: *Automata with learning.* Preliminary report.

The automata moves in an artificial environment having positions or states $q_i (i=1, \cdots, N_q)$. It has a repertory of moves that it can make, each called m_{ij} ($j=1, \cdots, N_i$). From state q_i by move m_{ij} it goes to a new uniquely determined state q_k, that is, $(q_i, m_{ij})=q_k$. Each state q_i is characterized by an aspect a_i having the value $+1$ or -1. The a_i is a "drive" in the psychological sense, and when a_i is positive the automata is active. In state q_i the automata initially randomly chooses an m_{ij} where all the m's have an equal probability. In the case $(q_i, m_{ij})=q_{i+1}$ whose a_{i+1} is negative (drive extinguished), then the probability is increased for choice m_{ij} when in state q_i. In (q_i, m_{ij}) there is a transfer relation such that when some $m_{i+1,k}$ of q_{i+1} has a probability greater than $2/N_{i+1}$, then the probability of taking m_{ij} in q_i is also increased. The automata as postulated can learn its way through a maze, learning from the goal backwards; it can remember the solution to two or more mazes; it forgets nonused information; and its behavior is not predictable. (Received September 5, 1950.)

465*t*. L. A. Zadeh: *On stability of linear varying-parameter systems.*

Starting with the definition of stability in the case of linear varying-parameter systems: a system is stable if and only if every bounded input produces a bounded output, it is shown that the necessary and sufficient condition for stability is that the impulsive response of the system $W(t, \tau)$ should belong to $L(0, \infty)$ for all t ($W(t, \tau)$ is the response at t to a unit impulse applied at $t-\tau$). The system function of a linear varying-parameter system is related to $W(t, \tau)$ through $H(s; t)=\int_0^\infty W(t, \tau)e^{-s\tau}d\tau$. From this it follows that the system function of a stable system is analytic in the right half and on the imaginary axis of the s-plane for all t. This result can be applied with advantage to the investigation of stability of linear varying-parameter systems. In particular, it yields useful criteria of stability for differential equations having periodic coefficients. (Received September 14, 1950.)

466*t*. L. A. Zadeh: *Initial conditions in linear varying-parameter systems.*

Consider a linear varying-parameter system N whose behavior is described by an nth order linear differential equation $L(p; t)v(t) = u(t)$. Let $u(t)$ be zero for $t<0$ and let the initial values of $v(t)$ and its derivatives be $v^{(\nu)}(0) = \alpha_\nu$ $(\nu = 0, 1, \cdots, n-1)$. Let $H(s; t)$ be the system function of N. When the system is initially at rest (that is, all α_ν are zero), the response of N to $u(t)$ may be written as $v(t) = \mathcal{L}^{-1}\{H(s; t)\, U(s)\}$ (see abstract 56-6-465). When, on the other hand, some of the α_ν are not zero, the expression for the response to a given input $u(t)$ becomes $v(t) = \mathcal{L}^{-1}\{H(s; t)[U(s) + \Delta(s)]\}$, where $\Delta(s)$ is a polynomial in s and p_0 given by $\Delta(s) = \{[L(s; 0) - Lp_0; 0)]/(s - p_0)\}v$ (p_0 represents a differential operator such that $p_0^\nu v = v^{(\nu)}(0) = \alpha_\nu$). $\Delta(s)$ is essentially the Laplace transform of a linear combination of delta-functions of various order (up to $n-1$) such that the initial values of the derivatives of the response of N to this combination are equal to α_ν. (Received September 14, 1950.)

TOPOLOGY

467*t*. A. L. Blakers and W. S. Massey: *Generalized Whitehead products.*

J. H. C. Whitehead has defined (Ann. of Math. vol. 42 (1941) pp. 409–428) a product which associates with elements $\alpha \in \pi_p(X)$ and $\beta \in \pi_q(X)$, an element $[\alpha, \beta] \in \pi_{p+q-1}(X)$. The authors show how to define three new products, as follows: (a) A product which associates with elements $\alpha \in \pi_p(A)$ and $\beta \in \pi_q(X, A)$, an element $[\alpha, \beta] \in \pi_{p+q-1}(X, A)$. (b) A product which associates with elements $\alpha \in \pi_p(A/B)$ and $\beta \in \pi_q(A \cap B)$, an element $[\alpha, \beta] \in \pi_{p+q-1}(A/B)$. Here the sets A and B are a covering of the space $X = A \cup B$, and $\pi_p(A/B)$ is the p-dimensional homotopy group of this covering which has been introduced by the authors (Bull. Amer. Math. Soc. Abstract 56-3-208). (c) Let $(X; A, B)$ be a triad (see A. L. Blakers and W. S. Massey, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) p. 323), then there is a product which associates with elements of $\pi_p(A/B)$ and $\pi_q(X, A \cap B)$ an element of $\pi_{p+q-1}(X; A, B)$. The bilinearity of these three new products is established under suitable restrictions, and relationships between the various products are proved. The behavior of the products under homomorphisms induced by a continuous map or a homotopy boundary operator is also studied. (Received August 30, 1950.)

468*t*. A. L. Blakers and W. S. Massey: *The triad homotopy groups in the critical dimension.*

Let $X^* = X \cup \xi_1^n \cup \xi_2^n \cup \cdots \cup \xi_k^n$ be a space obtained by adjoining the n-dimensional $(n>2)$ cells ξ_i^n to the connected, simply connected topological space X. Let $\xi^n = \xi_1^n \cup \xi_2^n \cup \cdots \cup \xi_k^n$, and $\dot{\xi}^n = X \cap \xi^n$. Assume that the space $\dot{\xi}^n$ is arcwise connected, and that the relative homotopy groups $\pi_p(X, \dot{\xi}^n)$ are trivial for $1 \leqq p \leqq m$, where $m \geqq 1$. Then it is known that the triad homotopy groups $\pi_q(X^*; \xi^n, X)$ are trivial for $2 \leqq q \leqq m+n-1$. The authors now show that under the assumption of suitable "smoothness" conditions on the pair $(X, \dot{\xi}^n)$ (for example, both X and $\dot{\xi}^n$ are compact A.N.R.'s), there is a natural isomorphism of the tensor product $\pi_n(\xi^n, \dot{\xi}^n) \otimes \pi_{m+1}(X/\dot{\xi}^n)$ onto the triad homotopy group $\pi_{m+n}(X^*; \xi^n, X)$. This isomorphism is defined by means of a generalized Whitehead product. The Freudenthal "Einhängung" theorems in the critical dimensions can easily be derived from this theorem;

also, some results of J. H. C. Whitehead follow as a corollary (see Ann. of Math. vol. 42 (1941) pp. 409–428, especially Lemma 4). The proof makes use of the functional cup product of Steenrod, and the theory of obstructions to deformations of continuous mappings. (Received August 30, 1950.)

469*t*. Tibor Radó: *On the identifications in singular homology theory.*

According to the classical formulation of singular homology theory, a singular p-cell of a topological space X is an aggregate $[(x_0, \cdots, x_p, T)]$ of a geometrical p-simplex $(x_0, \cdots, x_p)$ and of a continuous mapping $T: (x_0, \cdots, x_p) \to X$, subject to the following identifications. (1) If $(y_0, \cdots, y_p)$ is an even permutation of $(x_0, \cdots, x_p)$, then $[(x_0, \cdots, x_p), T] = [(y_0, \cdots, y_p), T]$. (2) If m is an affine map such that $m(y_i) = x_i$, $i = 0, \cdots, p$, then $[(x_0, \cdots, x_p), T] = [(y_0, \cdots, y_p), Tm]$. The main result in this paper is the theorem that the singular homology groups of X are unchanged (up to isomorphisms) if both of these identifications are eliminated, provided that the geometrical simplexes $(x_0, \cdots, x_p)$, $p = 0, 1, 2, \cdots$, are taken from a fixed Hilbert space. This theorem completes previous work by Eilenberg, who obtained an analogous result concerning the identification (1) alone. (Received September 5, 1950.)

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J. H. Bell, *The solution of a unilateral direct product matrix equation.* Abstract 55-11-466.

p. 1044, lines 2, 3, and 7 of the abstract. For "$\cdot x$" read "$\cdot \times$."

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Vice Presidents	W. T. Martin John von Neumann Hassler Whitney
Secretary	J. R. Kline
Associate Secretaries	J. W. Green T. R. Hollcroft G. B. Price J. W. T. Youngs
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Bulletin Editorial Committee	Nathan Jacobson R. E. Langer Deane Montgomery E. B. Stouffer
Transactions Editorial Committee	Saunders MacLane G. T. Whyburn Antoni Zygmund
Colloquium Editorial Committee	Einar Hille C. C. MacDuffee R. L. Wilder
Mathematical Reviews Editorial Committee	William Feller M. H. Stone Hassler Whitney
Mathematical Surveys Editorial Committee	Nelson Dunford A. W. Tucker J. L. Walsh
Representatives on Board of Editors of the American Journal of Mathematics	Richard Brauer L. M. Graves Hassler Whitney

Ex-Presidents

M. H. Stone
T. H. Hildebrandt
Einar Hille

Ex-Secretary

R. G. D. Richardson

Members At Large

To serve until December, 1949	*To serve until December, 1950*	*To serve until December, 1951*
H. W. Bode	J. L. Doob	Emil Artin
Samuel Eilenberg	R. L. Jeffery	Herbert Busemann
B. O. Koopman	J. B. Rosser	M. H. Heins
Gordon Pall	N. E. Steenrod	G. W. Mackey
J. W. Tukey	Alfred Tarski	L. C. Young

BOARD OF TRUSTEES

B. P. Gill[1]
T. H. Hildebrandt
W. R. Longley
R. G. D. Richardson
P. A. Smith
G. T. Whyburn

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[1] Succeeded R. G. D. Richardson, died July, 1949.

1950

MEMBERS OF THE COUNCIL

Officers

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President Elect	John von Neumann
Vice Presidents	E. J. McShane W. T. Martin R. L. Wilder
Secretary	J. R. Kline
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Treasurer	A. E. Meder, Jr.
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Colloquium Editorial Committee	Einar Hille C. C. MacDuffee R. L. Wilder
Mathematical Reviews Editorial Committee	William Feller Einar Hille Hassler Whitney
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Representatives on Board of Editors of the American Journal of Mathematics	Richard Brauer Samuel Eilenberg L. M. Graves

Ex-Presidents

M. H. Stone T. H. Hildebrandt

Members At Large

To serve until December, 1950	*To serve until December, 1951*	*To serve until December, 1952*
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R. L. Jeffery	Herbert Busemann	P. R. Halmos
J. B. Rosser	M. H. Heins	Marc Kac
N. E. Steenrod	G. W. Mackey	S. B. Myers
Alfred Tarski	L. C. Young	D. C. Spencer

BOARD OF TRUSTEES

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T. H. Hildebrandt		G. T. Whyburn

REPRESENTATIVES AND COMMITTEES OF THE SOCIETY

Representatives of the Society in the Division of Mathematical and Physical Sciences of the National Research Council:

1948–1949—R. V. Churchill, M. R. Hestenes, T. H. Hildebrandt, M. H. Ingraham, John von Neumann, M. H. Stone.
1949–1950—R. V. Churchill, M. R. Hestenes, T. H. Hildebrandt, M. H. Ingraham, Marston Morse, Oscar Zariski.
1950–1951—G. C. Evans, T. H. Hildebrandt, Einar Hille, M. H. Ingraham, Marston Morse, Oscar Zariski.
1951–1952—G. C. Evans, Einar Hille, J. R. Kline, Marston Morse, D. C. Spencer, Oscar Zariski.

Representatives on the Council of the American Association for the Advancement of Science:

1948—R. P. Agnew, W. M. Whyburn.
1949—R. P. Agnew, W. M. Whyburn.
1950—H. F. Bohnenblust, L. R. Ford.
1951—H. F. Bohnenblust, L. R. Ford.

Representatives on the Editorial Board of the Annals of Mathematics:

J. L. Doob, Norman Levinson, R. L. Wilder.

Representatives on the Editorial Board of the Duke Mathematical Journal:

C. B. Allendoerfer, R. P. Boas, Jr.

Representative on the American Year Book:

Hans Rademacher.

Representative on Advisory Board of Applied Mechanics Reviews:

Richard Courant.

Liaison Officer in Connection with Editoral Management of Quarterly of Appled Mathematics:

J. B. Rosser.

Colloquium Lecturers:

1948—Richard Brauer.
1949—G. A. Hedlund.
1951—Deane Montgomery.

Committee to Select Gibbs Lecturers for 1950 and 1951:

William Feller (Chairman), G. A. Hedlund, M. H. Heins.

Gibbs Lecturers:

1923—M. I. Pupin.
1924—Robert Henderson.
1925—James Pierpont.
1926—H. B. Williams.
1927—E. W. Brown.
1928—G. H. Hardy.
1929—Irving Fisher.
1930—E. B. Wilson.
1931—P. W. Bridgman.
1932—R. C. Tolman.
1934—Albert Einstein.
1935—Vannevar Bush.
1936—H. N. Russell.
1937—C. A. Kraus.
1939—Theodore von Kármán.
1941—Sewall Wright.
1943—Harry Bateman.
1944—John von Neumann.
1945—J. C. Slater.
1946—Subrahmanyan Chandrasekhar.
1947—P. M. Morse.
1948—Hermann Weyl.
1949—Norbert Wiener.
1950—G. E. Uhlenbeck.

Committee on Society Visiting Lectureship:

Hassler Whitney (Chairman), Richard Courant, H. S. Wall.

Visiting Lecturers:

1927–1928—Constantin Carathéodory, University of Munich.
1928–1929—Hermann Weyl, Zurich Technical School.
1929–1930—Enrico Bompiani, University of Rome.
1930–1931—Wilhelm Blaschke, University of Hamburg.
1931–1932—R. L. Moore, University of Texas.
1936–1937—Thirjkkannapuram Vijayaraghavan, University of Dacca.
1948–1949—Casimir Kuratowski, University of Warsaw.

Auditors for 1950:

T. F. Cope, B. P. Gill.

Committee on Publication of Moore Manuscript:

R. W. Barnard, E. W. Chittenden, L. M. Graves, T. H. Hildebrandt, M. H. Ingraham.

Committee on Role of Society in Mathematical Publication:

Einar Hille (Chairman), Garrett Birkhoff, G. C. Evans, J. R. Kline, R. E. Langer, A. W. Tucker.

Committee on Printing Contracts:

C. J. Rees (Chairman), R. A. Johnson, H. M. MacNeille, J. M. Thomas.

Committees to Select Hour Speakers:

For Annual and Summer Meetings: J. R. Kline (Chairman), A. A. Albert, G. A. Hedlund.
For Eastern Sectional Meetings: T. R. Hollcroft (Chairman), E. G. Begle, D. V. Widder.
For Western Sectional Meetings: J. W. T. Youngs (Chairman), Ralph Hull, Walter Leighton.
For Far Western Sectional Meetings: J. W. Green (Chairman), E. F. Beckenbach, M. R. Hestenes.
For Southeastern Sectional Meetings: W. M. Whyburn (Chairman), Tomlinson Fort, J. M. Thomas.

Committee on Places of Meetings:

R. L. Wilder (Chairman), R. W. Brink, J. M. Thomas.

Policy Committee for Mathematics:

Representatives of American Mathematical Society: Einar Hille, R. E. Langer, W. T. Martin, Marston Morse (Chairman).
Representative of Association for Symbolic Logic: A. E. Meder.
Representative of Institute of Mathematical Statistics: Henry Scheffé.
Representatives of Mathematical Association of America: L. R. Ford, H. M. Gehman, C. V. Newsom.
Secretary: J. R. Kline.

Committee on Applied Mathematics:

Walter Bartky (Chairman), G. C. Evans, K. O. Friedrichs, John von Neumann, William Prager, Eric Reissner.

Committee on Library and Housing:

T. H. Hildebrandt (Chairman), Arnold Dresden, B. P. Gill, J. R. Kline, Marston Morse, P. A. Smith.

Editorial Committee for Applied Mathematics Symposium Proceedings:

R. V. Churchill (Chairman), Eric Reissner, A. H. Taub.

Committee on Translation of Russian and Other Mathematical Articles:

J. V. Wehausen (Chairman), R. P. Boas, Samuel Eilenberg, D. H. Lehmer, William Prager, G. Y. Rainich, Irving Kaplansky.

INTERNATIONAL CONGRESS OF MATHEMATICIANS

Cambridge, Mass., August 30–September 6, 1950

Organizing Committee:

Garrett Birkhoff (Chairman), W. T. Martin (Vice Chairman), A. A. Albert, J. L. Doob, G. C. Evans, T. H. Hildebrandt, Einar Hille, J. R. Kline, Solomon Lefschetz, Saunders MacLane, Marston Morse, John von Neumann, Oswald Veblen, J. L. Walsh, Hassler Whitney, D. V. Widder, R. L. Wilder.

Editorial Committee:

L. M. Graves (Chairman), Einar Hille, P. A. Smith, Oscar Zariski.

Financial Committee:

John von Neumann (Chairman), W. L. G. Williams (Vice Chairman), J. L. Coolidge, B. P. Gill, M. H. Ingraham, A. E. Meder, Jr.

Secretariat:

J. R. Kline (Secretary), R. P. Boas (Associate Secretary).

Budget Committee:

W. T. Martin (Chairman), J. R. Kline, A. E. Meder, Jr., G. B. Price, Oswald Veblen, Oscar Zariski.

Cooperation Committee:

Hassler Whitney (Chairman), E. F. Beckenbach, Samuel Eilenberg, S. S. Wilks, J. W. T. Youngs.

Entertainment Committee:

L. H. Loomis (Chairman), C. R. Adams, Mrs. L. V. Ahlfors, Mrs. G. D. Birkhoff, J. A. Clarkson, Mrs. W. C. Graustein, F. B. Hildebrand, Evelyn M. Hull, J. R. Kline, E. R. Lorch, Mrs. W. T. Martin, E. B. Mode, G. A. O'Donnell, Mrs. H. B. Phillips, Helen G. Russell, J. H. Van Vleck, Mrs. J. L. Walsh, Mrs. D. V. Widder, Mrs. Norbert Wiener.

Publicity Committee:

G. W. Mackey (Chairman), A. A. Bennett, R. P. Boas, J. A. Clarkson, C. O. Oakley, R. M. Thrall.

FORMER PRESIDENTS

J. H. Van Amringe, 1889–1890.
Emory McClintock, 1891–1894.
G. W. Hill, 1895–1896.
Simon Newcomb, 1897–1898.
R. S. Woodward, 1899–1900.
E. H. Moore, 1901–1902.
T. S. Fiske, 1903–1904.
W. F. Osgood, 1905–1906.
H. S. White, 1907–1908.
Maxime Bôcher, 1909–1910.
H. B. Fine, 1911–1912.
E. B. Van Vleck, 1913–1914.
E. W. Brown, 1915–1916.
L. E. Dickson, 1917–1918.
Frank Morley, 1919–1920.
G. A. Bliss, 1921–1922.
Oswald Veblen, 1923–1924.
G. D. Birkhoff, 1925–1926.
Virgil Snyder, 1927–1928.
E. R. Hedrick, 1929–1930.
L. P. Eisenhart, 1931–1932.
A. B. Coble, 1933–1934.
Solomon Lefschetz, 1935–1936.
R. L. Moore, 1937–1938.
G. C. Evans, 1939–1940.
Marston Morse, 1941–1942.
M. H. Stone, 1943–1944.
T. H. Hildebrandt, 1945–1946.
Einar Hille, 1947–1948.

ENDOWMENT FUND

In 1923 an Endowment Fund was collected to meet the greater demands on the Society's publication program caused by the ever increasing number of important mathematical memoirs. Of this fund, which now amounts to some $71,000, a considerable proportion was contributed by members of the Society. Under the terms of the will of the late Robert Henderson, for many years a Trustee of the Society, the Society receives approximately $4,000 yearly. Upon the death of the other legatees, the Society will receive the entire principal of the estate for its Endowment Fund.

SPECIAL FUNDS

The Bôcher Memorial Prize

This prize was founded in memory of Professor Maxime Bôcher. It is awarded every five years for a notable research memoir in analysis which has appeared during the preceding five years in a recognized journal published in the United States or Canada; the recipient must be a member of the Society, and not more than fifty years old at the time of publication of his memoir.

First (Preliminary) Award, 1923: To G. D. Birkhoff, for his memoir *Dynamical systems with two degrees of freedom.*

Second Award, 1924: To E. T. Bell, for his memoir *Arithmetical paraphrases,* and to Solomon Lefschetz, for his memoir *On certain numerical invariants with applications to abelian varieties.*

Third Award, 1928: To J. W. Alexander, for his memoir *Combinatorial analysis situs.*

Fourth Award, 1933: To Marston Morse, for his memoir *The foundations of a theory of the calculus of variations in the large in m-space,* and to Norbert Wiener, for his memoir *Tauberian theorems.*

Fifth Award, 1938: To John von Neumann, for his memoir *Almost periodic functions and groups.*

Sixth Award, 1943: To Jesse Douglas, for his memoirs *Green's functions and* *the problem of Plateau, The most general form of the problem of Plateau,* and *Solution of the inverse problem of the calculus of variations.*

Seventh Award, 1948: To A. C. Schaeffer and D. C. Spencer for their memoir *Coefficients of Schlicht functions.*

The Frank Nelson Cole Prize in Algebra.
The Frank Nelson Cole Prize in the Theory of Numbers.

These prizes were founded in honor of Professor Frank Nelson Cole, on the occasion of his retirement as Secretary of the American Mathematical Society and Editor of the Bulletin, after twenty-five years of service; the fund was later doubled by his son, Charles A. Cole. They are awarded at five-year intervals for contributions to algebra and the theory of numbers, respectively, under restrictions similar to those for the Bôcher prize.

First Award, 1928: To L. E. Dickson, for his book *Algebra und ihre Zahlentheorie,* Zurich, 1927.

Second Award, 1931: To H. S. Vandiver, for his several papers on Fermat's last theorem published in the Transactions of the American Mathematical Society and the Annals of Mathematics during the last five years, with special reference to a paper entitled *On Fermat's last theorem,* which appeared in volume 31 of the Transactions.

Third Award, 1939: To A. A. Albert, for his papers on the construction of Riemann matrices, published in volumes 35 and 36 of the Annals of Mathematics.

Fourth Award, 1941: To Claude Chevalley, for his paper entitled *La théorie du corps de classes,* published in volume 41 of the Annals of Mathematics.

Fifth Award, 1944: To Oscar Zariski, for four papers on algebraic varieties, published in volumes 61 and 62 of the American Journal of Mathematics and volumes 40 and 41 of the Annals of Mathematics.

Sixth Award, 1946: To H. B. Mann for his paper entitled *A proof of the fundamental theorem on the density of sums of sets of positive integers,* published in volume 43 of the Annals of Mathematics.

Seventh Award, 1949: To Richard Brauer, for his memoir *On Artin's L-series with general group characters,* published in volume 48 of the Annals of Mathematics.

The Eliakim Hastings Moore Fund.

This fund was founded in 1922 in honor of Professor Eliakim Hastings Moore, on the occasion of the twenty-fifth anniversary meeting of the Chicago Section of the American Mathematical Society. The income from the fund is to be used at the discretion of the Council of the Society, for the publication of important mathematical books or memoirs, or the award of prizes.

The Marion Reilly Fund.

Dean Marion Reilly willed to the American Mathematical Society a portion of her estate to be used for the advancement of research in pure mathematics. The principal of this fund is $23,500.

The Ernest William Brown Fund.

From the estate of Professor Ernest William Brown, a fund of $1,000 is available, the interest on which, at the discretion of the Council, can be used for the furtherance of such mathematical interests as (a) the publication of important mathematical books, memoirs, and periodicals, and (b) lectures to be delivered on special occasions by invited guests of the Society.

The Robert Henderson Fund.

Upon his retirement from the Board of Trustees on December 31, 1940, Dr. Robert Henderson presented to the Society an unrestricted gift of $1,000 which the Trustees have set aside as the Robert Henderson Fund.

The James K. Whittemore Fund.

From the estate of Professor James K. Whittemore, a fund of $1,000 is available for use in the interest of mathematical research.

The Generva B. Hutchinson Fund.

A fund from the estate of the late Professor Generva B. Hutchinson for use, at the discretion of the Council and Trustees, in the advancement of mathematical research.

The Helen A. Merrill Fund

From the estate of the late Professor Helen A. Merrill, a fund of $650 is available for use in the interest of mathematical research.

INSTITUTIONAL MEMBERS

Acadia University, Wolfville, N.S., Canada.
University of Alabama, University, Ala.
Amherst College, Amherst, Mass.
University of Arizona, Tucson, Ariz.
Bell Telephone Laboratories, New York 14, N.Y.
Beloit College, Beloit, Wis.
Brigham Young University, Provo, Utah.
University of British Columbia, Vancouver, B.C., Canada.
Brooklyn College, Brooklyn 10, N.Y.
Brown University, Providence 12, R.I.
Bryn Mawr College, Bryn Mawr, Pa.
University of Buffalo, Buffalo 14, N.Y.
California Institute of Technology, Pasadena 4, Calif.
University of California, Berkeley 4, Calif.
University of California at Los Angeles, Los Angeles 24, Calif.
Carnegie Institute of Technology, Pittsburgh 13, Pa.
Case Institute of Technology, Cleveland 6, Ohio.
Catholic University of America, Washington 17, D.C.
University of Chicago, Chicago 37, Ill.
University of Cincinnati, Cincinnati 21, Ohio.
City College, New York 31, N.Y.
University of Colorado, Boulder, Colo.
Columbia University, New York 27, N.Y.
Connecticut College, New London, Conn.
Cornell University, Ithaca, N.Y.
Dartmouth College, Hanover, N.H.
University of Delaware, Newark, Del.
Duke University, Durham, N.C.
Duquesne University, Pittsburgh 19, Pa.
Equitable Life Insurance Company of Iowa, Des Moines 6, Iowa.
Georgetown University, Washington 7, D.C.
University of Georgia, Athens, Ga.
Gettysburg College, Gettysburg, Pa.
Harvard University, Cambridge 38, Mass.
Haverford College, Haverford, Pa.
Illinois Institute of Technology, Chicago 16, Ill.
University of Illinois, Urbana, Ill.
Immaculata College, Immaculata, Pa.
Indiana University, Bloomington, Ind.
Institute for Advanced Study, Princeton, N.J.
Iowa State College of Agriculture and Mechanic Arts, Ames, Iowa.
The State University of Iowa, Iowa City, Iowa.
The Johns Hopkins University, Baltimore 18, Md.
University of Kansas, Lawrence, Kan.
University of Kentucky, Lexington 29, Ky.
Kenyon College, Gambier, Ohio.
Lehigh University, Bethlehem, Pa.

Louisiana State University and Agricultural and Mechanical College, Baton Rouge 3, La.
McGill University, Montreal, Que., Canada.
University of Manitoba, Winnipeg, Man., Canada.
University of Maryland, College Park, Md.
University of Massachusetts, Amherst, Mass.
Massachusetts Institute of Technology, Cambridge 39, Mass.
Mathematical Association of America, Buffalo 14, N.Y.
Metropolitan Life Insurance Company, New York 10, N.Y.
Michigan State College of Agriculture and Applied Science, East Lansing, Mich.
University of Michigan, Ann Arbor, Mich.
University of Minnesota, Minneapolis 14, Minn.
University of Missouri, Columbia, Mo.
University of Nebraska, Lincoln 8, Neb.
University of New Mexico, Albuquerque, N.M.
New York University, New York 12, N.Y.
University of North Carolina, Chapel Hill, N.C.
Northwestern University, Evanston, Ill.
University of Notre Dame, Notre Dame, Ind.
Ohio State University, Columbus 10, Ohio.
Oklahoma Agricultural and Mechanical College, Stillwater, Okla.
University of Oklahoma, Norman, Okla.
Oregon State College, Corvallis, Ore.
University of Oregon, Eugene, Ore.
Pennsylvania State College, State College, Pa.
University of Pennsylvania, Philadelphia 4, Pa.
University of Pittsburgh, Pittsburgh 13, Pa.
Princeton University, Princeton, N.J.
Purdue University, Lafayette, Ind.
Rice Institute, Houston 1, Tex.
University of Rochester, Rochester 3, N.Y.
Rutgers University, New Brunswick, N.J.
The College of St. Thomas, St. Paul 5, Minn.
University of Saskatchewan, Saskatoon, Sask., Canada.
Smith College, Northampton, Mass.
Southern Illinois University, Carbondale, Ill.
Stanford University, Stanford, Calif.
Stevens Institute of Technology, Hoboken, N.J.
Swarthmore College, Swarthmore, Pa.
Sweet Briar College, Sweet Briar, Va.
Syracuse University, Syracuse 10, N.Y.
Temple University, Philadelphia 22, Pa.
University of Tennessee, Knoxville 16, Tenn.
Texas Christian University, Ft. Worth, Tex.
Texas Technological College, Lubbock, Tex.
Department of Pure Mathematics, University of Texas, Austin, Tex.
University of Toronto, Toronto 5, Ont., Canada.
Tulane University of Louisiana, New Orleans 15, La.
Vanderbilt University, Nashville 4, Tenn.

Virginia Polytechnic Institute, Blacksburg, Va.
University of Virginia, Charlottesville, Va.
Washington University, St. Louis 5, Mo.
University of Washington, Seattle 5, Wash.
Wayne University, Detroit 1, Mich.
Wellesley College, Wellesley 81, Mass.
Wells College, Aurora, N.Y.
Wesleyan University, Middletown, Conn.
Western and Southern Life Insurance Company, Cincinnati 1, Ohio.
Western Reserve University, Cleveland 6, Ohio.
Wheaton College, Norton, Mass.
University of Wichita, Wichita, Kan.
Williams College, Williamstown, Mass.
Yale University, New Haven 11, Conn.

ORGANIZATIONS COÖPERATING AS NONMEMBERS

National Research Council, Washington 25, D.C.
Rockefeller Foundation, New York 20, N.Y.
University of Wisconsin, Madison 6, Wis.

BY-LAWS OF THE AMERICAN MATHEMATICAL SOCIETY

ARTICLE I

OFFICERS

Section 1. There shall be a President, a President Elect (during the even-numbered years only), an Ex-president (during the odd-numbered years only), three Vice Presidents, a Secretary, four Associate Secretaries, a Treasurer and a Librarian.

Section 2. It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

Section 3. The Librarian shall have charge of arrangements for the exchange of the Society's publications.

ARTICLE II

BOARD OF TRUSTEES

Section 1. There shall be a Board of Trustees consisting of five Trustees elected by the Society in accordance with Article VII.

Section 2. The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

Section 3. The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the financial business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

ARTICLE III

EDITORIAL COMMITTEES

Section 1. There shall be seven Editorial Committees, as follows: a committee of two members for the *Bulletin;* committees of three members each for the *Proceedings,* the *Colloquium Publications, Mathematical Surveys* and *Mathematical Reviews;* a committee of four members jointly for the *Transactions* and the *Memoirs,* and one consisting of two representatives of the Society on the Board of Editors of the *American Journal of Mathematics.*

ARTICLE IV

COUNCIL

Section 1. The Council shall consist of fifteen members at large and the following ex officio members: the officers of the Society specified in Article I, the members of the Editorial Committees specified in Article III, any former Secretary for a period of two years following his term of office, and members of the Executive Committee who remain on the Council by the operation of Article VII, Section 4. Former Presidents and former Secretaries belonging to the Council in 1948 shall continue as members of the Council during the full terms specified in the By-Laws in effect on January 1, 1948, namely, life membership for former Secretaries who have served ten years or more and six years for former Presidents after completion of their terms of office.

Section 2. The Council shall formulate and administer the scientific policies of the Society and shall act in an advisory capacity to the Board of Trustees.

Section 3. In the absence of the Secretary from any meeting of the Council, one of the Associate Secretaries present may be designated as Acting Secretary for the meeting, either by written authorization of the Secretary, or, failing that, by majority agreement among the Associate Secretaries present.

Section 4. All members of the Council shall be voting members. The method for settling matters before the Council at any meeting shall be by majority vote of the members present. If the result of a vote is challenged, it shall be the duty of the presiding officer to determine the true vote by a roll call. In a roll call vote, each Council member shall vote only once (although he may be a member of the Council in several capacities), and he shall state before the vote in which capacity he votes. The group consisting of the four Associate Secretaries shall have one vote, and it shall be divided equally among those who vote as Associate Secretaries. Each of the six Editorial Committees shall have one vote, and it shall be divided equally among those who vote as members of the respective Editorial Committees. All other members of the Council shall have one vote each. Fractional votes shall be counted.

Section 5. Any group of members of the Council who have a total of five votes as defined in Section 4 of this Article shall constitute a quorum for the transaction of business at any meeting of the Council.

Section 6. Between meetings of the Council, business may be transacted by a mail vote. Votes shall be counted as in the case of a roll call vote as specified in Section 4 of this Article, "members present" being replaced by "members voting" An affirmative vote by mail on any proposal shall be declared if and only if (a) more than half of the total number of possible votes is received by the time announced for the closing of the polls, and (b) at least three-quarters of the votes received by then are affirmative. If members who have a total of five or more votes request postponement at the time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless either (1) at the discretion of the Secretary, the question is made the subject of a second vote by mail, in connection with which brief statements of reasons, for and against, are circulated or (2) the Council places the matter at issue before the Executive Committee for action.

Section 7. The Council may delegate to the Executive Committee (Article V) certain of its duties and powers. Between meetings of the Council the Executive Committee shall act for the Council on such matters and in such ways as the Council may specify. Nothing herein contained shall be construed as empowering the Council to divest itself of responsibility for formulating and administering the scientific policies of the Society.

ARTICLE V

EXECUTIVE COMMITTEE

Section 1. There shall be an Executive Committee of the Council, consisting of four elected members and the following ex officio members: the President, the Secretary, the President Elect (during even-numbered years), and the Ex-president (during odd-numbered years).

Section 2. The Executive Committee of the Council shall be empowered to act for the Council on matters which have been delegated to the Executive Committee by the Council. If three members of the Executive Committee request that any matter be referred to the Council, the matter shall be so referred. The Executive Committee shall be responsible to the Council and shall report its actions to the Council. It may consider the agenda for meetings of the Council and may make recommendations to the Council.

Section 3. Each member of the Executive Committee shall have one vote. An affirmative vote on any proposal before the Executive Committee shall be declared if and only if at least four affirmative votes are cast for the proposal. A vote on any proposal may be determined at a meeting of the Executive Committee, but it shall not be necessary to hold a meeting to determine a vote.

ARTICLE VI

EXECUTIVE DIRECTOR

Section 1. There shall be an Executive Director who shall be a paid employee of the Society. He shall have charge of the central office of the Society, and he shall be responsible for the general administration of the affairs of the Society in accordance with the policies that are set by the Board of Trustees and by the Council.

Section 2. The Executive Director shall be elected by the Council for a specified term and approved by the Board of Trustees. The terms and conditions of his employment shall be fixed by the Board of Trustees.

Section 3. The Executive Director shall work under the immediate direction of the Board of Trustees and of the Council, and in cooperation with the Secretary. He shall attend meetings of the Board of Trustees, of the Council, and of the Executive Committee, but he shall not be a member of any of these bodies.

ARTICLE VII

ELECTION AND TERMS OF OFFICERS

Section 1. The term of office shall be one year in the case of the President Elect and the Ex-president; two years in the case of the Trustees (with the exceptions stated in the next paragraph of this Section), the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the Librarian and the members of the Editorial Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. The term of office for elected members of the Executive

Committee shall be two years (with the exceptions stated in the last paragraph of this Section), two of the elected members retiring annually. In every case, however, the officials specified in Articles I, II, III, IV, and V, with the exception of the President Elect and the Ex-president, shall continue to serve until their successors have been duly elected and qualified.

The term of office in the case of the Trustees to be elected at the Annual Meeting to be held between the fifteenth of December, 1948, and the fifteenth of January, 1949, shall be three years in the case of two of the Trustees to be elected and shall be two years in the case of three of the Trustees to be elected. Upon the expiration of the terms of office of the Trustees elected at the Annual Meeting to be held between the fifteenth of December, 1948, and the fifteenth of January, 1949, the term of office of all Trustees shall be two years.

In the case of members elected to the Executive Committee at the first election following adoption of these By-Laws the terms of office shall be one year for two of the members so elected and two years for the other two. After the expiration of the one-year terms of office, the term of office for all elected members of the Executive Committee shall be two years as specified in the first paragraph of this Section.

The initial Editorial Committee for the *Proceedings* shall consist of three members, chosen by the Council from the existing Editorial Committee of the *Bulletin* They shall be withdrawn from the latter committee and shall complete their terms of office as members of the new committee. At the first election following adoption of this by-law, a second member shall be elected to the Editorial Committee of the *Bulletin* and a fourth member to the joint Editorial Committee of the *Transactions,* and the *Memoirs*.

Section 2. Election of officers specified in Article I (with the exception of the President and the Ex-president), Trustees, members of the Editorial Committees, and members at large of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member of the Society by the Secretary at least one month before the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the name of the voter and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain one or more names proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A plurality of all votes cast, whether cast in person or by mail, shall be necessary for election. In case of failure to secure a plurality for any office, the members present at the Annual Meeting shall choose by ballot among the members having the highest number of votes.

Section 3. At the end of his term of office, the President Elect shall become the President. At the end of his term of office, the President shall become the Ex-president.

Section 4. Within fourteen days after the election of officers at the Annual Meeting the Secretary shall send to all members of the Council for a mail vote a ballot containing two names for each place to be filled on the Executive Committee. The nominees shall be chosen by a committee appointed by the President. Members of the Council may vote for persons not nominated. Any member of the Council who is not an ex officio member of the Executive Committee (see Article V, Section 1) shall be eligible for election to the Executive Committee. In case a member is elected to the Executive Committee for a term extending beyond his regular term on the Council, he shall automatically continue as a member of the Council during his term on the Executive Committee.

Section 5. The President and Vice Presidents shall not be eligible for immediate re-election to their respective offices. A member at large or an ex officio member of the Council shall not be eligible for immediate election (or re-election) as a member at large of the Council.

Section 6. If the President of the Society die or resign while a President Elect is in office, the President Elect shall serve as President for the remainder of the year and thereafter shall serve his regular two-year term. If the President of the Society die or resign when no President Elect is in office, the Council, with the approval of the Board of Trustees, shall designate one of the Vice Presidents to serve as President for the balance of the regular presidential term. If the President Elect of the Society die or resign before becoming President, his office shall remain vacant until the next regular election of a President Elect, and the Society shall, at the next Annual Meeting, elect a President for a two-year term. If the Ex-president die or resign before the expiration of his term of office, the Council, with the approval of the Board of Trustees, shall designate a former President of the Society to serve as Ex-president during the remainder of the regular term of the Ex-presidency. Such vacancies as may occur at any time in the group consisting of the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, the Librarian, and the members of the Editorial Committees shall be filled by the Council with the approval of the Board of Trustees. If a member of an Editorial Committee takes temporary leave from his duties, that Committee shall nominate a substitute for consideration by the Council. The Council shall then elect a substitute. The Council shall fill from its own membership any vacancy in the elected membership of the Executive Committee.

Section 7. If any Trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

Section 8. If any member at large of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

ARTICLE VIII

Members and Their Election

Section 1. Election of members shall be by vote of the Council or of its Executive Committee.

Section 2. There shall be three classes of members, namely, ordinary, contributing and institutional.

Section 3. Application for admission to ordinary membership shall be made by the applicant on a blank provided by the Secretary, and shall be approved either by two ordinary members of the Society or by the institutional member whose nominee he is (see Article IX, Section 6). Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council approved by the Board of Trustees.

Section 4. An ordinary member may become a contributing member by paying the dues for such membership. (See Article IX, Section 3.)

Section 5. A university or college, or a firm, corporation, or association interested in the support of mathematics may be elected an institutional member.

ARTICLE IX

DUES AND PRIVILEGES OF MEMBERS

Section 1. Any person elected to ordinary membership in the Society under the provisions of Article VIII, Section 3, who is not a nominee of an institutional member (See Section 6 of this Article) shall be admitted to membership upon payment, within sixty days of the date of his election, of an initiation fee of five dollars and such dues as are payable under Section 2 of this Article.

Section 2. The annual dues of a person, not a nominee of an institutional member, who is elected to ordinary membership under the provisions of Article VIII, Section 3, shall be fourteen dollars, with the following exceptions: (1) during the first and second years of membership, the annual dues shall be nine dollars; during the third and fourth years of membership, the annual dues shall be eleven dollars and fifty cents; a person shall be considered to have completed his first year of membership on January 1 following his election; (2) the amount of dues may be altered by reciprocity agreements with other societies; (3) the Council may make special rulings in exceptional cases, with the approval of the Board of Trustees. Each new member shall pay in proportion to the unexpired fraction of the year at the time of his election.

Section 3. The minimum dues for a contributing member shall be twenty dollars per year. Members may, upon their own initiative, pay larger dues.

Section 4. The minimum dues of an institutional member shall depend on the amount of published material credited to that member in certain journals during a specific period. The formula for computing these dues shall be established from time to time by the Council subject to approval by the Board of Trustees. Institutions may pay larger dues than the computed minimum.

Section 5. Any member whose annual dues are at least one hundred dollars shall be designated as a Sustainer of the Society; any member whose annual dues are at least five hundred dollars shall be designated as a Patron of the Society.

Section 6. The privileges of an institutional member shall depend on its dues in a manner to be determined by the Council, subject to approval by the Board of Trustees. These privileges shall be in terms of Society publications to be received by the institution and of the number of persons it may nominate for ordinary membership in the Society.

Section 7. A person elected to ordinary membership as a nominee of an institutional member under the provisions of Section 6 of this Article and of Article VIII, Section 3, shall be admitted to membership immediately and shall not be required to pay an initiation fee. He shall not be required to pay dues so long as it is agreed that he is designated as a nominee of the institutional member under these provisions. If a nominee of an institutional member later becomes a dues-paying member, he shall pay dues at the rate of nine dollars a year for the remainder (if any) of his first two years of membership, eleven dollars and fifty cents a year for the remainder (if any) of his third and fourth years of membership, and fourteen dollars a year thereafter.

Section 8. After retirement from active service on account of age, any ordinary or contributing member who is not in arrears of dues and with membership extending over at least twenty years may, by giving proper notification to the Secretary, have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings but not the *Bulletin* and *Proceedings*.

Section 9. An ordinary member or a contributing member shall receive the *Bulletin* and *Proceedings*, except in the case of members whose dues are remitted under Section 8 of this Article.

Section 10. Nine dollars of the annual dues of those who receive the *Bulletin* and *Proceedings* under the provisions of this Article shall be allocated in payment therefor.

Section 11. The annual dues of ordinary and contributing members shall be due and payable at the beginning of each calendar year, at which time the Society shall submit a bill for these dues. If the annual dues of any member remain unpaid beyond a reasonable time, the Board of Trustees shall remove his name from the list of members, after due notice. If a member wishes to discontinue his membership at any time, he shall send a written resignation to the Society.

Section 12. Any member who became a life member before October 25, 1941, by the payment of a sum determined in accordance with actuarial principles shall have for life the status and privileges of an ordinary member without further payment of dues. No additional applications for life memberships will be accepted.

ARTICLE X

MEETINGS

Section 1. The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council. No matter of general business shall be considered at any meeting of the Society except the Annual Meeting, without the recommendation of the Council.

Section 2. Meetings of the Executive Committee may be called by the President; he shall call a meeting at any time upon the written request of two of its members.

Section 3. The Council shall meet at the Annual Meeting of the Society. Special meetings of the Council may be called by the President; he shall call a special meeting at any time upon the written request of five of its members. No special meeting of the Council shall be held unless written notice of it shall have been sent to all members of the Council at least ten days before the day set for the meeting.

Section 4. The Board of Trustees shall hold its regular Annual Meeting each year between December 15 and January 20, but not before the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each Trustee. He should call a meeting upon the receipt of a written request of two of the Trustees. Meetings for the transaction of business may also be held by common consent of all the Trustees.

Section 5. Papers intended for presentation at any meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

ARTICLE XI

PUBLICATIONS

Section 1. The Society shall publish an official organ called the *Bulletin of the American Mathematical Society.* It shall publish two journals, known as the *Transactions of the American Mathematical Society* and the *Proceedings of the American Mathematical Society.* It shall publish a series of mathematical papers known as the *Memoirs of the American Mathematical Society.* The object of the *Transactions, Proceedings,* and *Memoirs* is to make known important mathematical researches. It shall publish a periodical called *Mathematical Reviews,* containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called *Colloquium Publications* which shall embody in book form new mathematical developments. It shall publish a series of monographs called *Mathematical Surveys* which shall furnish expositions of the principal methods and results of particular fields of mathematical research. It shall also cooperate in the conduct of the *American Journal of Mathematics.*

Section 2. The editorial management of the publications of the Society listed in Section 1 of this article, and the participation of the Society in the editorial management of the *American Journal of Mathematics* shall be in charge of the respective Editorial Committees as provided in Article III, Section 1.

ARTICLE XII

AMENDMENTS

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

MEMBERS OF THE SOCIETY

October 1, 1950

The following list includes the name of every person who is a member of the Society as of October 1, 1950.

So far as information is available the listings are made up as follows: (a) Surname. A dagger before a surname indicates life membership. (b) Initials, or the given name when there is but one, except that the first given name of every woman member is given in full. Parentheses enclosing any part of a name indicate that that part is not generally used in correspondence. (c) Official title or position. (d) Name and address of institution or organization with which item (c) is associated. (e) The address to be used for mail if this is different from (d). The name of the city and postal district number, where necessary, are not repeated if they are the same as in (d). (f) Following the regular listing, information re military service or temporary appointments is given.

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Albert, A. H.
Peterson, O. J.
Tucker, C. B.
Hays
Schaefer, G. R.
Stopher, E. C.
Lawrence
Univ. of Kansas
Babcock, W.
Bell, P. O.
Black, F.
Bradt, R. N.
Brown, H. D.
Chowla, S.
Fisher, R. C.
Forman, J. W.
Jordan, H. E.
Larkin, J. R.
Marceau, R. L.
Moore, W. K.
Pihlblad, M. M.
Price, G. B.
Rasmussen, O. M.
Riley, J. D.
Schatten, R.
Scott, W. R.
Showalter, A. B.
Smith, G. W.
Stouffer, E. B.
Ulmer, G.
Wheeler, J. J.
Wolfe, F. L.
Wolontis, V. M.
Lindsborg
Marm, A.
Manhattan
Kansas State Coll. of Agric. and Applied Science
Babcock, R. W.
Franck, A.
Furman, A.
Greer, E.
Hachmeister, V. G.
Janes, W. C.
Mossman, T. A.
Northam, J. I.
Parker, S. T.
Sanger, R. G.
Sloat, F. B.
Stratton, W. T.
Young, P. M.
North Newton
Richert, D. H.
Pittsburg
Smith, R. G.
Salina
Arnoldy, M. N.
Topeka
Brown, B. M.
Putt, C. N.
Washburn Municipal Univ of Topeka
Eberhart, P.
Greene, L. Z.
Martinson, M. E.
Wakeeney
Gibson, R. W.
Wichita
Longenecker, J. V. S.
Univ. of Wichita
Read, C. B.
Reagan, L. M.
Winfred
Bernstorf, E. E.

KENTUCKY

Bowling Green
Yarbrough, H. M.
Covington
Frisch, M. E.
Danville
Robinson, W. J.
Lexington
Univ of Kentucky
Boron, L. F.
Boyd, P. P.
Church, E. T.
Cowling, V. F.
Downing, H. H.
Goodman, A. W.
Pence, S. E.
South, D. E.
Thompson, L. O.
Ward, J. A.
Louisville
Bullitt, W. M.
Nasareth Coll.
Fowler, M. C.
Ursuline Coll.
Schaeffer, M. R.
Univ. of Louisville
Fields, W. L.
Moore, W. L.
Stevenson, G.
Murray
Carman, M. G.
Richmond
Park, R. S.
Williamsburg
Boswell, J. M.
Winchester
Howard, A. S.
Stanford
Pettus, M.

LOUISIANA

Baton Rouge
Louisiana State Univ. and Agric. and Mech. Coll.
Bienvenu, L. B.
Broussard, R. L.
Karnes, H. T.
Rees, P. K.
Rickey, F. A.
Rutt, N. E.
Schenkman, E. V.
Townsend, B. B.
Trammell, G. J.
Wade, L. I.
Wang, H. C.
Southern Univ.
Campbell, W. B.
Hunte, B. E.
Kraft, F. S.
Titus, E.
Hammond
Davis, M. R.
Lafayette
Loflin, Z. L.
Lake Charles
Bradford, W. H.
Natchitoches
Carlton, L. V.
New Orleans
Evans, J. O.
Gadsden, C. P.
Tulane Univ. of Louisiana
Allan, R. E.
Beard, H. P.
Buchanan, H. E.
Derr, L. J.
Duren, W. L.
Feibleman, J. K.
Hersh, N. K.
Koch, R. J.
McArthur, C. W.
Many, A. E.
Morrison, D. R.
Pettis, B. J.
Riess, J. K.
Sanderson, J.
Thomson, J. F.
Uhl, H. R.
Wallace, A. D.
Ward, L. E.
Weiss, M. J.
Ruston
Temple, W. B.
Shreveport
Griffith, W. C.
McKnight, B.

MAINE

Brunswick
Bowdoin Coll.
Chirstie, D. E.
Hammond, E. S.
Holmes, C. T.
Kennebunk
Levinson, H. C.
Orono
Univ. of Maine
Comegys, E.
Kimball, S. H.
Simmons, G. F.
Waterville
Pinette, L. K.

MARYLAND

Aberdeen
de Maré, M. C.
Reklis, V. M.
Aberdeen Proving Ground
Ballistic Research Labs.
Bennett, F. D.
Carter, W. C.
Clippinger, R. F.
Cox, M. J.
Dederick, L. S.
Dimsdale, B.
Enequist, L. N.
Fritz, W. B.
Galbraith, A. S.
Garfinkel, B.
Giese, J. H.
Hart, B. I.
Juncosa, M. L.
Keirstead, R. E.
Kravitz, S.
Lewis, J. V.
Lotkin, M.
Reitwiesner, G. W.
Schnittke, R. I.
Smith, T. L.
Stein, A.
Treuenfels, P. M.
Wood, L. E.
Annapolis
Clements, G. R.
Torrance, E. M.
U.S. Naval Acad.
Abbott, J. C.
Bailey, R. P.
Ball, N. H.
Benac, T. J.
Betz, E. E.
Brady, C. P.
Conrad, W. A.
Craw, A. R.
Gras, E. C.
Hawkins, E.
Kells, L. M.
McLaughlin, K. F.
Mann, G. J.
Milkman, J.
Moore, T. W.
Morrow, R. C.
Palmquist, K. L.
Robinson, V. N.
Saslaw, S. S.
Scarborough, J. B.
Seekins, C. W.
Simpson, R. C.
Sohl, H. K.
Strange, W. J.
Strohl, G. R.
Swafford, E. G.
Tierney, J. A.
White, J. H.
Wilson, L. T.
U.S. Naval Postgrad. School
Bleick, W. E.
Campbell, R. C.
Church, R.
Faulkner, F. D.
Jennings, W.
Lockhart, B. J.
Pulliam, F. M.
Rawlins, C. H.
Root, R. E.
Torrance, C. C.
Baltimore
Carpenter, F. D.
Robinson, L. B.
Zierler, N.
Coll. of Notre Dame of Maryland
Karl, M. C.
Goucher Coll.
Torrey, M. M.
Johns Hopkins Univ.
Berning, P. H.
Carey, E. D.
Carstoiu, I.
Chow, W. L.
Clifford, A. H.
Cohen, A.
Hartman, P.
Haviland, E. K.
Huck, R.
Jenkins, J. A.
Larsen, T.
Lewis, D. C.
Light, F. W.
Littell, A. S.
Mannos, M.
Moore, W. E.
Morrill, W K.
Novikoff, A. B.
Rabinovitch, N. L.
Reed, L. J.
Shapiro, G.
Smith, A. C.
Wolfson, K. G.
Zilber, J. A.
Loyola Coll.
Dawkins, W. S.
Mt. St. Agnes Coll.
Thomas, M. P.
Morgan State Coll.
Mishoe, L. I.
Proctor, W. W.
Saunders, S. W.
Stephens, C. F.
Bethesda
Blankinship, W. A.
Montgomery Junior Coll.
Ripley, J. A.
Naval Medical Research Inst.
Cole, K. S.
Gerende, L. J.
Chestertown
Gordon, R. D.
Chevy Chase
Hollcroft, T.
College Park
Univ. of Maryland
Boyer, J. M.
Brigham, N. A.
Diaz, J. B.
Good, R. A.
Gorciu, V. G.
Hall, D. W.

Haywood, S.
Higgison, E. C.
Jackson, S. B.
Jarnagin, M. P.
Leutert, W. W.
Martin, M. H.
Meals, L. K.
Rawling, A. G.
Sharp, W. T.
Shepherd, J. C.
Thickstun, W. R.
Thorpe, A. C.
Vanderslice, J. L.
Vollmer, J. E.
Wagner, T. C. G.
Weinberger, H. F.
Weinstein, A.
Zarantonello, E. H.
Emmitsburg
McCullough, R. J.
Frederick
Maloney, C. J.
Silver Spring
Johns Hopkins Univ.
Cronvich, L. L.
Frenkiel, F. N.
Mitchell, A. K.
Munro, G. C.
Rich, R. P.
Sigley, D. T.
Sopka, J. J.
Whitman, P. M.
Naval Ordnance Lab.
Cheydleur, B. F.
Eddy, R. P.
Hyman, M. A.
Piersol, C. C.
Polachek, H.
Rosenbloom, J. H.
Seeger, R. J.
Shanks, D.
Solem, A. D.
Theilheimer, F.
Van Tuyl, A. H.
Zettler-Seidel, P. W.
Suitland
Morris, C. R.
Westminster
Spicer, C. A.
Woodstock
Koehler, C. F.

MASSACHUSETTS

Amherst
Amherst Coll.
Breusch, R. H.
Brown, B.
Eells, J.
Esty, T. C.
Sprague, A. H.
Univ. of Massachusetts
Cullen, H. F.
Rose, I. H.
Wagner, R. W.
Arlington
Widder, V. A.
Boston
Coolidge, T. J.
Dysart, R.
Fine, L. A.
Gould, A. B.
Miller, A. L.
Mooers, C. N.
Reis, F. B.
Rich, G. R.
Rines, D.
Boston Univ.
Bruce, R. E.
Giever, J. B.
Johanson, R. N.
Marcou, R. J.
Mode, E. B.
Scheid, F. J.
Sobczyk, A.
Syer, H. W.
Whitehead, K. B.
Harvard Univ.
Brown, T. H.
John Hancock Mutual Life Ins. Co.
Garabedian, H. A.
Hoskins, R. H.
Howell, A. C.
Northeastern Univ.
Brown, H. K.
Spear, J.
Bridgewater
Wood, F. L.
Cambridge
Arnowitt, R. L.
Franklin, C. W.
Cambridge Junior Coll.
Cohen, C.
Harvard Univ.
Ahlfors, L. V.
Ayoub, R. G.
Beatley, R.
Bergman, S.
Birkhoff, G.
Block, I. E.
Bridgman, P. W.
Browne, H. N.
Coolidge, J. L.
Cunningham, F.
Davis, P.
Fair, G. M.
Fischer, I.
Gleason, A. M.
Goldman, O.
Gorenstein, D.
Graves, R. L.
Hanson, J. E.
Hayden, S.
Huntington, E. V.
Kay, A. F.
Kemble, E. C.
Kneale, S. G.
Loomis, L. H.
Mackey, G. W.
Markus, L. F.
Middleton, D.
Mises, R. v.
Mosteller, F.
Murray, G. G.
Neustadter, S. F.
Osserman, R.
Papas, C. H.
Pollak, H. O.
Rogers, H. G.
Royden, H. L.
Rulon, P. J.
Schlesinger, E. C.
Shapley, H.
Sheffer, H. M.
Turanski, W. J.
Van Vleck, J. H.
Walsh, J. L.
Watson, N. T.
Weintraub, H.
Wermer, J.
Whitney, E. L.
Whitney, H.
Widder, D. V.
Wilson, E. B.
Wolfsohn, N. Z.
Young, D. M.
Zariski, O.
Zulon, J. A.
Massachusetts Inst. of Tech.
Ambrose, W.
Aumann, R. J.
Berghoefer, F. G.
Bott, G.
Browder, F. E.
Brown, A. A.
Carr, J. W.
Coddington, E. A.
Cohen, I. S.
Copeland, A. H., Jr.
Crandall, S. H.
Davis, R. B.
Dobbie, J. M.
Engel, J. H.
Engelman, C.
Finkelstein, D.
Finston, M.
Franklin, P.
Frey, E. J.
Geller, L.
Gillman, L.
Haas, V. B.
Harvey, G. G.
Hildebrand, F. B.
Hildebrandt, T. W.
Holt, F. S.
Hurewicz, W.
Kingsbury, R. S.
Laning, J. H.
Levinson, N.
Lin, C. C.
Lindgren, B. W.
Lorenz, E. N.
Luce, R. D.
Manger, W. P.
Martin, W. T.
Moon, P.
Oberbeck, T. E.
O'Neill, B.
Phillips, H. B.
Pitts, W.
Reissner, E.
Root, W. L.
Rudin, W.
Salem, R.
Selfridge, O. G.
Shapiro, H. S.
Shields, W. H.
Singer, I. M.
Springer, G.
Stark, J. M.
Struik, D. J.
Thomas, G. B.
Wadsworth, G. P.
Weiss, E.
Whitehead, G. W.
Wiener, N.
Woods, F. S.
Zeldin, S. D.
Chestnut Hill
Boston Coll.
Eiardi, A. J.
Haefeli, H. G.
O'Donnell, G. A.
Thomas, A. S.
Chicopee
Madden, T. M.
Concord
Parke, N. G.
Fitchburg
Bissinger, B. H.
Lynn
Bauer, P. S.

Medford
Tufts Coll.
Clarkson, J. A.
Fulton, D. G.
Graustein, M. C.
Penico, A. J.
Milton
Pocock, J. G.
Newton Centre
McClay, D. T.
Northhampton
Smith Coll.
Johnson, R. E.
McCoy, N. H.
O'Neill, A. F.
Rambo, S. M.
Norton
Garabedian, C. A.
Geiringer, H.
Pittsfield
Satterthwaite, F. E.
Washburne, A. C.
Southbridge
Boeder, P.
SouthHadley
Mt. Holyoke Coll.
Bates, G. E.
Kiokemeister, F.
Litzinger, M.
Springfield
Bowie, H. E.
Vineyard Haven
Dreier, T.
Waltham
Fein, L.
Riblet, H. J.
Wellesley
Wellesley Coll.
Ayer, M. C.
Copeland, L. P.
Maharam, D.
Russell, H. G.
Stark, M. E.
Young, M. M.
Weston
Burke, M. L.
O'Shea, E. F.
Williamstown
Williams Coll.
Agard, H. L.
Gordon, C. H.
Hardy, J. G.
Klein, J. S.
Mead, D. G.
Richmond, D. E.
Worcester
Wheeler, A. H.
Clark Univ.
Bumer, C. T.
Christian, R. R.
Stubbe, J. S.
Coll. of the Holy Cross
McBrien, V. O.
Smith, T. J.
Worcester Polytech. Inst.
Morley, R. K.
Rice, H.

MICHIGAN

Albion
Larsen, H. D.
Ann Arbor
Early, M. L.
Fowler, K. A.
Grennan, E. B.
Roettinger-Kaplan, I.
Univ. of Michigan
Bartels, R. C. F.
Bradshaw, J. W.
Brauer, R.
Brown, D. M.
Büchi, J. R.
Burks, A. W.
Burroughs, W. K.
Chover, J.
Churchill, R. V.
Coburn, N.
Coe, C. J.
Comstock, M. L.
Copeland, A. H.
Craig, C. C.
Crisler, E. H.
Darling, D. A.
Davis, H. C.
Dolph, C. L.
Downing, A. C.
Dushnik, B.
Dwyer, P. S.
Fischer, C. H.
Ford, W. B.
Greenberg, M. L.
Harary, F.
Hay, G. E.
Heider, L. J.
Hildebrandt, T. H.
Hopkins, L. A.
Jenner, W. E.
Jones, P. S.
Kaplan, W.
Kincaid, W. M.
Kuller, R. G.
Lauer, R. M.
Lee, J. R.
Leisenring, K. B.
LeVeque, W. J.
Livesay, G. R.
Lohwater, A. J.
Love, C. E.
McLaughlin, J. E.
Marx, I.
Miller, J. M.
Morkovin, V. M.
Moy, S. T. C.
Myers, S. B.
Nesbitt, C. J.
Norman, R. Z.
Nyswander, J. A.
Piranian, G.
Poor, V. C.
Raiffa, H.
Rainich, G. Y.
Rainville, E. D.
Rang, E. R.
Rauch, L. L.
Reade, M. O.
Resch, D.
Riordan, J. F.
Ritt, R. K.
Roberts, R. A.
Rothe, E. H.
Rothe, J. C.
Running, T. R.
Samelson, H.
Shively, R. L.
Smith, W. K.
Spencer, G. L.
Thompson, G. L.
Thrall, R. M.
Titus, C. J.
Tornheim, L.
Tsao, S. H.
Ullman, J. L.
Wechsler, M. T.
Wiegmann, N. A.
Wilder, R. L.
Wright, J. B.
Bay City
Ewing, M. M.
Berrien Springs
Specht, E. J.
Clarkston
Boothby, W. M.
Detroit
Deniston, R. F.
Fleiger, A. G.
Neale, A. B.
Yntema, T. O.
Detroit Inst. of Tech.
Loomis, C. K.
Monik, H.
Lawrence Inst. of Tech.
Bagby, L. C.
Univ. of Detroit
McCarthy, E. D.
Mehlenbacher, L. E.
Pixley, E. C.
Smith, W. F.
Wayne Univ.
Borgman, W. M.
Cohn, H.
Coral, M.
Crispin, J. W.
Dunford, M. J.
Eisenstadt, B. J.
Epstein, B.
Folley, K. W.
Harrison, G.
Jacobson, A. W.
Kaplan, S.
Morrow, D. C.
Nelson, A. L.
Pixley, H. H.
Southard, T. H.
East Lansing
Plant, L. C.
Rubenstein, M. A.
Michigan State Coll.
Bell, J. H.
Blyth, M. I.
Carr, R. E.
Frame, J. S.
Groenewoud, C.
Grove, V. G.
Hanson, H. A.
Harter, H. L.
Herzog, F.
Hill, J. D.
Katz, L.
Kelly, L. M.
Lapidus, L.
McMillan, O. T.
Nordhaus, E. A.
Payne, M. H.
Powell, J. E.
Stelson, H. E.
Stewart, B. M.
Sweetland, P. C.
Wells, C. P.
Zimmer, J. W.
Ferndale
Jones, C. D.
Houghton
Rodefer, R. A.
Kalamazoo
Peterson, J. K.
Walton, T. O.
Marquette
Spooner, C. C.

Milford
McNeal, R. L.
Nazareth
Dethier, B. E.
Morrison, M. J.
Pontiac
Dodson, H. W.
Ypsilanti
Michigan State Normal Coll.
Pate, R. S.
Weaver, T.
Univ. of Michigan
Brown, R. G.
Garfunkel, I. M.
Goode, H. H.
Mazurkiewicz, H. G.
Ritter, E. K.
Siegel, K. M.

MINNESOTA

Bemidji
Colson, H. D.
Collegeville
Kalinowski, W. C.
Duluth
Cothran, J. C.
Lindquist, C. B.
Mankato
Fleming, W.
Wells, W. J.
Minneapolis
Morgan, W. D.
Sheffield, E. T.
Univ. of Minnesota
Amundson, N. R.
Barnett, H. H.
Bearman, J. E.
Brink, R. W.
Bussey, W. H.
Cameron, R. H.
Carlson, E.
Donsker, M. D.
Evans, R. L.
Fischer, I. C.
Friberg, M. S.
Fulks, W. B.
Gelbaum, B. R.
Gibbens, G.
Graves, R. E.
Hall, N. A.
Hart, W. L.
Hatfield, C.
Hill, E. L.
Indritz, J.
Ito, W. H.
Kalisch, G. K.
Kirmser, P. G.
Koehler, F.
Lewis, D. R.
Loud, W. S.
McMillin, K. M.
Munro, W. D.
Nering, E. D.
Olmsted, J. M. H.
Quaid, L. J.
Shumway, R. R.
Simmons, D. H.
Stoner, I.
Thompson, J. E.
Tingley, A. J.
Turrittin, H. L.
Wang, C. Y.
Warschawski, S. E.
Yarnell, J. L.
Moorhead
Mundhjeld, S.
Northfield
Carleton Coll.
May, K. O.
St. Olaf Coll.
Carlson, C. S.
Norlie, O. M.
St. Paul
Ohnsorg, F. R.
Coll. of St. Thomas
Bush, L. E.
Norris, M. J.
Sheridan, L. W.
Smith, F. C.
Zemlin, R. A.
Hamline Univ.
Culmer, W. J. A.
Macalester Coll.
Camp, E. J.
Scherer, R. H.
Winona
Seiler, L. de la S.

MISSISSIPPI

Biloxi
Spears, W. K.
Blue Mountain
Gillis, M. E.
Cleveland
Walters, E. B.
Columbus
Erickson, R. W.
Hattiesburg
Gerneth, D. C.
Holly Springs
Caldwell, G. O.
Jackson
Mitchell, B. E.
Winn, W. S.
Okolona
Gardner, C. D.
State College
Mississippi State Coll.
Kanter, L. H.
Ollivier, A.
Pettis, C. R.
Walker, B. M.
University
Univ. of Mississippi
Bickerstaff, T. A.
Hume, A.
Miller, R. A.
Spragens, W. H.

MISSOURI

Cape Girardeau
Michel, R. J.
Columbia
Univ. of Missouri
Betz, H.
Blumenthal, L. M.
Burcham, P. B.
Ewing, G. M.
Gaddum, J. W.
Griffith, H. C.
Haynes, N. A.
Stamey, W. L.
Taam, C. T.
Utz, W. R.
Westfall, W. D. A.
Zemmer, J. L.
Fayette
Helton, F. F.
Fulton
Ekstrom, R. E.
Jefferson City
Talbot, W. R.
Kansas City
Castellani, M.
Dengler, M. A.
Doyle, W. C.
Lackay, M. P.
Melcher, G.
Wagner, F. J.
Kirksville
Kline, V. W.
Rolla
Erkiletian, D. H.
St. Louis
Green, F. W.
Haimo, D. T.
Fontbonne Coll.
Lewis, M. T.
Harris Teachers Coll.
Marth, E. C.
Willerding, M. F.
Maryville Coll.
Kernaghan, M.
St. Louis Coll of Pharmacy and Allied Sciences
DeFoe, O. K.
St. Louis Univ.
Allen, R. W.
Andrews, J. J.
Bushkovitch, A. V.
Elder, J. D.
Regan, F.
Vezeau, W. A.
Washington Univ.
Couch, W. A.
Downs, T. L.
Dunkel, O.
Elliott, H. M.
Haimo, F.
Harrison, C. E.
Hartstein, H. D.
Hirschman, I. I.
Kahal, R.
Leighton, W.
Middlemiss, R. R.
Nehari, Z.
Rider, P. R.
Roever, W. H.
Schaerf, H. M.
Sholander, M.

Thron, W. J.
Van Schaack, G. B.
Varney, R. N.
Springfield
Graves, W. L.
H'Doubler, F. T.
Owchar, M.
Warrensburg
Akers, L. W.
West Plains
Turner, V. D.

MONTANA

Bozeman
Montana State Coll.
Antosiewicz, H. A.
Hurst, J. W.
Livers, J. J.
Lowney, R. E.
Roscoe, M. K.
Uhrich, G. E.
Whitesitt, J. E.
Butte
Smith, A. J.
Garrison
Canning, J.
Miles City
Olson, I. L.
Missoula
Pott, F. H.
Montana State Univ.
Carey, E. F. A.
Chatland, H.
Coffey, D. E.
Duquette, A. L.
Lennes, N. J.
Marsaglia, G.
Merrill, A. S.
Ostrom, T. G.

NEBRASKA

Crete
Oleson, L. C.
Fremont
Moenter, R. L.
Lincoln
Union Coll.
Ogden, E. B.
Univ. of Nebraska
Basoco, M. A.
Brenke, W. C.
Buck, C. C.
Camp, C. C.
Clarke, F. M.
Collins, O. C.
Doole, H. P.
Gaba, M. G.
Halfar, E.
Jehle, H.
Leavitt, W. G.
Lenser, W. T.
Parks, K. L.
Ribeiro, H. B.
Runge, L. L.
Seifert, G.
Omaha
Becker, H. W.
Bettinger, A. K.
Earl, J. M.

NEVADA

Reno
Univ. of Nevada
Beesley, E. M.
Davis, E. A.
Demers, M. R.

NEW HAMPSHIRE

Durham
Univ. of New Hampshire
Ames, D. B.
Crabtree, J. B.
Slobin, H. L.
Somers, R. H.
Exeter
Saltonstall, W. G.
Hanover
Morgan, F. M.
Dartmouth Coll.
Brown, B. H.
Doyle, T. C.
Durfee, W. H.
Fraser, W. C. G.
King, A. L.
Mathewson, L. C.
Perkins, F. W.
Robinson, R.
Silverman, L. L.

NEW JERSEY

Bayonne
Gutmann, P.
Belmar
Signal Corps
Borsuk, J.
Epstein, I. J.
Selfridge, R. G.
Camden
Weyland, C. E.
Wolfgang, W. G
Chathem
Hoyt, R. S.
Clifton
Cotellessa, R. F.
Convent
O'Neill, A. C.
East Orange
Gilbert, N. M.
Elberon
Wynne, W. M.
Ft. Monmouth
Bernhard, H. A.
Norwood, L. R.
Glassboro
McCobb, R. H.
Haddon Heights
Williams, J. B.
Hoboken
Lipschutz, M. A.
Stevens Inst of Tech.
Davis, H. N.
Deimel, R. F.
Rose, N. J.
Jersey City
Ayres, H. C.
Waldinger, H. V.
Lawrenceville
Kiernan, P. J.
Long Branch
Rosenthal, J. E.
Wulf, L. M.
Madison
Drew Univ.
Battin, I. L.
Glass, W. A.
Greenspan, B.
Maplewood
Hazeltine, A
Montclair
Jacobus, D. S.
Morristown
Critchlow, F. G.
Murray Hill
Bell Telephone Labs., Inc.
Bode, H. W.
Dietzold, R. L.
Farley, B. G.
Gilbert, E. N.
Gray, M. C
Hamming, R. W.
Hebbert, C. M.
Kaplan, E. L.
Ling, D. P.
McMillan, B.
Olmstead, P. S.
Prim, R. C.
Raisbeck, G.
Schelkunoff, S. A.
Shannon, C. E.
Shewhart, W. A.
Zobel, O. J.
Newark
Mosesson, Z. I.
Neifeld, M. R.
Mutual Benefit Life Ins. Co
Kirk, D. B.

Rhodes, E. E.
Rotter, P. T.
Thompson, J. S.
Newark Coll. of Eng.
Fithian, J. H.
Hoffmann, P. O.
Jaffe, W. J.
Molina, E. C.
Rutgers Univ.
Jacoby, A. R.
New Brunswick
Rutgers Univ.
Barlaz, J.
Bartholomay, A. F.
Bunyan, L. H.
Clark, F. E.
Cohn, R. M.
Court, L. M.
Dutka, J.
Fender, F. G.
Firestone, C. D.
Gilbert, S. A.
Grant, H. S.
Hazard, K. E.
LeLeiko, M.
Makarov, A. G.
Meder, A. E.
Morris, R.
Nelson, C. A.
Ott, E. R.
Phelps, C. R.
Rauch, H. E.
Robertson, M. S.
Saalfrank, C. W.
Starke, E. P.
Walter, R. M.
Zimmerberg, H. J.
Nutley
Assadourian, F.
Paulsboro
Hicks, J. S.
Princeton
Chang, H. H. L.
Elliott, J.
Mettler, J. W.
Nickerson, H. K.
Inst. for Advanced Study
Alexander, J. W.
Bigelow, J. H.
Bott, R.
Charney, J. G.
Dyson, V. H.
Fuglede, B.
Gödel, K.
Goldstine, H. H.
Heller, A.
Hoffman, A. J.
Hu, S. T.
Kadison, R. V.
Kaufman, B.
Kodaira, K.
Lepson, B.
Moise, E. E.
Montgomery, D.
Morse, M.
von Neumann, J.
Newburgh, J. D.
Putnam, C. R.
Ramanathan, K. G.
Selberg, A.
Specker, E. P.
Veblen, O.
Weyl, H.
Princeton Univ.
Adams, E. P.
Adamson, I. T. A. C.
Adem Chahin, J.
Artin, E.
Bargmann, V.
Blackett, D. W.
Blanchfield, R. C.
Bochner, S.
Calabi, E.
Chen, K. T.
Church, A.
Eckler, A. R.
Eisenhart, L. P.
Feller, W.
Fox, R. H.
Fuller, F. B.
Gilbert, W. M.
Gulden, S. L.
Hausner, M.
Herz, C. S.
Joseph, L.
Kemeny, J. G.
Lang, S.
Lefschetz, S.
Link, R. F.
Lyndon, R. C.
McCarthy, J.
Milnor, J. W.
Nash, J. F.
Papakyriakopoulos, C. D.
Sampson, J. H.
Shapley, L. S.
Spencer, D. C.
Steenrod, N. E.
Stephan, F. F.
Stephenson, G.
Suppiger, E. W.
Tate, J. T.
Teichmann, T.
Tores Diaz, G.
Tucker, A. W.
Tukey, J. W.
Wightman, A. S.
Wigner, E. P.
Wilks, S. S.
Woodbury, M. A.
Red Bank
Newman, P.
South Orange
Stanwick, C. A.
Seton Hall Coll.
Elston, F. G.
Herschdorfer, M.
Laffan, J. C.
Summit
McMillan, A. W.
Teaneck
Meyer, H. K.
Rayher, E.
Trenton
Delate, E. J.
Klein, J.
New Jersey State Teachers Coll.
Goldstein, W.
Shuster, C. N.
Trenton Junior Coll.
Ferester, A. H.
Upper Montclair
Boggs, E. M.
Campbell, G. A.
New Jersey State Teachers Coll.
Davis, D. R.
Humphreys, T. R.
Mallory, V. S.
West Orange
Edison, T. M.

NEW MEXICO

Albuquerque
New Mexico School of Mines
Cassity, C. R.
Mitchell, W. L.
Univ. of California
Boldyreff, A. W.
Schutzberger, H.
Univ. of New Mexico
Beach, J. W.
Buell, C. E.
Hendrickson, M. S.
Hildner, R. C.
LaPaz, L.
Mitchell, M.
Las Vegas
Rodgers, T. G.
Los Alamos
Galentine, P. G.
Ulam, S. M.
Univ. of California
Allen, W. A.
Anderson, W.
Calkin, J. W.
Carlson, B. G.
Everett, C. J.
Goldstein, M.
Hammer, P. C.
von Holdt, R. E.
Mark, J. C.
Peck, L. G.
Stark, R. H.
White, G. N.
White, T. J.
Portales
Propes, E. A.
Raton
Kloss, R. E.
Roswell
Roudebush, W. C.
State College
Kramer, M. S.

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Newsom, C. V.
Thompson, W. R.
New York State Coll. for Teachers
Beaver, R. A.
Birchenough, H.
Lester, C. A.
Alfred
Freund, J. E.
Annandale-on-Hudson
Tremblay, C. J.
Aurora
Wells Coll.
Carroll-Rusk, E.
Djorup, M. B.
Hollcroft, M. P.
Hollcroft, T. R.
Baldwin
Grove, C. C.

Beacon
Sherman, J.
Bethpage
Gross, G. L.
Mack, C. E.
Binghamton
Greene, U. S.
Bronxville
Manning, A. W.
Strodt, R. C.
Brooklyn
Feinstein, A.
Gerst, I.
Gumbel, E. J.
Herschfeld, A.
Honig, J.
Hopkins, M. W.
Kramer-Lassar, E. E.
Kravetz, S.
Landau, S.
LaSala, L. A.
Levine, L. M.
Maddaus, I.
Miller, J. L.
Perlstein, M.
Salkind, C.
Shenitzer, A.
Wayne, A.
Brooklyn Coll.
Boeker, M. D.
Borofsky, S.
Boyer, C. B.
Fleisher, E.
Forman, W.
Griffin, H. M.
Henry, A. H.
Johnson, R. A.
Kennison, L. S.
Kieval, H. S.
Kormes, J. P.
Landers, A. W.
Levenson, M. E.
Maria, A. J.
Maria, M. H.
Moore, L. T.
Osterberg, A.
Prenowitz, W.
Richardson, M.
Singer, J.
Washnitzer, G.
Wolfe, J. M.
Woodbridge, M. Y.
Long Island Univ.
Koditschek, P.
Lieber, L. R.
Ritterman, M. B.
Seckler, B. D.
Polytech. Inst of Brooklyn
Belsky, A.
Fenn, I. H.
Fialkow, A. D.
Forray, M. J.
Foster, R. M.
Hutchinson, L. C.
Kalish, A.
Katz, J. L.
Klamkin, M. S.
Kovitz, A. T.
Lowe, R. B.
Morduchow, M.
Russell, J. P.
Sachs, B.
Schacknow, A. B.
Terzuoli, A. J.
Ullman, R.
Weber, E.
Whitford, D. E.
Wolfe, H.
Pratt Inst.
Beckman, F. S.
Cowles, W. H. H.
Helme, G. C.
Moore, A. H.
Thompson, J. E.
St. Francis Coll.
Appuhn, W. E. F.
Dice, W. F.
Gabriel, R. F.
Quinn, L.
St. John's Univ.
Cassidy, W. F.
Germino, E. J.
Phelan, K. S.
Robusto, C. C.
Sarno, A. H.
Tolle, L. F.
St. Joseph's Coll. for Women
Byrne, M. C.
Kadlack, M. A.
Buffalo
Benson, L.
Cramer, R. H.
Podmele, T. L.
Scott, M. L.
Walker, G. W.
Bell Aircraft Corp.
Bates, M. R.
Dixon, G. H.
Halpern, F. R.
Isenberg, J. S.
Saltarelli, G. X.
Welmers, E. T.
D'Youville Coll.
Maloney, M. M.
Univ. of Buffalo
Baeumler, H. W.
Behrns, V. N.
Gehman, H. M.
Gough, L.
Lapidus, B.
McArtney, J. M.
Montague, H. F.
Montgomery, M. D.
Schneckenburger, E. R.
Seiden, E.
Welmers, I. W.
Canton
Limpert, J. V.
Peters, R. M.
Cleveland
Morenus, E. M.
Clinton
Hamilton Coll.
van Alstyne, J. P.
Ferry, F. C.
Gere, B. H.
Corning
Lewis, J. A.
Shaffer, D. B.
Cortland
Benedick, J. W.
Elmira
Suffa, M. C.
Endicott
Andrus, W. E.
State Univ. of New York
Aulbach, H.
Kent, J. R. F.
Wright, F. M.
Far Rockaway
Moore, L.
Flushing
Eckweiler, H. J.
Patterson, J. M.
Ripton, T. B.
Wasserman, C. K.
Queens Coll.
Archibald, R. G.
Brown, A. B.
Cohen, L. W.
Cope, T. F.
Eaton, J. E.
Feld, J. M.
Hirsh, J. E.
Hoffmann, B.
Raudenbush, H. W.
Sard, A.
Sullivan, M. M.
White, M. E.
Wouk, A.
Zippin, L.
Garden City
Bowden, J.
Game, R. I.
Geneva
Colls. of the Seneca
Beinert, R. L.
Durfee, W. H.
Hubbs, H. N.
Great Neck
Stewart, S. W.
Hamilton
Munshower, C. W.
Wardwell, J. F.
Hempstead
Hofstra Coll.
Hawthorne, F. S.
Hove, E. M.
Ollmann, L. F.
Stabler, E. R.
Wolf, H.
Hicksville
Goldman, I. L.
Ithaca
Miller, E. A.
Cornell Univ.
Agnew, R. P.
Berterman, J. E.
Blackman, J.
Carver, W. B.
Fuchs, W. H.
Hunt, G. A.
Hurwitz, W. A.
Iwanowski, R. M.
Kac, M.
Kuo, Y. H.
Lucy, H. M.
Malti, M. G.
Morton, J. E.
Murdock, W. L.
Olum, P.
Pollard, H.
Rogers, J. C.
Rosser, J. B.
Walker, R. J.
Williams, C. S.
Yood, B.
Jamaica
Kelly, J. B.
Kiamesha Lake
Slud, M. H.
Lake Success
Hauser, A. A.
Riley, M. H.
Laurelton
Buck, H. J.
Long Island City
Zabb, N. J.
Loudonville
Lynch, B. A.
Mullen, C.
Mineola
Huntley, H. B.
Mt. Kisco
Blake, E. M.
Mt. Vernon
Wetzel, R. A.
New Lebanon
Pflaum, C. W.
Newburgh
Snook, T. E.

New Paltz
Pickard, W. L.
Swain, R. L.
New York
Arnold, J. W.
Askovitz, S. I.
Baker, E. G.
Bal, K. Y.
Becker, M. F.
Berger, A. H.
Bernstein, A.
Biot, M. A.
Boehm, F.
Bowden, M.
Brind, A.
Burgess, L.
Byerly, R. W.
Cooperman, P.
Crane, C. H.
Deutsch, R. A.
Dwork, B. M.
Fall, J.
Flexner, W. W.
Freier, J. B.
Galuten, A.
Glusman, S.
Gotz, W.
Hallett, W. N.
Halperin, M.
Hardin, R. I.
Holdman, R. E.
Ingram, W. H.
Jeffries, J. B.
Joffe, S. A.
Kagno, I. N.
Kallir, J. R.
Karlin, M.
Katz, W.
Kennedy, R. C.
Keyser, S. Y.
Kimball, B. F.
Kiss, S. A.
Kormes, M.
Lewis, J. H.
Lowell, S. C.
Meyerson, H. B.
Miller, B. W.
Miloslavsky, G. C.
Mitchell, H. B.
Morrison, N.
Neuman, S.
Nodelman, H. M.
Parsons, C. H.
Perlstein, S. R.
Polk, M. L.
Prakash, A.
Preston, B.
Proschan, F.
Roosevelt, G. E.
Ruderman, H. D.
Salerno, J.
Salkind, W.
Sasuly, M.
Senior, J. L.
Shook, R. C.
Simon, L. G.
Singer, D.
Smith, D. J.
Sternberg, W.
Tanzola, J. J.
Thompson, N. S.
Vitale, R. L.
Vogel, B. B.
Weaver, W.
Wernick, W.
Williams, E. C.
Williams, H. B.
Williams, R.
Wohl, S. S.
Wright, R. E.
American Math. Society
Hull, E. M.
Kellar, M. E.
MacNeille, H. M.
American Telephone and Telegraph Co.
Belcher, D. R.
Halbert, K. W.
Hodgkinson, W.
Bell Telephone Labs., Inc.
Fry, T. C.
Jones, P. C.
MacColl, L. A.
Mead, S. P.
Riordan, J.
Sunde, E. D.
Vaage, E. F.
Wise, W. H.
Cathedral Coll.
Kovach, E. M.
City Coll.
Barber, S. F.
Bergmann, H. G.
Chernofsky, M. I.
Cocuzza, P. J.
Cortell, R.
Drabkin, I. E.
Dwork, J. S.
Ehrenpreis, L
Fagerstrom, W. H.
Feder, W.
Flatto, L.
Freilich, G.
Garrison, G. N.
Gill, B. P.
Hinman, H. H.
Hubert, W. G.
Hurwitz, S.
Linehan, P. H.
MacEwen, D. M.
Nathan, D. S.
Newman, D. J.
Post, E. L.
Robinson, S. L.
Schwartz, A.
Sklansky, Irving
Updike, G. Z.
Weingarten, H.
Wirth, H. P.
Wright, H. N.
Zeig, J. J.
Coll. of Mt. St. Vincent
Conlan, M. L.
O'Brien, C. M.
Columbia Univ.
Al-Dhahir, M. W.
Anderson, T. W.
Ascher, R. N.
Aurora, S.
Auslander, M.
Bazer, J.
Blum, E. K.
Bolton, G. L.
Brown, R. H.
Carlson, P. G.
Chevalley, C. C.
Chung, K. L.
Cooley, J. W.
Eilenberg, S.
Elgot, C. C.
Epstein, J.
Epstein, M. P.
Ewing, M.
Fass, A. L.
Fehr, H. F.
Fite, W. B.
Gavurin, L. L.
Gleissner, G. H.
Gourin, E.
Harish-Chandra
Hillman, A.
Jardetzky, W. S.
Kasner, E.
Kiefer, J. F.
Kolchin, E. R.
Koopman, B. O.
Krikorian, A.
Kvick, P. G.
Laderman, J.
Langley, C. W.
Lashof, R. K.
Ledley, R. S.
Lehmann, E L.
Lehrer, T. A.
Levene, H.
Levi, H.
Lochwood, M. de F.
Lorch, E. R.
McGaughy, E.
Mann, I.
Mindlin, R. D.
Mittleman, D.
Morse, B. J.
Mullins, G. W.
Murray, F. J.
Nagel, E.
Newman, J. J.
Northcott, J. A.
Peacock, T. E.
Percus, J. K.
Pevsner, A.
Plithides, C. G.
Poor, C. L.
Raney, G. N.
Raussen, J.
Ritt, J. F.
Savage, I. R
Scheffé, H.
Siceloff, L. P.
Smith, P. A.
Stein, S. K. B.
Steinhardt, F.
Stephens, H. W.
Sternberg, D.
Strodt, W.
Supnick, F.
Sutton, P. M.
Taylor, R. L.
Tsang, H. C.
Upton, C. B.
Wald, A.
Webster, J. B.
Weiner, J. H.
Wertheimer, F.
Wolf, O.
Wolfowitz, J.
Yadoff, O.
Zadeh, L. A.
Cooper Union
Chessin, P.
Lehmann, C. H.
Miller, F. H.
Econometric Inst., Inc.
Boschan, P.
Roos, C. F.
Equitable Life Assurance Society of the U.S.
Brown, W. C.
Clarkson, H. E.
Murphy, R. D.
Steinhaus, H. W.
Fordham Univ.
Curran, P. M.
Kirby, A. R.
Kubis, J. F.
Lynch, J. J.
Nash, J. H.
Steele, D. A.
Franklin School
Carson, E R. I.
Stevens, N. H.

Hunter Coll.
Anderson, R. L.
Aroian, L. A.
Bradley, A. D.
Bushey, J. H.
Bushey, Jewell H.
Darkow, M. D.
Eisele, C.
Guggenbuhl, L.
Hill, L. S.
Kutman, H. K.
Landers, M. K.
Lipsey, S. I.
Norris, N.
Salwen, C. B.
Tuller, A.
Weisner, L.
Whelan, A. M.
Hunter Coll. High School
Allegri, L.
Cooper, E. M.
Hydrocarbon Research, Inc.
Katz, S.
Peiser, A. M.
International Business Machines Corp.
Eckert, W. J.
Herrick, H. L.
Hurd, C. C.
Mason, D. R.
Quarles, D. A.
Seeber, R. R.
Manhattan Coll.
Cockshott, A. H.
Hall, F. C.
Hynard, E. O.
McMahon, F. A.
Marano, J. A.
O'Connor, A. R.
Taylor, F. B.
Ventriglia, A. E.
Welch, B. A.
Manhattan Life Ins. Co.
Greenwood, J. A.
Porter, E. A.
Manhattanville Coll. of the Sacred Heart
Kirby, M. D.
Mira, J. A.
Metropolitan Life Ins. Co.
Bassford, H. R.
Brady, G. V.
Campbell, G. C.
Cathcart, A.
Davis, M. E.
Hohaus, R. A.
Lew, E. A.
Mutual Life Ins. Co.
Learson, R. J.
Monsky, M.
New School of Social Research
Altwerger, S. I.
Kogbetliantz, E. G.
New York Univ.
Adler, C. F.
Altshuler, B.
Artin, N. J.
Baron, L.
Berg, P. W.
Berkowitz, J.
Bernardi, S. D.
Bernstein, F.
Blank, A. A.
Brown, A.
Callahan, W. R.
Coler, M. A.
Cooley, H. R.
Courant, R.
Dietsche, H. J.
Douglis, A.
Dressler, R. F.
Friedman, B.
Friedrichs, K. O.
Giddings, H. A.
Grad, H.
Graham, P. H.
Hett, J. H.
Hirsch, W. M.
Isaacson, E.
John, F.
John, F. W.
Karp, S. N.
Kline, M.
Kolodner, I. I.
Lax, A. C.
Lax, P. D.
McConnell, R. K.
Magnus, W.
Miller, K. S.
Mulligan, J. H.
Nirenberg, L.
Noether, G. E.
Payne, C. K.
Peters, A. S.
Pohle, F. V.
Putnam, R. G.
Reddick, H. W.
Rehberg, C. F.
Ritter, I. F.
Rose, M. E.
Roth, S. G.
Shapiro, H. N.
Spinelli, R. A.
Stoker, J. J.
Tilley, A.
Touart, C. N.
Wang, C. T.
Wolinsky, A.
Yanosik, G. A.
Reeves Instrument Co., Inc.
Brock, P.
Fifer, S.
Teleregister Corp.
Phelps, R. J.
Warren, J. E.
Walter Hervey Junior Coll.
Hammer, C.
Yeshiva Univ.
Block, D.
Ginsburg, J.
Greitzer, S. L.
Lisman, H.
Lowan, A. N.
Niagara University
Banks, G. B.
Oneonta
Callahan, E. B.
Paul Smiths
Ladd, D. C.
Plattsburg
Bush, K. A.
Fleshler, A. D.
Poughkeepsie
Vassar Coll
Asprey, W. A.
Baker, F. E.
Dolciani, M. P.
McDonald, J.
Newton, A. V.
Smith, G.
Wells, M. E.
Rochester
Betz, W.
MacCheyne, G. W.
Eastman Kodak Co.
Chesna, J.
Herzberger, M. J.
Marchand, E. W.
Morris, R. H.
Univ. of Rochester
Atkins, H. P.
Bagemihl, F.
Barton, D. C.
Bernstein, D. L.
Danese, A. E.
Gale, A. S.
Gunderson, N. G.
Klimczak, W. J.
Komm, H.
Miller, D. S.
Randolph, J. F.
Seidel, W.
Watkeys, C. W.
Rocky Point
Carter, P. S.
St. Bonaventure
Scheier, M. A.
Saratoga Springs
Williams, M. E.
Schenectady
Polowy, H.
General Elec. Co.
Alger, P. L.
Blanco, B. C.
Crawford, J. E.
Poritsky, H.
Weil, H.
Union Coll.
Burkett, F. J. H.
Farrell, O. J.
Fox, A. H.
Garis, C. F. F.
Morse, D. S.
Snyder, A. D.
Staten Island
Murray, W. R.
Norman, P. B.
Stony Brook
Berry, W. J.
Syracuse
Cowgill, A. P.
Ingersoll, B. M.
Syracuse Univ.
Bennett, H. C.
Bergmann, P. G.
Bers, L.
Carroll, I. S.
Cohen, E.
Cole, N.
Exner, R. M.
Finn, R. S.
Gelbart, A.
Gilbert, P. W.
Goheen, H. E.
Hemmingsen, E.
Hofstra, R. B.
Kibbey, D. E.
Loewner, C.
Ludwig, E. A.
Milgram, A. N.
Morgan, K. A.
Mostow, G. D.
Pardee, O. O.
Penez, J. L.
Pierce, W. A.
Protter, M. H.
Pu, F. P.
Rice, H. G.
Rosenbloom, P. C.
Rosskopf, M. F.
Sanders, J.
Smith, R. F.
Snover, J. E.
Stokes, R. W.
Wetherell, E.

Zatzkis, H.
Troy
Rensselaer Polytech. Inst.
Allen, E. B.
Biggerstaff, J. S.
Campbell, J. D.
Gibson, J. C.
Gravalos, F. G.
Hall, C. F.
Hendler, A. S.
Huston, R. E.
Jones, A. W.
Maly, D.
Spiegel, M. R.
Vrooman, S. I.
Warnock, W. G.
Upton
Chernick, J.
Utica
Greeley, J. B.
West Haverstraw
Nicholl, L.
West Point
Ellis, H. V.
Matheson, J. D.
White Plains
Downey, J. F.
Wilkins, J. E.
Willseyville
Seely, C. E.
Wyoming
Hartnell, G.

NORTH CAROLINA

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Black Mountain
Dehn, M. W.
Chapel Hill
Vause, R. Z.
Univ. of North Carolina
Brauer, A. T.
Browne, E. T.
Cameron, E. A.
Drucker, B. M.
Hershner, I. R.
Hobbs, A. W.
Hotelling, H.
Jones, F. B.
Lasley, J. W.
Mackie, E. L.
Mann, W. R.
Miller, H. C.
Minton, P. D.
Nelson, A. C.
Robbins, H.
Smith, H.
Wahab, J. H.
Whyburn, W. M.
Wong, Y. K.
Charlotte
Nicholson, W. A.
Cullowhee
Hinds, A. K.
Durham
Pegram, A. M.
Winter, E. P.
Duke Univ.
Carlitz, L.
Dressel, F. G.
Elliott, W. W.
Estill, M. E.
Fulton, L. M.
Gergen, J. J.
Hickson, A. O.
Roberts, J. H.
Silva, J. A.
Thomas, J. M.
North Carolina Coll. at Durham
Browne, M. L.
High Point
Adams, L.
Greensboro
Barton, H.
Lewis, A. L.
Mars Hill
Edwards, M.
Morresville
Templeton, M. M.
Raleigh
Univ. of North Carolina
Bullock, R. C.
Cell, J. W.
Clarkson, J. M.
Fisher, H. A.
Levine, J.
Mumford, C. G.
Nahikian, H. M.
Park, H. V.
Sutton, P. P.
Red Springs
Van Dyke, L. A.
Salisbury
Catawba Coll.
Dearborn, D. C.
Douglas, C. H.
Welch, A. E.
Tryon
Dimick, C. E.
Wake Forest
Gentry, I. C.
Winston-Salem
Curlee, A. T.

NORTH DAKOTA

Dickinson
Thompson, J. C.
Grand Forks
Staley, R. C.
Fargo
North Dakota Agric. Coll.
Mentzer, R. L.
North Dakota State Coll.
Arena, F. J.
Smith, I. W.

OHIO

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Purdy, J. F.
Univ. of Akron
Lipscombe, W. H.
Mauch, M. E.
Alliance
Clark, W. G.
Athens
Ohio Univ.
Denbow, C. H.
Goedicke, V. A.
Marquis, R. H.
Reed, F. W.
Starcher, G. W.
Berea
Hansen, A. G.
Bowling Green
Bowling Green State Univ.
Krabill, D. M.
Ogg, F. C.
Overman, J. R.
Tinnappel, H. E.
Cincinnati
Catenaro, W. A.
Folz, C. H.
Our Lady of Cincinnati Coll.
Belai, L. G.
Lukacs, E.
Rice, M. E.
Univ. of Cincinnati
Baiada, E.
Barnett, I. A.
Blair, D.
Brand, L.
Dunholter, R. J.
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Herwitz, P. S.
Justice, H. K.
Kersten, H. J.
Lipsich, H. D.
Lubin, C. I.
Merriman, G.
Miller, H. L.
Moore, C. N.
Smith, E. S.
Smith, T. F.
Szász, O.
Taylor, W. C.
White, E. F.
Yowell, E. I.
Xavier Univ.
Marcaccio, W.
Stechschulte, V. C.
Cleveland
Bosshart, J. E.
Burwell, W. R.
Case Inst. of Tech.
Brown, O. E.
Clark, R. A.
Green, L. J.
Guenther, P. E.
Leone, F. C.
McCuskey, S. W.
Morris, M.
Parker, F. D.
Rinehart, R. F.
Saltzer, C.
Thomas, C. F.
Treon, M.
Fenn Coll.
Haskins, E. E.
Lemon, J. A.
Topp, C. W.
John Carroll Univ
Kremer, B. P.
Rust, C. H.

National Advisory Com. for Aeronautics
Costello, G. R.
Livingood, J. N. B.
Wasserman, R.
Western Reserve Univ.
Albers, L. U.
Beth, R. A.
Musselman, J. R.
Simon, W. G.
Waddell, M. C.
Columbus
Bell, J. L.
Grimble, H. B.
Battelle Memorial Inst.
Bell, J. C.
Chase, M. N.
Miesse, C. C.
Capital Univ.
Wildermuth, R. B.
Coll. of St. Mary of the Springs
Mulligan, C. A.
Ohio State Univ.
Adney, J. E.
Bamforth, F. R.
Bareis, G. M.
Beatty, H. M.
Cashwell, E. D.
Hall, M.
Harp, H. G.
Helsel, R. G.
Herstein, I. N.
Holyoke, T. C.
Jones, M. E.
Kuhn, H. W.
Lazar, N.
Levine, N.
Lewis, D. J.
Mann, H. B.
Martin, E. W.
Mendenhall, R. V.
Mickle, E. J.
Miller, L. H.
Moll, J. L.
Myers, W. M.
Pepper, P. M.
Radó, T.
Rasor, S. E.
Rechard, O. W.
Reichelderfer, P. V.
Ryser, H. J.
Stanger, P. C.
Tikson, M.
Toops, H. A.
Tozzer, V. M.
West, C. T.
Whitney, D. R.
Dayton
Univ. of Dayton
Schraut, K. C.
Wright-Patterson Air Force Base
Astrachan, M.
Block, R. B.
Carson, A. B.
Casey, K. V.
Davidson, A.
Downing, R. H.
Gatewood, B. E.
Gephart, L.
Gorn, S.
Holtom, C.
Rawhouser, R.
Defiance
Godfrey, E. L.
Delaware
Rowland, S. A.
Gambier
Kenyon Coll.
Berg, W. D.
Nikodým, O. M.
Silverman, E.
Transue, W. R.
Granville
Denison Univ
Kato, C.
Wetzel, M. D.
Wiley, F. B.
Hiram
Clarke, E. H.
Olney, H.
Kent
Kent State Univ.
Brooks, F.
Dressler, B. B.
Harshbarger, F.
Iwanchuk, R. Y.
Jasper, S. J.
Jenkins, E. D.
Lowenstein, L. L.
Manchester, R. E.
Olson, E. J.
Riggs, C. L.
Lebanon
Brafman, F.
Marietta
Bennett, T.
Sandt, J. E.
Marion
Leach, C. E.
Mt. St. Joseph
Dimond, A. T.
Molloy, M. C.
Newark
Wiley, R. B.
New Concord
Knight, L. C.
Oberlin
Yeaton, M. J.
Oberlin Coll.
Cairns, W. D.
Graff, F. G.
Sinclair, M. E.
Tuckerman, B.
Vance, E. P.
Yeaton, C. H.
Oxford
Miami Univ.
Anderson, W. E.
Bryant, R. W.
Pollard, H. S
Sangren, W. C.
Spenceley, G. W.
Western Coll.
Allen, H. W.
Tappan, A. H.
Springfield
Hahn, S. W.
Tripp, M. O.
South Euclid
Garvin, M. C.
Toledo
Blackall, C. J.
Zeiders, H. L.
Wilberforce
Curry, D. S.
Wilmington
Cassel, C. W.
Morgan, W. B.
Wooster
Coll. of Wooster
Fobes, M. P.
Williamson, C. O.
Yanney, B. F.
Yellow Springs
Antioch Coll.
Churchill, E.
Myatt, D. J.
Rempfer, R. W.
Youngstown
Burt, H. H.
Youngstown Coll.
Ellis, F. M.
Potor, V. A.

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Mackin, B. L.
Bartlesville
Rice, R. B.
Bethany
Greer, E. V.
Enid
Johnson, W. G.
Langston
Glover, I. E.
Norman
LaFon, G. Q.
Univ. of Oklahoma
Andree, R. V.
Bernhart, A.
Brixey, J. C.
Chen, Y. W.
Court, N. A.
Goffman, C.
Grau, A. A.
Hassler, J. O.
Huff, W. N.
Labarre, A. E.
McFarland, D.
Mathis, H. F.
Meacham, E. D.
Spears, O. S.
Springer, C. E.
Oklahoma City
Diamond, L. E.
Perry
Keith, R. J.
Shawnee
Harris, I. H.
Stillwater
Oklahoma Agric. and Mech. Coll.
Aronszajn, N.
Barnett, J.
Clark, C. J.
Diamond, A. H.
Hamilton, O. H.
Lewis, P. E.
McKinsey, J. C. C.
Nemerever, W. J.
Pedrick, G. B.
Smith, H. W.
Zeichner, A.
Ziebur, A. D.
Tulsa
Carter Oil Co.
Cook, E. A.
Ricker, N.
Shreve, D. R
Univ. of Tulsa
Carter, H. N.
Goss, R. N.
Roth, W. E.
Weatherford
Linscheid, H. W.

OREGON

Coos Bay
Meiler, J. G.
Corvallis
Oregon State Coll.
Arnold, B. H.
Brewer, B. W.
Clark, C. L.
Ehlers, F. E.
Eves, H.
Gysbers, J. C.
Hostetter, I. M.
Li, J. C. R.
Lonseth, A. T.
Manning, R.
Milne, W. E.
Poole, A. R.
Price, J. F.
Saunders, R. B.
Stone, W. M.
Eugene
Univ. of Oregon
Cheo, L. P. H.
Civin, P.
Dixon, W. J.
Ghent, K. S.
McAllister, C. R.
Massey, F. J.
Maxfield, M. W.
Moursund, A. F.
Niven, I.
Peterson, T. S.
Rothaus, O. S.
Shepherd, W. L.
Wood, F. E.
Young, F. H.
Freewater
Hill, H. A.
Portland
Reed Coll.
Griffin, F. L.
Rosenbaum, L. J.
Rosenbaum, R. A.
Wend, D. V. V.
Williams, L. B.
Univ. of Portland
Vassallo, W.
Salem
Garfin, L.
Luther, C. F.

PENNSYLVANIA

Allentown
Deck, L. J.
Koehler, T. L.
Altoona
Herpel, C.
Annville
Erickson, R. L.
Scholz, J. P.
Bethlehem
Lehigh Univ.
Beale, F. S.
Beer, F. P.
Blakers, A. L.
Chellevold, J. O.
Cutler, E. H.
Goldberg, S.
Hailperin, T.
Kenny, B. C.
Lamson, K. W.
Latshaw, V. V.
Pitcher, E.
Raynor, G. E.
Schecter, S.
Shook, C. A.
Smail, L. L.
Snyder, H. H.
Spohn, R. H.
Stoll, R. R.
Wilansky, A.
Bloomsburg
Waldron, M. E.
Bryn Mawr
Bryn Mawr Coll.
Burton, L. J.
Lehr, M.
Oxtoby, J. C.
Oxtoby, J. S
Pell-Wheeler, A.
California
Atkins, F. E.
Carlisle
Ayres, F.
Kuebler, R. R.
Chambersburg
Wilson Coll.
Davison, C. I.
Disert, M. C.
Johnson, R. F.
Loh, Z.
Robinson, J. E.
Chester
Jaffe, H.
Dallas
Garrahan, M. I.
Easton
Fuller, D. L.
Lafayette Coll.
Smith, J. C.
Tartler, A.
East Pittsburgh
Westinghouse Elec. Corp.
Comenetz, G.
Harder, E. L.
Slepian, J.
Elkins Park
Moser, C. E.
Erie
Fasenmyer, M. C.
Gettysburg
Arms, R. A.
Glenshaw
Smith, M. F.
Glenside
Patterson, J. O.
Greensburg
McNeil, M. G.
Greenville
Harter, N. W.
Grove City
Montgomery, R. E.
Harrisburg
Karpov, P. C.
Haverford
Haverford Coll.
Allendoerfer, C. B.
Oakley, C. O.
Thomsen, D. L.
Wilson, A. H.
Immaculata
Hafner, M. R.
Piccirillo, M. S.
Jenkintown
McKee, J. A.
Vansant, H. E.
Johnsville
Keen, W. H.
Porter, J. W.
Kutztown
Knedler, P. A.
Lancaster
Haag, V. H.
Western, D. W.
Lansdowne
Glenn, O. E.
La Plume
Oister, C.
Trapani, B. J.
Latrobe
Yahner, C. G.
Lester
Carner, F. E.
Lewisburg
Gold, J. S.
Miller, W. I.
Lincoln University
Gruenzweig, S.
McKeesport
Kuo, K. C.
McKees Rocks
Arnold, H. C.
Meadville
Allegheny Coll.
Steen, F. H.
Sturley, E. A.
Tanimoto, T. T.
Merion Station
Lawler, E. D.
Oreland
George, T. S.
Philadelphia
Clark, J. R.
Constable, M. L.
Crouse, C. W.
Durand, J. C.
Fudge, H. G.
Jacoby, K. S.
Jones, L. G. F.
Krucklin, R. E.
McDonough, D. L.
Meier, P.
Morton, F. J.
Munster, A. C.
Petrie, G. W.
Sherman, J.
Simpson, F. R.
Singleton, R. R.
Smart, O. M.
Van Andel, J. R.
Drexel Inst. of Tech.
Bickel, R. J.
Davis, J. E.
DiDomenica, E. M.
McNeary, S. S.
Reese, H. D.
Rosen, S.
Franklin Inst.
Brown, H. H.
Hestenes, A. D.
Houghton, D. B.
Schieber, L.
LaSalle Coll.
Connelly, D.
Lewis, G.
St. Joseph's Coll.
Fedorovsky, B. M.
Temple Univ.
Casale, E. C.
Hostinsky, L. A.
Lawton, W. S.
Pitcairn, J.
Schild, A.

Slook, T. H.
Wurster, M. A.
Univ. of Pennsylvania
Aissen, M. I.
Anderson, R. D.
Beal, F. W.
Calloway, J. M.
Campbell, H. E.
Caris, P. A.
Dragonette, L. A.
Epstein, B.
Evans, H. B.
Fine, N. J.
Fischer, W. R.
Goff, J. A.
Gottschalk, W. H.
Kamel, H.
Kline, J. R.
Knobelauch, E. A.
Lehner, J.
Lufkin, H. M.
Martin, R. M.
Papoulis, A.
Patterson, G. W.
Rademacher, H.
Raleigh, J.
Rosen, D.
Ruderfer, H.
Safford, F. H.
Schafer, R. D.
Schlain, E. L.
Schoenberg, I. J.
Schub, P.
Schweigert, G. E.
Shugert, S. P.
Tomber, M. L.
Whitney, A. M.
Witmer, E. E.
Zabronsky, H.
Pittsburgh
Elkins, T. A.
Fitzpatrick, R.
Frankel, E. T.
Leifer, H. R.
Mullan, C. E.
Zora, V. A.
Carnegie Inst. of Tech.
Adler, F. T.
Charnes, A.
Chiarulli, P.
Dines, L. L.
Duffin, R. J.
Handelman, G. H.
Heins, A. E.
Malvern, L. E.
Meacham, R. C.
Moskovitz, D.
Neelley, J. H.
Olds, E. G.
Rosenbach, J. B.
Sacks, L.
Saibel, E. A.
Schild, A.
Whitman, E. A.
Duquesne Univ.
Hailperin, C. B.
O'Donnell, R. E.
Ostrofsky, M.
Schwartz, B. L.
Mellon Inst. of Industrial Research
Bowman, J. R.
Briant, R. C.
Pennsylvania Coll. for Women
Calkins, H.
Univ. of Pittsburgh
Barsotti, I.
Brady, W. G.
Culver, M. M.
Elyash, E. S.
Hovey, B. K.
Laush, G.
Martin, J. E.
Montroll, E. W.
Taylor, J. S.
Westinghouse Elec. Corp.
Garabedian, H. L.
Taylor, J. J.
Wallach, S.
Reading
Dunkle, E. R.
Scranton
Walton, M. B.
Selinsgrove
Robinson, G. M.
Sharon
Greene, R. L.
State College
Mautner, F. M.
Owens, H. B.
Pennsylvania State Coll.
Akutowicz, E. J.
Barone, H. G.
Blau, J. H.
Cohen, T.
Curry, H. B.
Davids, N.
Dickinson, D. J.
Frink, A. H.
Frink, O.
Gordon, W. O.
Hagen, B. L.
Harrington, W. J.
Johnson, E.
Kinney, J. R.
Konhauser, J. D. E.
Krall, H. L.
Laubach, W.
Mautner, F. I.
Ormsby, E. F.
Owens, F. W.
Sheffer, I. M.
Story, H. F.
Wonn, R. W.
Swarthmore
Franklin Inst.
Miller, B. L.
Swann, W. F. G.
Swarthmore Coll.
Brinkmann, H. W.
Carruth, P. W.
Dresden, A.
Marriott, R. W.
Schafer, A. T.
Washington
Patterson, B. C.
Shaub, H. C.
Wayne
Feather, E. H.
Snyder, A. K.
Wilkes-Barre
Hull, M. E.
Richards, T. R.

PUERTO RICO

Hato Rey
Bobonis, A.
Mayagüez
Univ. of Puerto Rico
García, M.
Morales, J. D.
Sánchez-Díaz, R.
Rio Piedras
Capo, B. G.
Univ. of Puerto Rico
Bocanegra-Saldaña, O.
Colón, J. A.
Laborde-Montaner, P.

RHODE ISLAND

East Providence
Foyle, G. A.
Kingston
Stauffer, J. R. K.
Rhode Island State Coll.
Cummings, M. H.
Haggerty, G. B.
Pease, E. M. J.
Pawtucket
Raines, L. R.
Portsmouth
Taliaferro, R. C.
Providence
Adams, R. B.
Albert, R. G.
Wehausen, J. V.
Brown Univ.
Adams, C. R.
Archibald, R. C.
Bennett, A. A.
Carlen, M. E.
Carrier, G. F.
Clarke, A. B.
Federer, H.
Gale, D.
Gildea, R. A. J.
Gilman, R. E.
Glauz, R. D.
Heins, M. H.
Jones, H. T.
Jónsson, B.
van der Kulk, W.
Lee, E. H.
Levy, S. L.
Lindsay, R. B.
Manning, H. P.
Martin, E. C.
Massey, W. S.
Moore, J. C.
Neugebauer, O. E.
Peck, J. E. L.
Prager, W.
Schatz, J. A.

Smith, E. C.
Spencer, D. E.
Stahlman, W. D.
Stewart, F. M.
Truell, R.
Wagner, D. H.
Wheeling, R. F.
Providence Coll.
Gallagher, E. H.
McKenney, J. L.
West Warwick
Brant, L.

SOUTH CAROLINA

Charleston
The Citadel
Conner, W. J.
Dye, L. A.
Sutton, C. S.
Coll. of Charleston
DeFrancesco, H. F.
Clemson
Clemson Agric. Coll.
Bryan, N. R.
Coker, E. C.
Harden, J. C.
Johnson, R. B.
Wilson, W. A.
Clinton
Sherratt, W. A.
Columbia
Univ. of South Carolina
Hedberg, E. A.
Jackson, J. B.
Lee, H.
Lytle, R. A.
Novak, J. D.
Robinson, L. V.
Weber, B. R.
Weber, W. W.
Williams, W. L.
Greenville
Bowen, L. H.
Greenwood
Cothran, F. E.
Orangeburg
Williams, E. S.
Spartanburg
Patten, W. E

SOUTH DAKOTA

Brookings
South Dakota State Coll.
MacDougal, H. B.
Walder, O. E.
Wente, I. L.
Vermillion
Beaty, M. H.

TENNESSEE

Chattanooga
Effron, M. W.
Watson, M.
Clarksville
Bright, S. K.
Cookeville
Bradshaw, C. L.
Hart, J. J.
Knoxville
Univ. of Tennessee
Albert, G. E.
Baum, W. R.
Blackstock, M. C.
Brown, H. E.
Cooley, J. A.
Eagle, E. L.
Eaves, E. D.
Ficken, F. A.
Givens, W.
Harrold, O. G.
Lee, H. L.
Marr, J. M.
Miller, D. D.
Posey, E. E.
Rivers, M. E.
Sinclair, A.
Snyder, W. S.
Wilson, R. L.
Memphis
Memphis State Coll.
Williams, J. F.
Southwestern at Memphis
MacQueen, M. L.
Tiller, G. L.
Nashville
Fisk Univ.
Boyd, E.
Lorch, L.
Vanderbilt Univ.
Boyce, M. G.
Hyden, J. A.
Miser, W. L.
Ratner, L. T.
Shanks, E. B.
Oak Ridge
Carbide and Carbon Chemicals Corp.
Chelius, L. G.
Garrett, G. A.
Fairchild Engine and Airplane Corp
Edmonson, N.
Rubin, T.
Oak Ridge National Lab.
Coveyou, R. R.
Dismuke, N. M.
Householder, A. S.
Nelson, M. L.
Sewanee
Bruton, G. S.

TEXAS

Abilene
Burnam, J. E.
Kean, H. P.
Mullings, M. E.
Arlington
Lassen, L. L.
Austin
Barrett, L. K.
Bassam, M. A. A.
Decherd, M. E.
Schmied, R. W.
Univ. of Texas
Ball, B. J.
Barrett, J. H.
Batchelder, P. M.
Bell, E. L.
Bright, H. F.
Burgess, C. E.
Clarkson, M. H.
Craig, H. V.
Dyer, S. E.
Ettlinger, H. J.
Greenwood, R. E.
Hamstrom, M. E.
Hurt, J. M.
Kittrell, L. B.
Lane, R. E.
Lubben, R. G.
MacNerney, J. S.
Moore, R. L.
Morris, H. A.
Parker, E. T.
Porcelli, P.
Porter, G. H.
Porter, M. B.
Smith, S. E.
Titt, E. W.
Vandiver, H. S.
Wall, H. S.
Brownsville
Crawford, J. N.
College Station
Agric. and Mech. Coll. of Texas
Basye, R. E.
Beeman, W. E.
Binney, J. H.
Daum, J. A.
Hurt, J. T.
Kent, J. T.
Klipple, E. C.
Koss, W. E.

McCulley, W. S.
Moore, B. C.
Tittle, M. E.
Commerce
Wright, C. B.
Corpus Christie
Moseley, M. P.
Dallas
McDermott, E.
Sorrells, R. C.
Wood, H. A.
Southern Methodist Univ.
Seale, R. Q.
Starr, D. W.
Denton
North Texas State Teachers Coll.
Brown, M. C.
Copp, G.
Parrish, H. C.
Texas State Coll. for Women
Ashburn, A. W.
Miller, H. C.
El Paso
Cates, G. T.
González, J. M.
Ft. Worth
Colquitt, L. A.
Galveston
Wells, N. W.
Georgetown
Whitmore, R. M.
Wilcox, M. E.
Gordon
Keown, E. R.
Houston
Eberman, J. W.
Harrje, H. J.
Weinstein, A. K.
Humble Oil and Refining Co.
Rust, W. M.
Slotnick, M. M.
Rice. Inst.
Bray, H. E.
Brunk, H. D.
Dean, A. C.
Houston, W. V.
Lovett, E. O.
MacLane, G. R.
Pfeiffer, P. E.
Reiter, H. J.
Ulrich, F. E.
Texas State Univ. for Negroes
Parson, R. H.
Westberry, J. E.
Univ. of Houston
Abernethy, J. J.
Aucoin, A. A.
Grover, B. B.
Newhouse, A.
Rader, C. B.
Wright, M.
Huntsville
Querry, J. W.
Kingsville
Dorroh, J. L.
Oliphant, E. A.
Lubbock
Michie, J. N.
Texas Tech. Coll.
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Lindsay, J. W.
Moser, L.
Parker, B.
Marlin
Falconer, B. L.
Mercedes
Shaw, W. F.
Plainview
McCoy, D.
San Antonio
Gough, M. de L.
Hallmark, W. J.
Hill, M. L.
San Marcos
Dibrell, R. A.
Tulloch, L. H.
Sherman
Dean, T. S.
Waco
Williams, W. J.

UTAH

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Hunsaker, N. C.
Tingey, V. H.
Provo
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Salt Lake City
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Univ. of Utah
Biesele, F. C.
Chamberlin, R. E.
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Henriques, A. S.
Hodges, R. S.
Horsfall, I. O.
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VERMONT

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Bristol
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Univ. of Vermont and State Agric. Coll.
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Fishback, W. T.
Larrivee, J. A.
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Montpelier
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Northfield
McConnon, M.

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Virginia Polytech. Inst.
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Lawrence, V. S.
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Charlottesville
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Univ. of Virginia
Botts, T. A.
Dresser, F. T.
Floyd, E. E.
Kasriel, R. H.
Klee, V. L.
Linfield, B. Z.
Lowdenslager, D. B.
McShane, E. J.
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Pace, W. E.
Schlauch, L. R.
Whyburn, G. T.
Whyburn, L. S.
Williams, R. F.
Dahlgren
Naval Proving Ground
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Bramble, C. C.
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Ross, C.
Danville
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McBee, E. L.
Ferrum
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Ft. Monroe
Atanasoff, J. V.
Fredericksburg
Burns, G. P.
Carter, H. C.
Hampton
National Advisory Com. for Aeronautics
Andrews, T. B.
de Moraes, C. A.
Sherman, W. L.

Hampton Institute
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Hollins College
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Hopewell
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Lexington
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Virginia Military Inst.
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Paxton, E. K.
Lynchburg
Lynchburg Coll.
Bahous, C. R.
Randolph-Macon Woman's Coll.
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Norfolk
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Sweet Briar
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Univ. of Richmond
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Stetson, J. M.

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Bell, R. F.
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Olympia
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Doyle, W.
Pullman
State Coll. of Washington
Brenner, J. L.
Butler, L. G.
Clement, P. A.
Cooke, K. L.
Hacker, S. G.
Heacock, R. R.
Knebelman, M. S.
Vatnsdal, J. R.
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Bennett, C. A.
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Hammond, E. E.
Jones, L. G.
Kinerk, H. E.
Moore, M. R.
Turner, M. H.
Seattle Pacific Coll.
Beegle, B. L.
Univ. of Washington
Avann, S. P.
Ball, R. W.
Ballantine, C. R.
Ballantine, F. W.
Ballantine, J. P.
Beaumont, R. A.
Birnbaum, Z. W.
Brownell, F. H.
Bryan, W. S.
Byrne, J. R.
Chapman, D. G.
Cramlet, C. M.
Davidson, M.
Dekker, D. B.
Haller, M. E.
Hewitt, E.
Hildebrand, J. L.
Jerbert, A. R.
Kingston, J. M.
Lee, C. Y.
Leipnik, R. B.
McFarlan, L. H.
Meyer, M. R.
Mo, Y.
Owen, D. B.
Paulson, E.
Peterson, R. P.
Tang, N. Y.
Tingey, F. H.
Winger, R. M.
Wisner, R. J.
Yagi, F.
Zuckerman, H. S.
Spokane
Ryan, D. R.
Tacoma
Goman, E. G.
Vancouver
Kuykendall, V.
Walla Walla
Hutchings, W. L.

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Tye, B.
Bluefield
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Huntington
Knifley, L. M.
Sandlin, W. T.
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West Virginia State Coll.
Aheart, A. N.
Crawley, A. B.
King, M. C.
Ponds, J. W.
Keyser
Iverson, P. F.
Lost City
Moyer, H. W.
Morgantown
West Virginia Univ.
Cunningham, A. B.
Davis, H. A.
Peters, I. D.
Pollock, H. S.
Reynolds, C. N.
Turner, B. M.
Vest, M. L.
Philippi
Sellers, W. H.

WISCONSIN

Appleton
Berry, A. C.
Stewart, J. C.
Beloit
Beloit Coll.
Conwell, H. H.
Finch, J. V.
Huffer, R. C.
Green Bay
Zarling, L. B.
Madison
Buck, E. F.
Carlson, K. H.
Ericksen, W. S.
Knaplund, P. W.
Wysocki, T. J.
Univ. of Wisconsin
Allen, F. E.
Ames, W. F.
Arnold, K. J.
Axt, P.
Bing, R. H.
Bruck, R. H.
Buck, R. C.
Buehler, R. J.
Campbell, D. W.
Colvin, B. H.
Dinneen, G. P.
Donoghue, W. F.
Eberlein, W. F.
Evans, H. P.
Fleming, S. G.
Fleming, W. H.
Friedman, H. A.
Fulkerson, D. R.
Fuller, L. E.
Fullerton, R. E.
Henriksen, M.
Ingraham, M. H.
Johnson, W. E.

Kleene, S. C.
Kleinfeld, E.
Langer, R. E.
Larson, C. W.
MacDuffee, C. C.
March, H. W.
Mayor, J R.
Mitchell, B E.
Owens, O. G.
Renno, J. G.
Rose, G. F.
Ross, J. D.
Schaeffer, A. C.
Slaby, H. T.
Smith, K. T.
Sokolnikoff, E. S.
Sokolowsky, D.
Steel, R. G. D.
Thomas, G. H. M.
Wendt, A.
Young, L. C.
Milton
Loomer, A. P.
Milwaukee
Beckwith, E. R.
Krug, R. E.
Vass, J. I.
Marquette Univ.
Moeller, A. C.
Pettit, H. P.
Talacko, J. V.
Wilczewski, J.
Mt. Mary Coll.
Vaudreuil, M. F.
State Teachers Coll.
Norris, R. E.
Robbins, O. M.
Univ. of Wisconsin in Milwaukee
Bardell, R. H.
Kenney, J. F.
Marden, M.
Parkinson, G. A.
Schuld, E. G.
Spitzbart, A.
Weeks, R. J.
Wolf, L. A.
Oshkosh
Bristow, L.
Racine
May, A. E.
Ripon
Alexander, A.B.
Divinsky, N.J.
Sheboygan
Battig, L.
West DePere
DeCleene, L. A. V.
Williams Bay
Chandrasekhar, S.
Smith, B.

WYOMING

Casper
Bailey, A. W.
Lamont
Bellamy, B. C.
Laramie
Univ. of Wyoming
Rechard, O. H.
Smith, W. N.
Varineau, V. J.

CANADA

ALBERTA
Calgary
Millar, J. G.
Edmonton
Univ. of Alberta
Campbell, J. W.
Cook, A. J.
Fostvedt, T. M.
Jacka, R. C.
Jordan, H. A.
Wyman, M.
BRITISH COLUMBIA
Vancouver
Univ. of British Columbia
Buchanan, D.
Derry, D.
Gage, W. H.
Hull, T. E.
James, R. D.
Jennings, S. A.
Leimanis, E.
McGregor, J. L.
Moyls, B. N.
Murdoch, D. C.
Simons, W H.
MANITOBA
Winnipeg
Univ. of Manitoba
Lawson, J. W.
McEwen, W. H.
Mendelsohn, N. S.
NEW BRUNSWICK
Fredericktón
Univ. of New Brunswick
Edwards, L. P.
Rothberger, F.
Staal, R. A.
Sackville
Crawford, W. S. H.
St. Joseph's
Poitras, A.
NOVA SCOTIA
Halifax
Adshead, J. G.
Walmsley, C.
ONTARIO
Chalk River
Kushneriuk, S. A.
Hamilton
Lehrman, S.
McMaster Univ.
Bankier, J. D.
Britton, F. R.
Findlay, W.
Johns, A. E.
McCallion, W. J.
Sumner, D. B.
Kingston
Queen's Univ.
Ellis, H. W.
Halperin, I.
Jeffery, R. L.
Miller, N.
Scheidegger, A. E.
London
Univ. of Western Ontario
Cole, R. H.
Kingston, H. R.
Magee, G. R.
Ottawa
MacPhail, M. S.
Rottenberg, J. A.
Toronto
DeLury, D. B.
Grant, A. A.
Pounder, D. W.
Waddell, M. E. G.
Wolfenden, H. H.
Univ. of Toronto
Beatty, S.
Burk, J. D.
Coleman, A. J.
Coxeter, H. S. M.
Crosby, W. J. R.
Fox, G. E. N.
Gilchrist, L.
Heaps, H. S.
Kassimatis, C.
Krieger, C. C.
Lorentz, G.
Pounder, I. R.
Robinson, G. de B.
Stanton, R. G.
Thompson, W. B.
Tutte, W. T.
Webber, W. J.
QUEBEC
Montreal
Benjamin, I.
Nichols, W. A.
Loyola Coll.
O'Connor, R. E.
McGill Univ.
Duffie, J. A. H.
Rosenthall, E.
Shaw, A. N.
Williams, W. L. G.
Univ. of Montreal
Gauthier, A.
Kozakiewicz, W.
L'Abbé, M
Stock, R.
SASKATCHEWAN
Saskatoon
Univ. of Saskatchewan
Ferns, H. H.
Grosswald, E.
Scherk, P.
Vigder, J. S.

AFRICA

ALGERIA
Algiers
Bourion, G.
EGYPT
Alexandria
Doss, R.
Cairo
Dana, V.
Dreyfus, D. M.
Guiza
Mostafa, H. H.
SOUTH AFRICA
Cape Town
Househam, K. O.
Pauc, C. Y.
Johannesburg
Tannenbaum, B.

ASIA

CHINA
Anshan
Chen, P. P.
Canton
Lu, C. H. L.
Changsha
Wu, S. C.
Hangchow
Chen, K. K.
Loo, C. T.
Hong Kong
Wong, Y. C.
Nanking
Chiang, L. F.
Peiping
Tsing Hua Univ.
Chien, W. Z.
Hua, L. K.
Min, S. H.
Tuan, H. F.
Shanghai
Hsü, H. T.
Shantung
Hsieh, L. F.
INDIA
Aligarh
Shah, S. M.
Annamalainagar
Rao, K. N.
Bangalore
Rao, B. S. M.
Bombay
Tata Inst. of Fundamental Research
Chandrasekharan, K.
Kosambi, D. D.
Levi, F. W. D.
Masani, P. R.
Delhi
Kalra, S. N.
Seth, B. R.
Madras
Ananda-Rau, K.
Vijayaraghavan, T.
Nagpur
Shrikhande, S. S.
New Delhi
Chand, P.
Seth, G. R.
Poona
Paranjpye, R. P.
Waltair
Minakshisundaram, S.
INDOCHINA
Saïgon
Wachs, S.
ISRAEL
Jerusalem
Jabotinsky, E. T.
Preiser, T. H.
Hebrew Univ.
Amirà, B. A.
Dvoretzky, A.
Fekete, M.
Fraenkel, A. A.
Levitzki, J.
Motzkin, T. S.
Schiffer, M. M.
Rehovot
Gillis, J.
Pekeris, C. L.
Tel-Aviv
Klein, M.
JAPAN
Kyûsyû
Kondô, M.
LEBANON
Beirut
Kennedy, E. S.
Mulholland, H. P.
MALAYA
Singapore
Oppenheim, A.
PAKISTAN
Peshkawar
Siddiqi, M R.
USSR
Moscow
Draim, N. A.
TURKEY
Ankara
Uluçay, C.
Istanbul
Berker, R.
Lanckton, A. L.
Zonguldak
Demir, H.

AUSTRALASIA

AUSTRALIA
Adelaide
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Brisbane
Raybould, E. H.
Hobart
Jaeger, J. C.
Löwig, H. F. J.
Melbourne
Univ. of Melbourne
Behrend, F. A.
Cherry, T M.
Schwerdtfeger, H. W. E.
Perth
Weatherburn, C. E.
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Univ. of Sydney
Carslaw, H. S.
Room, T. G.
Thorne, H. H.
NEW ZEALAND
Auckland
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Christ Church
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Dunedin
Martyn, W. J.

CENTRAL AND SOUTH AMERICA

ARGENTINA
Buenos Aires
Guglialmelli, J. P.
Naval School
DeCesare, E. A.
Univ. of Buenos Aires
Baidaff, B. I.
Barral-Souto, J.
Blaquier, J.
Bula, C. A.
Cotlar, M.
Gonzalez Dominguez, A.
Cuyo
Cabezas, P. G.
LaPlata
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Univ. of LaPlata
Cesco, R. P.
Duranona y Vedia, A.
Florio, J. S.
Machado, E. A. M.
Ricabarra, R. A.
Sagastume Berra, A. E.
Trejo, C. A.
Rosario
Univ. of Rosario
Gaspar, F. L.

Levi, B.
Santaló, L. A.
San Luis
Pérsico, H. A.
Turrin, G.
BRAZIL
Recife
Freire, L. B.
Rio de Janeiro
de Carvalho, C. A.
Dacorso Netto, C.
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Mello e Souza, J. C.
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Catholic Univ.
de Silve, L. O. T.
F. D. Inst. of Education
Quintella, A. N. M.
National School of Eng
Peixoto, M. C.
Peixoto, M. M.
Univ. of Brasil
Abdelhay, J.
Abdelhay, M. Y. de M. N.
Monteiro, A. A.
Mousinho, M. L. M.
Nachbin, L.
de Oliveira Castro, F. M.
Rocha, P. S.
São Paulo
Tech. Univ. of Aeronautics
Murnaghan, F. D.
Univ. of São Paulo
Breves Filho, J. A.
Castrucci, B.
Catunda, O.
Dias, C. L. da S.
Farah, E.
Furquim de Almeida, F.
Gomide, E. F.
Lima Filho, G. S.
Salvador
Santas, A. M. V.
COLOMBIA
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COSTA RICA
San José
Trejos-Fernández, J. J.
CUBA
Havana
González, M. O.
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MEXICO
Mexico
Medina, A.
Schoenbaum, E.
National Polytech Inst
Vallarta, M. S.
Univ. of Mexico
Barajas, A.
González-Baz, E.
Graef Fernandez, C.
Romero-Juárez, A.
Monterrey
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Thullen, P.
PERU
Lima
de Losada y Puga, C.
Secada, C. F.
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Goa, C. S.
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BELGIUM
Brussels
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Brünn
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Cartwright, M. L.
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Heyman, J.
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Mordell, L. J.
Pars, L. A.
Rankin, R. A.
Smithies, F.
Durham
Burchnall, J. L.
Willmore, T. J.
Englefield Green
McCrea, W. H.
Exeter
Hayman, W. K.
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Page, B. S.
Perrott, I. B.
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